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MULTIPLE SOLUTIONS FOR A Q-LAPLACIAN EQUATION ON AN ANNULUS

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Abstract. In this article, we study the q-Laplacian equation

$$-\Delta_q u = \big||x|-2\big|^a u^{p-1}, \quad 1<|x|<3,$$

where $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ and $q>1$. We prove that the problem has two solutions when a is large, and has two additional solutions when p is close to the critical Sobolev exponent $q^* = \frac{Nq}{N-q}$. A symmetry-breaking phenomenon appears which shows that the least-energy solution cannot be radial function.

1. Introduction

This article concerns the q-Laplacian equation

$$\begin{gathered} -\Delta_q u = \Phi_a u^{p-1} \quad \text{in } \Omega, \\ u>0 \quad \text{in } \Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Delta_q u = div(|\nabla u|^{q-2}\nabla u)$, $\Omega = \{x\in\mathbb{R}^N | 1<|x|<3\}$ is an annulus in $\mathbb{R}^N$, $N\geq 3$, $a>0$, $p>q>1$ and Φ_a is the radial function

$$\Phi_a(x) = \big||x|-2\big|^a.$$

Equation (1.1) is an extension of the problem

$$\begin{gathered} -\Delta_q u = |x|^a u^{p-1} \quad \text{in } |x|<1, \\ u=0 \quad \text{on } |x|=1. \end{gathered} \tag{1.2}$$

Equation (1.2) can be seen as a natural extension to the annular domain Ω of the celebrated Hénon equation with Dirichlet boundary conditions

$$\begin{gathered} -\Delta u = |x|^a u^{p-1} \quad \text{in } |x|<1, \\ u=0 \quad \text{on } |x|=1. \end{gathered} \tag{1.3}$$

This equation was proposed by Hénon in [13] when he studied rotating stellar structures.

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For $a > 0$, $2 < p < 2^* = 2N/(N-2)$, we generalize (1.3) to the case of q-Laplacian. In fact, the weight function $\Phi_a(x)$ reproduces on Ω a similar qualitative behavior of $|\cdot|^a$ on the unit ball B in $\mathbb{R}^N$.

A standard compactness argument shows that the infimum

$$\inf_{0\neq u\in H_0^1(B)} \frac{\int_B |\nabla u|^2 dx}{\big(\int_B |x|^a |u|^p dx\big)^{2/p}} \tag{1.4}$$

is attained for $2 < p < 2^*$ and any $a > 0$. Ni [16] proved that the infimum

$$\inf_{0\neq u\in H_{0,\mathrm{rad}}^1(B)} \frac{\int_B |\nabla u|^2 dx}{\big(\int_B |x|^a |u|^p dx\big)^{2/p}} \tag{1.5}$$

is attained for any $2 < p < 2^* + 2a/(N-2)$ by a function in $H^1_{0,\mathrm{rad}}(B)$, the space of radial $H^1_0(B)$ functions. Therefore, radial solutions of (1.3) also exist for supercritical exponents p. Indeed, $H^1_{0,\mathrm{rad}}(B)$ shows a power-like decay away from the origin (as a result of the Strauss Lemma, see [1, 21]) that combines with the weight $|x|^a$ and provides the compactness of the embedding $H^1_{0,\mathrm{rad}}(B) \subset L^p(B)$ for any $2 < p < 2^* + 2a/(N-2)$.

When $a > 0$, Smets, Su and Willem obtained some symmetry-breaking results for (1.3) in [23]. They proved that minimizers of (1.4) could not be radial, at least for a sufficiently large. Consequently, (1.3) had at least two solutions when a was large (see also [24]).

Serra [20] proved that (1.3) had at least one nonradial solution when $p = 2^*$, and in [4] Badiale and Serra obtained the existence of more than one solutions to (1.3) also for some supercritical values of p. These solutions are nonradial and they are obtained by minimization under suitable symmetry constraints.

Cao and Peng [8] proved that, for p sufficiently close to 2^*, the ground-state solutions of (1.3) possessed a unique maximum point whose distance from ∂B tended to zero as $p \to 2^*$. And they also proved the same results in the case of q-laplacian of the Hénon equation (see [9]).

This result was improved in [18], where multi-bump solutions for the Hénon equation with almost critical Sobolev exponent p were found, by applying a finite-dimensional reduction. These solutions are not radial, though they are invariant under the action of suitable subgroups of $O(N)$, and they concentrate at boundary points of the unit ball B in $\mathbb{R}^N$ as $p \to 2^*$. However, the role of a is a static one (for more results for $p \approx 2^*$, see also [19]).

When the weight disappeared; i.e. $a = 0$, Brezis and Nirenberg proved in [3] that the ground state solution of $-\Delta u = u^p$ in $H^1_0(\Omega)$ was not a radial function. Actually, they proved that both a radial and a nonradial (positive) solution arise as $p \approx 2^*$.

When the weight in (1.1) disappeares; i.e. $a = 0$, Li and Zhou [14] proved that existence of multiple solutions to the p-Laplacian type elliptic problem

$$\begin{gathered} -\Delta_p u(x) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.6}$$

where Ω was a bounded domain in $\mathbb{R}^N$ $(N > 1)$ with smooth boundary $\partial\Omega$, and $f(x,u)$ went asymptotically in u to $|u|^{p-2}u$ at infinity.

When $q = 2$ in (1.1), Calanchi, Secchi and Terraneo [10] obtained multiple solutions for a Hénon-like equation on $1 < |x| < 3$. For more results about asymptotic estimates for solutions of the Hénon equation with a large, one can see [5, 6].

This paper is mainly motivated by [10]. We want to extend the results in [10] to a general q-Laplacian problem. We consider the critical points of

$$R_{a,p} = \frac{\int_\Omega |\nabla u|^q dx}{\big(\int_\Omega \Phi_a |u|^p dx\big)^{q/p}}, \quad u \in W_0^{1,q}(\Omega)\backslash\{0\}, \tag{1.7}$$

which is the Rayleigh quotient associated with (1.1).

Our main results are as follows.

Theorem 1.1. *Assume that $p \in (q, q^*)$. For a large enough, any ground state $u_{a,p}$ is a nonradial function.*

Theorem 1.2. *Assume that $a > 0$. For p close to q^* the quotient $R_{a,p}$ has at least two nonradial local minima.*

Theorem 1.3. *There exist $\overline{a} > 0$ and $q < \overline{p} < q^*$ such that for all $a \geq \bar{a}$ and $\bar{p} \leq p < q^*$, it results that the mountain-pass level c(defined by (2.14))is a critical value for $R_{a,p}$ and it is attained by a nonradial function.*

To the best of our knowledge, the results we obtain in this article are new. To prove Theorem 1.1, the key is to give the energy estimates of both ground solutions and radial symmetry solutions when p is fixed. Applying these estimates, we can prove that ground solution is not radially symmetric when $a \to \infty$. Using the same arguments in [10], we can prove Theorem 1.2 and Theorem 1.3. We would like to point out that our problem is more complicated than the problem in [10].

This paper is organized as follows. In Section 2 we prove our main results. In Section 3 we describe the behavior of ground-state solution of (1.1) when $p < q^*$ is fixed and $a \to +\infty$. Although the conclusion is not as precise as in the case $p \to q^*$, we can nevertheless show that a sort of concentration near the boundary $\partial\Omega$ still appears.

2. Proofs of main results

2.1. **Proof of Theorem 1.1.** Denote

$$S_{a,p} = \inf_{u \in W_0^{1,q}(\Omega)\backslash\{0\}} R_{a,p}(u). \tag{2.1}$$

It is easy to prove that up to a scaling for p subcritical $S_{a,p}$ is attained by a function $u_{a,p}$ that satisfies (1.1). To prove that for a is large any solution $u_{a,p}$ is not radial first we need an estimate from above of $S_{a,p}$.

Lemma 2.1. *Assume that $p \in (q, q^*)$. There exists $\bar{a}$ such that for $a \geq \bar{a}$,*

$$S_{a,p} \leq C a^{q-N+\frac{qN}{p}}. \tag{2.2}$$

Proof. We use the same techniques as in [23]. Let ϕ be a smooth function with support in the unit ball B. Let us consider $\phi_a(x) = \phi(a(x - x_a))$, where $x_a = (3 - \frac{1}{a}, 0, \dots, 0)$. Since ϕ_α has support in the ball $B(x_a, \frac{1}{a})$, by the change of variable $y = a(x - x_a)$, we obtain

$$\int_\Omega \Phi_a \phi_a^p(x)dx = \int_{B(x_a,\frac{1}{a})} \big||x| - 2\big|^a \phi_a^p(x)dx \geq \big(1 - \frac{2}{a}\big)^a a^{-N} \int_B \phi^p(y)dy. \tag{2.3}$$

Furthermore,

$$\int_\Omega |\nabla\phi_a|^q dx = a^q \int_\Omega |\nabla\phi(a(x-x_a))|^q dx = a^{q-N}\int_B |\nabla\phi|^q dx. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$R_{a,p}(\phi_a) = \frac{\int_\Omega |\nabla\phi_a|^q dx}{\big(\int_\Omega \Phi_a(x)\phi_a^p dx\big)^{q/p}} \le Ca^{q-N+\frac{qN}{p}}.$$

Hence we obtain

$$S_{a,p} \le R_{a,p}(\phi_a) \le Ca^{q-N+\frac{qN}{p}}$$

for all a large enough. □

Let $W^{1,q}_{0,\mathrm{rad}}(\Omega)$ be the space of radially symmetric functions of $W^{1,q}_0(\Omega)$. In the sequel, we denote $u(x) = u(|x|)$ for $u \in W^{1,q}_{0,\mathrm{rad}}(\Omega)$.

Consider the minimization problem

$$S^{\mathrm{rad}}_{a,p} = \inf_{u\in W^{1,q}_{0,\mathrm{rad}}(\Omega)\setminus\{0\}} R_{a,p}(u). \tag{2.5}$$

It is well known that any minimizers of (2.5) can be scaled so as to be solutions of (1.1). Thus, we will use freely this fact in the sequel.

In the following lemma, we obtain an estimate of the energy $S^{\mathrm{rad}}_{a,p}$ as $a \to \infty$.

Lemma 2.2. *Let $p > q$. As $a \to \infty$, there exist two constants C_1, C_2 depending on p such that*

$$0 < C_1 \le \frac{S^{\mathrm{rad}}_{a,p}}{a^{q-1+\frac{q}{p}}} \le C_2 < +\infty. \tag{2.6}$$

Proof. Let $\phi \in C_0^\infty(\Omega)$ be a positive, radial function, and set

$$\phi_a(x) = \phi_a(|x|) = \phi(a(|x| - 3 + 3/a)).$$

Then

$$\int_\Omega \Phi_a\phi_a^p dx \ge \big(1-\frac{2}{a}\big)^a a^{-1}\int_\Omega \phi^p dx$$

and

$$\begin{aligned}
\int_\Omega |\nabla\phi_a|^q dx &= \omega_{N-1}\int_{3-\frac{2}{a}}^3 (\phi_a'(r))^q r^{N-1} dr \\
&= \omega_{N-1}\int_1^3 a^q(\phi'(s))^q\Big(\frac{s}{a}+3-\frac{3}{a}\Big)^{N-1} a^{-1} ds \\
&= a^{q-1}\omega_{N-1}\int_1^3 (\phi'(s))^q\Big(\frac{s+3a-3}{sa}\Big)^{N-1} s^{N-1} ds \\
&\le 3^{N-1}a^{q-1}\int_\Omega |\nabla\phi|^q dx,
\end{aligned}$$

since $1 \le \frac{s+3a-3}{sa} \le 3$. Therefore,

$$R^{\mathrm{rad}}_{a,p}(\phi_a) = \frac{\int_\Omega |\nabla\phi_a|^q dx}{\big(\int_\Omega \Phi_a(x)\phi_a^p dx\big)^{q/p}} \le C(a,p)a^{q-1+\frac{q}{p}},$$

where

$$C(a,p) = \frac{3^{N-1}\int_\Omega |\nabla\phi|^q dx}{(1-\frac{2}{a})^{qa/p}\big(\int_\Omega \phi^p dx\big)^{q/p}},$$

for any $p > q$ and $a > 1$, $C(a,p) \le C_2$. So we obtain $S^{\rm rad}_{a,p} \le C_2 a^{q-1+\frac{q}{p}}$.

To find the lower bound C_1, we will do some scaling. Let us define the functions $\phi_1 : [1,2] \to [1,2]$ and $\phi_2 : [2,3] \to [2,3]$ as follows:

$$\phi_1(r) = 2 - (2-r)^b, \quad \phi_2(r) = 2 + (r-2)^b,$$

where $b \in (0,1)$ will be chosen later. It is obvious that we can obtain a piecewise C^1 homeomorphism $\phi : [1,3] \to [1,3]$ by gluing ϕ_1 and ϕ_2. Now, for any radial function $u \in W^{1,q}_0(\Omega)$, setting $\upsilon(\rho) = u(\phi(\rho))$ and choosing $b = 1/(a+1)$, we obtain

$$\begin{aligned}
&\int_\Omega \Phi_a(x)\big|u(|x|)\big|^p dx \\
&= \omega_{N-1} \int_1^3 \Phi_a(r)|u(r)|^p r^{N-1} dr \\
&\le 3^{N-1}\omega_{N-1} \int_1^3 \Phi_a(r)|u(r)|^p dr \\
&= 3^{N-1}\omega_{N-1}\Big(\int_1^2 \Phi_a(\phi_1(\rho))|\upsilon(\rho)|^p \phi_1'(\rho) d\rho + \int_2^3 \Phi_a(\phi_2(\rho))|\upsilon(\rho)|^p \phi_2'(\rho) d\rho\Big) \\
&= 3^{N-1}\omega_{N-1} b \int_1^3 |\upsilon(\rho)|^p d\rho
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega |\nabla u|^q dx &= \omega_{N-1} \int_1^3 |u'(r)|^q r^{N-1} dr \\
&\ge \omega_{N-1} \int_1^3 |u'(r)|^q dr \\
&= \omega_{N-1}\Big(\int_1^2 |\upsilon'(\rho)|^q \frac{1}{|\phi_1'(\rho)|^{q-1}} d\rho + \int_2^3 |\upsilon'(\rho)|^q \frac{1}{|\phi_2'(\rho)|^{q-1}} d\rho\Big) \\
&= \omega_{N-1} \frac{1}{b^{q-1}} \int_1^3 |\upsilon'(\rho)|^q |\rho-2|^{(1-b)(q-1)} d\rho \\
&\ge \omega_{N-1} \frac{1}{b^{q-1}} \int_1^3 |\upsilon'(\rho)|^q |\rho-2|^{(q-1)} d\rho.
\end{aligned}$$

Therefore,

$$R_{a,p}(u) \ge C a^{q-1+\frac{q}{p}} \inf_{\upsilon \in W^{1,q}_0(\Omega)\setminus\{0\}} \frac{\int_1^3 |\upsilon'(\rho)|^q |\rho-2|^{(q-1)} d\rho}{\big(\int_1^3 |\upsilon(\rho)|^p d\rho\big)^{q/p}}, \tag{2.7}$$

where C depends only on N. To end the proof, we will show that the right-hand side of (2.7) is greater than zero. This follows from some general Hardy-type inequality (see [17, Theorem 11.4]), but we present here an elementary proof for the sake of completeness. Indeed, given $\upsilon \in W^{1,p}_{0,{\rm rad}}(\Omega)$, for $\rho \in [1,2]$, we can write

$$\begin{aligned}
|\upsilon(\rho)| &= |\upsilon(\rho) - \upsilon(1)| \\
&\le \int_1^\rho |\upsilon'(t)||2-t|^{(q-1)/q} \frac{1}{|2-t|^{(q-1)/q}} dt \\
&\le \Big(\int_1^\rho |\upsilon'(t)|^q |2-t|^{q-1} dt\Big)^{1/q} \Big(\int_1^\rho \frac{1}{|2-t|} dt\Big)^{(q-1)/q}
\end{aligned}$$

$$\leq \Big(\int_1^3 |\upsilon'(t)|^q |2-t|^{q-1} dt \Big)^{1/q} \Big(-\ln|2-\rho| \Big)^{(q-1)/q}.$$

Hence,

$$\begin{aligned}
\int_1^2 |\upsilon(\rho)|^p d\rho &\leq \Big(\int_1^3 |\upsilon'(t)|^q |2-t|^{q-1} dt \Big)^{p/q} \int_1^2 \Big(-\ln|2-\rho| \Big)^{\frac{p(q-1)}{q}} d\rho \\
&= \Big(\int_1^3 |\upsilon'(t)|^q |2-t|^{q-1} dt \Big)^{p/q} \int_0^\infty t^{\frac{p(q-1)}{q}} e^{-t} dt \\
&\leq \Gamma\Big(\frac{p(q-1)+q}{q}\Big) \Big(\int_1^3 |\upsilon'(t)|^q |2-t|^{q-1} dt \Big)^{p/q}
\end{aligned}$$

and in a similar way,

$$\int_2^3 |\upsilon(\rho)|^p d\rho \leq \Gamma\Big(\frac{p(q-1)+q}{q}\Big) \Big(\int_1^3 |\upsilon'(t)|^q |2-t|^{q-1} dt \Big)^{p/q}.$$

Therefore,

$$\int_1^3 |\upsilon'(t)|^q |2-t|^{q-1} dt \geq \Big(\int_1^3 |\upsilon(\rho)|^p d\rho \Big)^{q/p} \frac{1}{2^{q/p} \Gamma\big(\frac{p(q-1)+q}{q}\big)^{q/p}}.$$

This implies that the infimum in (2.7) is strictly positive. There exists a constant $C_1 > 0$ such that $S_{a,p}^{\rm rad} \geq C_1 a^{q-1+\frac{q}{p}}$. $\square$

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. Note that $q - N + \frac{q}{p}N < q - 1 + \frac{q}{p}$. Then it follows from (2.2) and (2.6) that $S_{a,p} < S_{a,p}^{\rm rad}$ when a is large. So any ground state $u_{a,p}$ is a nonradial function. $\square$

2.2. **Proof of Theorem 1.2.** Now we consider any minimum $u_{a,p}$. The following proposition describes the profile of $u_{a,p}$ as $p \to q^*$.

Proposition 2.3. *Let $p \in (q, q^*)$ and $a > 0$. Any minimum $u_{a,p}$ of $R_{a,p}(u)$ in $W_0^{1,q} \backslash \{0\}$ satisfies for some $x_0 \in \partial\Omega$,*

(1) *$|\nabla u_{a,p}|^q \to \mu\delta_{x_0}$ weakly in sense of measure as $p \to q^*$;*
(2) *$|u_{a,p}|^{q^*} \to \nu\delta_{x_0}$ weakly in sense of measure as $p \to q^*$,*

where $\mu > 0$ and $\nu > 0$ are such that $\mu \geq S_{0,q^*}\nu^{q/q^*}$ and δ_x is the Dirac mass at x.

Since the result can be proved by using the same arguments in [8], with some minor modifications, we omit its proof here.

Remark 2.4. Proposition 2.3 implies that any ground state solution concentrates in a single point at the boundary as $p \to q^*$ and consequently this solution is not radial. This symmetry breaking can be also proved by using a continuation argument as in [3]. Indeed, $\lim_{p\to q^*} S_{a,p} = S_{0,q^*}$, and since $S_{0,q^*} < S_{0,q^*}^{\rm rad}$ we conclude as in [3] that ground state of $S_{a,p}$ cannot be radially symmetric as $p \to q^*$.

Let

$$\Omega^- = \big\{ x \in \mathbb{R}^N : 1 < |x| < 2 \big\}, \quad \Omega^+ = \big\{ x \in \mathbb{R}^N : 2 < |x| < 3 \big\},$$

$$\Sigma = \big\{ u \in W_0^{1,p}(\Omega)\backslash\{0\} : \int_{\Omega^-} |\nabla u|^q dx = \int_{\Omega^+} |\nabla u|^q dx \big\},$$

$$\wedge = \big\{u \in W_0^{1,p}(\Omega) : \int_{\Omega^-} |\nabla u|^q dx > \int_{\Omega^+} |\nabla u|^q dx\big\}.$$

We already know that any global minimizer of $R_{a,p}$ yields a first solution $u_{a,p}$. From Proposition 2.3 we know that this solution concentrates at precisely one point of the boundary $\partial\Omega$. Noting that this boundary has two connected components, we will minimize $R_{a,p}$ over the set Λ of $W_0^{1,q}$ functions which concentrate at the opposite component of boundary. We need a careful estimate to show that minimizers fall inside the interior of Λ.

To obtain a second local minimizer, we assume without loss of generality that any $u_{a,p}$ concentrates at some point on the sphere $|x| = 3$. After a rotation, we can even assume that any $u_{a,p}$ concentrates at the point $(3, 0, \dots, 0)$.

Lemma 2.5. *If $a > 0$, then there exists $\delta > 0$ such that*

$$\liminf_{p\to q^*} T_{a,p} > S_{0,q^*} + \delta, \tag{2.8}$$

where $T_{a,p} = \inf_{u\in\Sigma} R_{a,p}(u)$.

Proof. First we prove that $T_{a,p}$ is attained by a function $\upsilon_{a,p} \in \Sigma$. Consider a minimizing sequence $\{u_n\}$ for $T_{a,p}$. We can apply the homogeneity of $R_{a,p}$ and assume that $\int_\Omega |\nabla u_n|^q dx = 1$. Passing to a subsequence, u_n converges to $\upsilon = \upsilon_{a,p}$ weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^s(\Omega)$, for all $s \in (q, q^*)$. What we have to check is that $\upsilon \in \Sigma$ (proving that the convergence of u_n to υ is strong). From the strong convergence in $L^s(\Omega)$ we have that

$$R_{a,p}(\upsilon) \le \frac{1}{\big(\int_\Omega \Phi_a(x)|\upsilon|^p dx\big)^{q/p}} = T_{a,p} \tag{2.9}$$

and particularly $\upsilon \neq 0$. We may assume that $\upsilon \ge 0$ in Ω. For the sake of contradiction, assume that

$$\int_{\Omega^+} |\nabla \upsilon|^q dx < \frac{1}{2}.$$

Fix a nonnegative smooth function $\psi_1 \in C_0^\infty(\Omega^+)$, $\psi_1 \neq 0$ and $\delta \ge 0$. Setting $u = \upsilon + \delta\psi_1$ from the positivity of υ and ψ_1, we have, for $\delta > 0$,

$$\int_\Omega \Phi_a(x)|\upsilon|^p dx < \int_\Omega \Phi_a(x)|u|^p dx. \tag{2.10}$$

Now,

$$\int_{\Omega^+} |\nabla u|^q dx = \int_{\Omega^+} |\nabla\upsilon + \delta\nabla\psi_1|^q dx, \tag{2.11}$$

if we define $f_1 : [0, +\infty] \to \mathbb{R}$ by

$$f_1(\delta) = \int_{\Omega^+} |\nabla\upsilon + \delta\nabla\psi_1|^q dx,$$

we know that f_1 is continuous and $f_1(0) < \frac{1}{2}$, $\lim_{\delta\to\infty} f_1(\delta) = +\infty$. Hence there exists $\delta_1 > 0$ with $f_1(\delta_1) = 1/2$. We can reason in an analogous way if $\int_{\Omega^-} |\nabla\upsilon|^q dx < \frac{1}{2}$ in order to find $\delta_2 \ge 0$ and $\psi_2 \ge 0$ such that $\int_{\Omega^-} |\nabla(\upsilon + \delta_2\psi_2)|^q dx = 1/2$.

From (2.10), this implies that there exists $\omega = \upsilon + \delta_1\psi_1 + \delta_2\psi_2 \in \Sigma$ such that $R_{a,p}(\omega) < T_{a,p}$, which yields a contradiction.

Finally we must have that $\upsilon_{a,p} \in \Sigma$ is a minimum point. Moreover for any $a > 0$ and $q < p < q^*$, we have $T_{a,p} \geq S_{a,p}$. We want to prove that the inequality is strict at least for $p \to q^*$. In fact assume on the contrary that

$$\liminf_{p \to q^*} T_{a,p} = \liminf_{p \to q^*} R_{a,p}(\upsilon_{a,p}) = S_{0,q^*}.$$

By the definition of S_{0,q^*} and Hölder inequality we obtain, for a subsequence $p = p_k \to q^*$,

$$\begin{aligned}
S_{0,q^*} &\leq \frac{\int_\Omega |\nabla \upsilon_{a,p}|^q dx}{\left(\int_\Omega |\upsilon_{a,p}|^{q^*} dx\right)^{\frac{q}{q^*}}} \\
&\leq |\Omega|^{\frac{(q^*-p)q}{q^* p}} \frac{\int_\Omega |\nabla \upsilon_{a,p}|^q dx}{\left(\int_\Omega |\upsilon_{a,p}|^p dx\right)^{q/p}} \\
&\leq |\Omega|^{\frac{(q^*-p)q}{q^* p}} \frac{\int_\Omega |\nabla \upsilon_{a,p}|^q dx}{\left(\int_\Omega \Phi_a(x) |\upsilon_{a,p}|^p dx\right)^{q/p}} = S_{0,q^*} + o(1),
\end{aligned}$$

since the weight satisfies $\Phi_a \leq 1$. In particular, we obtain

$$\frac{\int_\Omega |\nabla \upsilon_{a,p}|^q dx}{\left(\int_\Omega |\upsilon_{a,p}|^{q^*} dx\right)^{\frac{q}{q^*}}} \to S_{0,q^*}$$

and $\upsilon_{a,p}$ is a minimizing sequence of S_{0,q^*}.

By the same argument as Cao and Peng [8, Theorem 1.1], we can prove that $\upsilon_{a,p}$ concentrates at precisely one point of the boundary $\partial\Omega$. This yields a contradiction since $\int_{\Omega^+} |\nabla \upsilon_{a,p}|^q dx = \int_{\Omega^-} |\nabla \upsilon_{a,p}|^q dx$. □

Now we consider the points

$$x_{0,\varepsilon} = x_0 = \left(3 - \frac{1}{|\ln \varepsilon|}, 0, \dots, 0\right), \quad x_{1,\varepsilon} = x_1 = \left(1 + \frac{1}{|\ln \varepsilon|}, 0, \dots, 0\right),$$

and the function

$$U(x) = \frac{1}{(1 + |x|^{\frac{q}{q-1}})^{(N-q)/q}}.$$

We recall that S_{0,q^*} is not achieved on any proper subset of $\mathbb{R}^N$, and that it is independent of Ω. However, it is known that $S_{0,q^*}(\mathbb{R}^N)$ is achieved, and all the minimizers can be written in the form

$$\mathcal{U}_{\theta,y} = \frac{1}{(\theta^2 + |x - y|^{\frac{q}{q-1}})^{(N-q)/q}}, \quad \theta > 0,\ y \in \mathbb{R}^N.$$

We set

$$U_\varepsilon^i(x) = \varepsilon^{-\frac{N-q}{q}} U\Big(\frac{x - x_i}{\varepsilon^{(q-1)/q}}\Big) = \frac{1}{(\varepsilon + |x - x_i|^{\frac{q}{q-1}})^{\frac{N-q}{q}}}$$

and denote by $\psi_i (i = 0, 1)$ two cut-off functions such that $0 \leq \psi_i \leq 1$, $|\nabla \psi_i| \leq C |\ln \varepsilon|$ for some constant $C > 0$, and

$$\psi_i = \begin{cases} 1 & \text{if } |x - x_i| < \frac{1}{2|\ln \varepsilon|}, \\ 0 & \text{if } |x - x_i| \geq \frac{1}{|\ln \varepsilon|}. \end{cases}$$

The following lemma shows that the truncated functions

$$u_\varepsilon^i = \psi_i(x) U_\varepsilon^i(x), \quad i = 0, 1, \tag{2.12}$$

are almost minimizers for S_{0,q^*}. Since it is an easy modification of the arguments of [8], we omit the proof of this fact.

Lemma 2.6. *If $a > 0$, then*

$$\lim_{p\to q^*} R_{a,p}(u_\varepsilon^i) = S_{o,q^*} + K(\varepsilon), \tag{2.13}$$

with $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$.

As a direct consequence of Lemma 2.6, we obtain the following result.

Corollary 2.7. $S_{0,q^*} = S_{\alpha,q^*}$.

Proof. On one hand, $S_{0,q^*} \le S_{a,q^*}$ since $\Phi_a(|x|) \le 1$. On the other hand by Lemma 2.6, we have

$$R_{a,q^*}(u_\varepsilon^i) = \lim_{p\to q^*} R_{a,p}(u_\varepsilon^i) = S_{0,q^*} + K(\varepsilon),$$

which implies that $S_{0,q^*} + K(\varepsilon) \ge S_{a,q^*}$ for every $\varepsilon > 0$. Letting $\varepsilon \to 0$, we infer $S_{0,q^*} \ge S_{a,q^*}$. Therefore $S_{0,q^*} = S_{a,q^*}$. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $u_{a,p}$ be a ground state solution. Let us suppose that it concentrates on the outer boundary. The infimun of $R_{a,p}$ on $\bar\Lambda$ is attained. However it cannot be attained on the boundary $\partial\Lambda = \Sigma$. In fact, from Lemma 2.5, we obtain

$$\inf_{\Sigma} R_{a,p}(u) > S_{0,q^*} + \delta, \quad \text{as } p \to q^*$$

and

$$\inf_{\Lambda} R_{a,p}(u) \le R_{a,p}(u_\varepsilon^1) \to S_{0,q^*} + K_1(\varepsilon), \quad \text{as } p \to q^*,$$

since $u_\varepsilon^1 \in \Lambda$ for ε sufficiently small. Then the infimum is attained in an interior point of Λ and is therefore a critical point of $R_{a,p}$. □

2.3. **Proof of Theorem 1.3.** Now we prove the existence of a third nonradial solution, in the previous section we proved the existence of two solutions of (1.1) which were local minima of Rayleigh quotient for p near q^*. We would expect another critical point of $R_{a,p}$ located in some sense between these minimum points.

For ε sufficiently small let $u_\varepsilon^i = \psi_i(x)U_\varepsilon^i(x), i \in \{0,1\}$, be defined as in (2.12). We will verify that $R_{a,p}$ has the mountain-pass geometry. Let us introduce the mountain-pass level

$$c = c(a,p) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} R_{a,p}(\gamma(t)), \tag{2.14}$$

where

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,q}(\Omega)) : \gamma(0) = u_\varepsilon^0, \gamma(1) = u_\varepsilon^1\}$$

is the set of continuous paths joining u_ε^0 with u_ε^1. We claim that c is a critical value for $R_{a,p}$.

We start to prove that c is larger, uniformly with respect to ε, than the values of the functional $R_{a,p}$ at the points u_ε^0 and u_ε^1.

Lemma 2.8. *Set $M_\varepsilon = \max\{R_{a,p}(u_\varepsilon^0), R_{a,p}(u_\varepsilon^1)\}$. There exists $\sigma > 0$ such that $c \ge M_\varepsilon + \sigma$ uniformly with respect to ε.*

Proof. We prove that there exists σ such that for all $\gamma \in \Gamma$,

$$\max R_{a,p}(\gamma(t)) \geq M_\varepsilon + \sigma.$$

A simple continuity argument shows that for every $\gamma \in \Gamma$ there exists t_γ such that $\gamma(t_\gamma) \in \Sigma$, where

$$\Sigma = \big\{u \in W_0^{1,p}(\Omega)\backslash\{0\} : \int_{\Omega^-} |\nabla u|^q dx = \int_{\Omega^+} |\nabla u|^q dx\big\}.$$

In fact the map

$$t \in [0,1] \mapsto \int_{\Omega^+} |\nabla \gamma(t)|^q dx - \int_{\Omega^-} |\nabla \gamma(t)|^q dx$$

is continuous and it takes a negative value at $t = 0$ and a positive value at $t = 1$. It follows from Lemma 2.5 that for p near q^* there exists $\delta > 0$ such that

$$\max_{t\in[0,1]} R_{a,p}(\gamma(t)) \geq R_{a,p}(\gamma(t_\gamma)) \geq \inf_{u\in\Sigma} R_{a,p}(u) \geq S_{0,q^*} + \delta.$$

On the other hand, for ε small enough, we have

$$M_\varepsilon < S_{0,q^*} + \frac{\delta}{2}.$$

This completes the proof. □

By the previous estimates, we can show that c is a critical level for $R_{a,p}$. As a result a further nonradial solution to (1.1) arises.

Proof of Theorem 1.3. By the previous results, we can apply a deformation argument (see [1, 22]) to prove that c is a critical level and it is achieved (since the PS condition is satisfied) by a function υ. By the asymptotic estimate (2.6) for the radial level $S_{a,p}^{\rm rad}$, we know that there exists a constant C independent of p such that

$$S_{a,p}^{\rm rad} \geq C a^{q-1+\frac{q}{p}}.$$

Particularly, we obtain $S_{a,p}^{\rm rad} \to +\infty asa \to +\infty$. Therefore we can choose a_0 such that

$$S_{a,p}^{\rm rad} \geq 3S_{0,q^*} \quad \forall a \geq a_0.$$

Define $\zeta \in \Gamma$ by $\zeta(t) = tu_\varepsilon^1 + (1-t)u_\varepsilon^0$ for all $t \in [0,1]$, and let $\tau \in [0,1]$ be such that

$$R_{a,p}(\zeta(\tau)) = \max_{t\in[0,1]} R_{a,p}(\zeta(t)).$$

Noting that u_ε^0 and u_ε^1 have disjoint support one has, for ε small enough, we have

$$\begin{aligned}
R_{a,p}(\upsilon) &= c \leq R_{a,p}(\zeta(\tau)) \\
&= \frac{\int_\Omega |\nabla(\tau u_\varepsilon^1 + (1-\tau)u_\varepsilon^0)|^q dx}{\big(\int_\Omega \Phi_a|\tau u_\varepsilon^1 + (1-\tau)u_\varepsilon^0|^p dx\big)^{q/p}} \\
&= \frac{\int_\Omega \tau^q|\nabla u_\varepsilon^1|^q dx + \int_\Omega (1-\tau)^q|\nabla u_\varepsilon^0|^q dx}{\big(\tau^p\int_\Omega \Phi_a|u_\varepsilon^1|^p dx + (1-\tau)^p\int_\Omega \Phi_a|u_\varepsilon^0|^p dx\big)^{q/p}} \\
&\leq \frac{\tau^q\int_\Omega |\nabla u_\varepsilon^1|^q dx}{\big(\tau^p\int_\Omega \Phi_a|u_\varepsilon^1|^p dx\big)^{q/p}} + \frac{(1-\tau)^q\int_\Omega |\nabla u_\varepsilon^0|^q dx}{\big((1-\tau)^p\int_\Omega \Phi_a|u_\varepsilon^0|^p dx\big)^{q/p}} \\
&= R_{a,p}(u_\varepsilon^1) + R_{a,p}(u_\varepsilon^0)
\end{aligned}$$

$$\begin{aligned} &\le 2M_\varepsilon < 3S_{0,q^*} \\ &\le S^{\rm rad}_{a,p}. \end{aligned}$$

□

3. Behavior of the ground-state solution for a large

In this section, we mainly analyze a ground state solution as $a \to +\infty$. Even in this case this solution tends to "concentrate" at the boundary $\partial\Omega$. However, this concentration is much weaker than concentration as $p \to q^*$.

We use the notation $C(\rho_1, \rho_2) = \{x \in \mathbb{R}^N | \rho_1 < |x| < \rho_2\}$. Let δ be sufficiently small (say $\delta < \frac{1}{2}$) and φ be a smooth cut-off function such that $0 \le \varphi \le 1$ with

$$\varphi(x) = \begin{cases} 1, & x \in C(1, 1+\delta) \cup C(3-\delta, 3), \\ 0, & x \in C(2-\delta, 2+\delta). \end{cases} \tag{3.1}$$

From now on, since $p \in (q, q^*)$ is fixed we denote a ground state solution of problem (1.1) $u_{a,p}$ with u_a.

Lemma 3.1. *Let u_a be such that $R_{a,p}(u_a) = S_{a,p}$. If φ is defined in* (3.1)*, then*

$$R_{a,p}(\varphi u_a) = S_{a,p} + o(S_{a,p}) as a \to +\infty. \tag{3.2}$$

Proof. By the homogeneity of $R_{a,p}$, we may assume $\int_\Omega |\nabla u_a|^q dx = 1$. We will prove it by two steps.

Step 1. We claim that

$$\int_\Omega \Phi_a (\varphi u_a)^p dx = \int_\Omega \Phi_a u_a^p dx + o\Big(\int_\Omega \Phi_a u_a^p dx\Big). \tag{3.3}$$

Actually, if we assume

$$\limsup_{\alpha\to\infty} \frac{\int_\Omega \Phi_a u_a^p (1-\varphi^p) dx}{\int_\Omega \Phi_a u_a^p dx} = b > 0,$$

which implies that, up to some subsequence,

$$\frac{\int_\Omega \Phi_a u_a^p (1-\varphi^p) dx}{\int_\Omega \Phi_a u_a^p dx} > \frac{b}{2} > 0.$$

Since $1 - \varphi^p \equiv 0$ on $C(1, 1+\delta) \cup C(3-\delta, 3)$, we have

$$\begin{aligned} \int_\Omega \Phi_a u_a^p (1-\varphi^p) dx &= \int_{C(1+\delta, 3-\delta)} \Phi_a u_a^p (1-\varphi^p) dx \\ &\le (1-\delta)^a \int_\Omega u_a^p (1-\varphi^p) dx \\ &\le (1-\delta)^a \int_\Omega u_a^p dx. \end{aligned}$$

Hence,

$$\int_\Omega u_a^p dx \ge (1-\delta)^{-a} \int_\Omega \Phi_a u_a^p (1-\varphi^p) dx.$$

Thus,

$$\frac{\int_\Omega u_a^p dx}{\int_\Omega \Phi_a u_a^p dx} \ge (1-\delta)^{-a} \frac{\int_\Omega \Phi_a u_a^p (1-\varphi^p) dx}{\int_\Omega \Phi_a u_a^p dx} \ge (1-\delta)^{-a} \frac{b}{2}. \tag{3.4}$$

Since $S_{a,p}^{p/q} = (\int_\Omega \Phi_a u_a^p dx)^{-1}$, (3.4) can be written as

$$S_{a,p}^{p/q} \geq \frac{b}{2} \frac{(1-\delta)^{-a}}{\int_\Omega u_a^p dx} \geq \frac{b}{2}(1-\delta)^{-a} S_{0,p}^{p/q},$$

where

$$S_{0,p} = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^q dx}{\left(\int_\Omega u^p dx\right)^{q/p}}.$$

On the other hand, from (2.2), it follows that

$$S_{a,p}^{p/q} \leq C a^{p - \frac{pN}{q} + N},$$

which gives a contradiction for a large. Hence (3.3) is true.

Step 2. Now we prove that

$$\int_\Omega |\nabla(\varphi u_a)|^q dx = \int_\Omega |\nabla u_a|^q dx + o(1) = 1 + o(1). \tag{3.5}$$

It is easy to prove that u_a satisfies the problem

$$\begin{gathered} -\Delta_q u_a = S_{a,p}^{p/q} \Phi_a u_a^{p-1} \quad \text{in } \Omega, \\ u_a > 0 \quad \text{in } \Omega, \\ u_a = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.6}$$

Since $\int_\Omega |\nabla u_a|^q dx = 1$, up to subsequences, as $a \to \infty$, we have that:

$$u_a \to u \text{ weakly in } W_0^{1,q}(\Omega), \text{ and strongly in } L^s(\Omega) \text{ ė. in } \Omega.$$

Now we prove that $u = 0$. In fact, multiplying problem (3.6) by a smooth function ϕ with $\operatorname{supp}\phi \Subset \Omega$ and integrate, we obtain

$$\int_\Omega |\nabla u_a|^{q-2} \nabla u_a \nabla\phi dx = \int_\Omega S_{a,p}^{p/q} \Phi_a u_a^{p-1} \phi dx \to 0, \quad \text{as } a \to +\infty, \tag{3.7}$$

since, by (2.2), $S_{a,p}^{p/q}\Phi_a \to 0$ uniformly on $\operatorname{supp}\phi$ and u_a is uniformly bounded in L^s for $q < s < q^*$. Hence $\int_\Omega |\nabla u|^{q-2}\nabla u \nabla\phi dx = 0$ for all $\phi \in C_0^\infty(\Omega)$. Since $u \in W_0^{1,q}(\Omega)$, this implies that $u = 0$.

Note that

$$\begin{aligned} &\Big| \int_\Omega |\nabla u_a|^q dx - \int_\Omega |\nabla(\varphi u_a)|^q dx \Big| \\ &= \Big| \int_\Omega |\nabla u_a|^q dx - \int_\Omega |\nabla u_a \varphi + \nabla\varphi u_a|^q dx \Big| \\ &\leq \int_\Omega |\nabla u_a|^q (1 - \varphi^q) dx + \int_\Omega |\nabla\varphi u_a|^q dx \\ &\quad + C \int_\Omega \Big| |\nabla u_a \varphi|^{q-1} \nabla\varphi u_a + \nabla u_a \varphi |\nabla\varphi u_a|^{q-1} \Big| dx. \end{aligned} \tag{3.8}$$

Due to the strong convergence in L^s for all $s \in (q, q^*)$, the last terms tend to zero. To estimate the term $\int_\Omega |\nabla(u_a)|^q (1 - \varphi^q) dx$, we multiply (3.6) by $u_a(1-\varphi^q) = u_a\eta$ and integrate. Since $\operatorname{supp}\eta = \operatorname{supp}(1-\varphi^q) \Subset \Omega$, we have

$$\int_\Omega |\nabla u_a|^{q-2} \nabla u_a \nabla(u_a \eta) dx = \int_\Omega S_{a,p}^{p/q} \Phi_a u_a^p \eta dx.$$

Therefore,

$$
\begin{aligned}
\Big|\int_\Omega |\nabla u_a|^q \eta dx\Big| &\le \Big|\int_\Omega |\nabla u_a|^{q-2}\nabla u_a \nabla\eta\cdot u_a dx\Big| + \Big|\int_\Omega S_{a,p}^{p/q}\Phi_a u_a^p \eta dx\Big| \\
&\le \|\nabla\eta\|_\infty \int_{\operatorname{supp}\eta} |\nabla u_a|^{q-1}|u_a|dx + \Big|\int_{\operatorname{supp}\eta} S_{a,p}^{p/q}\Phi_a u_a^p\eta dx\Big| \to 0.
\end{aligned}
$$

Hence (3.5) holds. □

In Lemma 3.1 we proved that the infimum of the Rayleigh quotient $R_{a,p}$ is essentially achieved by the function ϕu_a. From the definition of ϕ, we can decompose $\phi u_a = u_{a,1} + u_{a,2}$, where $u_{a,1}$ vanishes in $C(2-\delta,3)$ and $u_{a,2}$ vanishes in $C(1,2+\delta)$. The following proposition is the key step in order to prove that the function ϕu_a concentrates at the boundary.

Proposition 3.2. *Let $\phi u_a = u_{a,1} + u_{a,2}$, where $\operatorname{supp} u_{a,1} \subset C(1,2-\delta)$ and $\operatorname{supp} u_{a,2} \subset C(2+\delta,3)$, and $\lambda_a = \int_\Omega \Phi_a u_{a,1}^p dx / \int_\Omega \Phi_a u_{a,2}^p dx$. If $\lim_{n\to\infty}\lambda_{a_n} = L$ for a sequence $a_n \to \infty$, then either $L = 0$ or $L = \infty$.*

Remark 3.3. For the quantity $\lambda_a = \int_\Omega \Phi_a u_{a,1}^p dx / \int_\Omega \Phi_a u_{a,2}^p dx$, we cannot exclude the case $\limsup_{a\to\infty}\lambda_a = +\infty$ and $\liminf_{a\to\infty}\lambda_a = 0$. If a uniqueness result for the minimizer u_a were known, then it would be easy to infer that $a \mapsto u_a$ is continuous. Hence λ_a would be continuous too, and we could replace both the lower and the upper limit by a unique limit. Generally, one does not expect such a uniqueness property for any p and any a. However, when $p \approx q^*$ we conjecture that the uniqueness argument of [19] may be used to our setting.

Proof. By the definition of $u_{a,1}$ and $u_{a,2}$ we have

$$
R_{a,p}(\varphi u_a) = \frac{\int_\Omega |\nabla u_{a,1}|^q dx + \int_\Omega |\nabla u_{a,2}|^q dx}{\big(\int_\Omega \Phi_a u_{a,1}^p dx + \int_\Omega \Phi_a u_{a,2}^p dx\big)^{q/p}}. \tag{3.9}
$$

Since u_a is a positive solution, we can say that $\lambda_a > 0$. We get

$$
\begin{aligned}
&R_{a,p}(\varphi u_a) \\
&= \frac{\int_\Omega |\nabla u_{a,1}|^q dx + \int_\Omega |\nabla u_{a,2}|^q dx}{\big(\lambda_a \int_\Omega \Phi_a u_{a,2}^p dx + \int_\Omega \Phi_a u_{a,2}^p dx\big)^{q/p}} \\
&= \frac{\int_\Omega |\nabla u_{a,1}|^q dx}{(\lambda_a+1)^{q/p}\big(\int_\Omega \Phi_a u_{a,2}^p dx\big)^{q/p}} + \frac{\int_\Omega |\nabla u_{a,2}|^q dx}{(\lambda_a+1)^{q/p}\big(\int_\Omega \Phi_a u_{a,2}^p dx\big)^{q/p}} \\
&= \frac{\lambda_a^{q/p}\int_\Omega |\nabla u_{a,1}|^q dx}{(\lambda_a+1)^{q/p}\big(\int_\Omega \Phi_a u_{a,1}^p dx\big)^{q/p}} + \frac{\int_\Omega |\nabla u_{a,2}|^q dx}{(\lambda_a+1)^{q/p}\big(\int_\Omega \Phi_a u_{a,2}^p dx\big)^{q/p}}.
\end{aligned} \tag{3.10}
$$

By the definition of $S_{a,p}$ each quotient $R_{a,p}(u_{a,1})$ and $R_{a,p}(u_{a,2})$ in the last term is greater than or equal to $S_{a,p}$. Therefore by Lemma 3.1 and equation (3.9) we obtain

$$
S_{a,p} + o(S_{a,p}) \ge \frac{1+\lambda_a^{q/p}}{(\lambda_a+1)^{q/p}} S_{a,p}. \tag{3.11}
$$

We notice that the function $g(x) = \frac{1+x^{q/p}}{(x+1)^{q/p}}$ is strictly greater than 1 for every $x > 0$, $g(0) = 1$ and $g(x) \to 1$ as $x \to +\infty$. Further it is increasing in $[0,1]$ and decreasing in $[1,+\infty)$ and $\max_{x>0} g(x) = g(1) = 2^{1-q/p}$. Let $L \in \Lambda$ and $\{a_n\}$ a sequence such that $\lambda_{a_n} \to L$ as $n \to +\infty$. Taking to the limit in (3.11), we obtain that $1 \geq \frac{1+L^{q/p}}{(L+1)^{q/p}}$ and so either $L = +\infty$ or $L = 0$. $\square$

Corollary 3.4. *With the notation of Proposition 3.2, for any sequence $\{a_n\}$ such that $\lambda_{a_n} \to 0$ one has*

$$\lim_{n\to+\infty} \frac{\int_\Omega |\nabla u_{a_n,1}|^q dx}{\int_\Omega |\nabla u_{a_n,2}|^q dx} = 0. \tag{3.12}$$

Proof. Denote

$$\frac{\int_\Omega |\nabla u_{a,1}|^q dx}{\int_\Omega |\nabla u_{a,2}|^q dx} = \kappa_a$$

and suppose that $\limsup_{n\to\infty} \kappa_{a_n} > 0$. Passing to subsequences, $\kappa_{a_n} > \kappa > 0$ for some κ. Therefore we have

$$\begin{aligned}
S_{a_n,p} + o(S_{a_n,p}) &= \frac{\int_\Omega |\nabla u_{a_n,1}|^q dx + \int_\Omega |\nabla u_{a_n,2}|^q dx}{\big(\int_\Omega \Phi_{a_n} u_{a_n,1}^p dx + \int_\Omega \Phi_{a_n} u_{a_n,2}^p dx\big)^{q/p}} \\
&= \frac{(1+\kappa_{a_n}) \int_\Omega |\nabla u_{a_n,2}|^q dx}{(\int_\Omega \Phi_{a_n} u_{a_n,2}^p dx)^{q/p} (1+\lambda_{a_n})^{q/p}} \\
&\geq R_{a_n,p}(u_{a_n,2}) \frac{1+\kappa}{1+o(1)} \\
&\geq (1+\kappa) S_{a_n,p} + o(S_{a_n,p}),
\end{aligned}$$

which is a contradiction. Therefore,

$$\kappa_{a_n} = \frac{\int_\Omega |\nabla u_{a_n,1}|^q dx}{\int_\Omega |\nabla u_{a_n,2}|^q dx} \to 0.$$

$\square$

The following result is an immediate consequence of the previous results.

Proposition 3.5. *For any a_n such that $\lambda_{a_n} \to 0$,*

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_{a_n,1}|^q dx = 0. \tag{3.13}$$

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