Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 162, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NONEXISTENCE OF ASYMPTOTICALLY FREE SOLUTIONS TO NONLINEAR SCHRÖDINGER SYSTEMS

NAKAO HAYASHI, CHUNHUA LI, PAVEL I. NAUMKIN

Abstract. We consider the nonlinear Schrödinger systems

$$
\begin{aligned}
-i\partial_t u_1 + \frac{1}{2}\Delta u_1 &= F(u_1, u_2),\\
i\partial_t u_2 + \frac{1}{2}\Delta u_2 &= F(u_1, u_2)
\end{aligned}
$$

in n space dimensions, where F is a p-th order local or nonlocal nonlinearity smooth up to order p, with $1 < p \le 1 + \frac{2}{n}$ for $n \ge 2$ and $1 < p \le 2$ for $n = 1$. These systems are related to higher order nonlinear dispersive wave equations. We prove the non existence of asymptotically free solutions in the critical and sub-critical cases.

1. Introduction

We study the nonexistence of asymptotically free solutions for the nonlinear dispersive wave equations

$$
\begin{aligned}
(\partial_t^2 + \frac{1}{4}\Delta^2)u &= \lambda|\partial_t u|^{p-1}\partial_t u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\\
u(0,x) &= u_0(x), \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
(\partial_t^2 + \frac{1}{4}\Delta^2)v &= \mu\nabla(|\nabla v|^{p-1}\nabla v), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\\
v(0,x) &= v_0(x), \partial_t v(0,x) = v_1(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\tag{1.2}
$$

where $\lambda, \mu \in \mathbb{C}$, $1 < p \le 1 + \frac{2}{n}$ for $n \ge 2$, and $1 < p \le 2$ for $n = 1$. When we consider the large time asymptotic behavior of solutions for the above equations, it is known that the critical power of the nonlinearity p is $1+\frac{2}{n}$, so that $1 < p < 1+\frac{2}{n}$ is called the sub-critical one.

Related to (1.1) and (1.2), the equations

$$
\begin{aligned}
(\partial_t^2 + (-\Delta)^m)u &= \lambda(-\Delta)^{\frac{m-1}{4}+\frac{m}{2}}(u^p), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n,\\
u(0,x) &= u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}^n
\end{aligned}
\tag{1.3}
$$

2000 *Mathematics Subject Classification.* 35Q55.
Key words and phrases. Dispersive nonlinear waves; asymptotically free solutions.

Submitted November 16, 2011. Published September 21, 2012.

and

$$\begin{gathered}(\partial_t^2+(-\Delta)^m)u=\lambda(-\Delta)^{\frac{m-1}{4}}P(\partial_t u,(-\Delta)^{\frac{m}{2}}u),\quad (t,x)\in\mathbb{R}\times\mathbb{R}^n,\\ u(0,x)=u_0(x),\quad \partial_t u(0,x)=u_1(x),\quad x\in\mathbb{R}^n,\end{gathered}\tag{1.4}$$

were studied in [7], with $m\geq 1$, and P a homogeneous polynomial of order p in two variables. If p is an integer and satisfies

$$p>1+\frac{2}{n-1},$$

with $n\geq 2$, it was shown in [7] that (1.3) and (1.4) have a unique global solution for small regular data. The number $1+\frac{2}{n-1}$ is the well-known critical exponent for the nonlinear wave equation. However, taking into account the time decay rates of solutions to the linear problem for (1.3) or (1.4) with $m\neq 1$, the critical exponent $1+\frac{2}{n-1}$ should be replaced by $1+\frac{2}{n}$. Indeed, a closely related problem to (1.4) written as

$$\begin{gathered}(\partial_t^2+\frac{1}{m^2}(-\partial_x^2)^m)u=\lambda|\partial_t u|^{p-1}\partial_t u,\quad (t,x)\in\mathbb{R}\times\mathbb{R},\\ u(0,x)=u_0(x),\partial_t u(0,x)=u_1(x),\quad x\in\mathbb{R},\end{gathered}\tag{1.5}$$

with $0<m\leq 2$, $m\neq 1$, $\lambda\in\mathbb{C}$, and $p>3$ for $0<m<1$, $p>2+m$ for $1<m\leq 2$ was studied in [5], and the existence of asymptotically free solutions was shown. Thus $p=3$ is a critical exponent for the problem (1.5) from the point of view of the scattering problem. Other problems related to (1.2) in one dimension were studied in [10] (see also the literature cited therein for the case of the initial-boundary value problems). Therefore, we call $p=1+\frac{2}{n}$ the critical exponent and $1<p<1+\frac{2}{n}$ sub-critical exponents for our problems (1.1) and (1.2).

Equation (1.1) may be transformed into a system of nonlinear Schrödinger equations. In fact, let us define new dependent variables by

$$u_1=\Big(i\partial_t+\frac{1}{2}\Delta\Big)u,\quad u_2=\Big(-i\partial_t+\frac{1}{2}\Delta\Big)u.$$

Now (1.1) becomes

$$\begin{pmatrix}(-i\partial_t+\frac{1}{2}\Delta)u_1\\ (i\partial_t+\frac{1}{2}\Delta)u_2\end{pmatrix}=\begin{pmatrix}F(u_1,u_2)\\ F(u_1,u_2)\end{pmatrix},\tag{1.6}$$

where

$$F(u_1,u_2)=-2^{-p}i\lambda|u_1-u_2|^{p-1}(u_1-u_2),$$

since $u=-(-\Delta)^{-1}(u_1+u_2)$ and $\partial_t u=\frac{1}{2i}(u_1-u_2)$. We write (1.6) in the form

$$\mathbf{Lu}=\mathbf{F}(\mathbf{u}),\tag{1.7}$$

where

$$\begin{gathered}\mathbf{L}=\begin{pmatrix}\overline{L}&0\\0&L\end{pmatrix}=\begin{pmatrix}-i\partial_t+\frac{1}{2}\Delta&0\\0&i\partial_t+\frac{1}{2}\Delta\end{pmatrix},\\ \mathbf{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix},\quad \mathbf{a}=\begin{pmatrix}1\\1\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}1\\-1\end{pmatrix},\\ \mathbf{F}(\mathbf{u})=-i\mathbf{a}2^{-p}\lambda|(\mathbf{b}\cdot\mathbf{u})|^{p-1}(\mathbf{b}\cdot\mathbf{u}).\end{gathered}$$

Similarly, if we define

$$v_1=|\nabla|^{-1}(-i\partial_t-\frac{1}{2}\Delta)v,\quad v_2=|\nabla|^{-1}(i\partial_t-\frac{1}{2}\Delta)v,$$

where $|\nabla| = (-\Delta)^{1/2}$, $|\nabla|^{-1} = (-\Delta)^{-1/2}$ and

$$G(v_1, v_2) = -\mu \frac{\nabla}{|\nabla|}\Big(|\frac{\nabla}{|\nabla|}(v_1+v_2)|^{p-1}\frac{\nabla}{|\nabla|}(v_1+v_2)\Big),$$

then (1.2) can be reduced to the following system of nonlinear Schrödinger equations with nonlocal nonlinearities

$$\begin{pmatrix}(-i\partial_t + \frac{1}{2}\Delta)v_1\\ (i\partial_t + \frac{1}{2}\Delta)v_2\end{pmatrix} = \begin{pmatrix}G(v_1,v_2)\\ G(v_1,v_2)\end{pmatrix}. \tag{1.8}$$

We write this equation in the form

$$\mathbf{L}\mathbf{v} = \mathbf{G}(\mathbf{v}) \tag{1.9}$$

with

$$\mathbf{v} = \begin{pmatrix}v_1\\ v_2\end{pmatrix}, \quad \mathbf{G}(\mathbf{v}) = -\mu\mathbf{a}\frac{\nabla}{|\nabla|}\Big(|\frac{\nabla}{|\nabla|}(\mathbf{a}\cdot\mathbf{v})|^{p-1}\frac{\nabla}{|\nabla|}(\mathbf{a}\cdot\mathbf{v})\Big).$$

Multiplying both sides of (1.7) by $\begin{pmatrix}-\overline{u_1}\\ \overline{u_2}\end{pmatrix}$, taking the imaginary parts and integrating in space, we obtain

$$\frac{d}{dt}(\|u_1\|_{\mathbf{L}^2}^2 + \|u_2\|_{\mathbf{L}^2}^2) = 2^{1-p}(\mathrm{Re}\lambda)\|u_1 - u_2\|_{\mathbf{L}^{p+1}}^{p+1}. \tag{1.10}$$

Therefore there exist $C > 0$, independent of $t > 0$ such that

$$\|u(t)\|_{\mathbf{L}^2} \le C \tag{1.11}$$

if $Re(\lambda) \le 0$. The Strichartz estimate and (1.11) imply that there exists a unique global solution of (1.7) for $1 < p < 1 + \frac{4}{n}$ such that

$$\mathbf{u} = \begin{pmatrix}u_1\\ u_2\end{pmatrix} \in \Big((\mathbf{C}\cap\mathbf{L}^\infty)(\mathbf{R};\mathbf{L}^2)\cap\mathbf{L}^\beta_{\mathrm{loc}}(\mathbb{R};\mathbf{L}^{p+1})\Big)^2,$$

where $\beta = \frac{4}{n}\frac{p+1}{p-1}$ (see Appendix 5). Note that the identity (1.10) can be written as

$$\frac{d}{dt}(\|\partial_t u\|_{\mathbf{L}^2}^2 + \frac{1}{4}\|\Delta u\|_{\mathbf{L}^2}^2) = 2(\mathrm{Re}\lambda)\|\partial_t u\|_{\mathbf{L}^{p+1}}^{p+1}.$$

In the same manner, multiplying both sides of (1.9) by $\begin{pmatrix}-\overline{v_1}\\ \overline{v_2}\end{pmatrix}$, taking the imaginary parts and integrating in space, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|v_1\|_{\mathbf{L}^2}^2 + \|v_2\|_{\mathbf{L}^2}^2)\\ &= 2\mathrm{Re}\Big(i\mu\int_{\mathbb{R}^n}|\frac{\nabla}{|\nabla|}(v_1+v_2)|^{p-1}\Big(\frac{\nabla}{|\nabla|}(v_1+v_2)\Big)\Big(\frac{\nabla}{|\nabla|}(\overline{v_1}-\overline{v_2})\Big)\,dx\Big),\end{aligned}$$

from which we obtain the estimate

$$\frac{d}{dt}\Big(\|v_1\|_{\mathbf{L}^2}^2 + \|v_2\|_{\mathbf{L}^2}^2\Big) \le 2|\mu|\|\frac{\nabla}{|\nabla|}(v_1+v_2)\|_{\mathbf{L}^{p+1}}^p\|\frac{\nabla}{|\nabla|}(v_1-v_2)\|_{\mathbf{L}^{p+1}}. \tag{1.12}$$

Estimate (1.12) is not sufficient to ensure the existence of global solutions to (1.9). We again multiply both sides of (1.9) by $\begin{pmatrix}-\partial_t\overline{v_1}\\ \partial_t\overline{v_2}\end{pmatrix}$, take the real parts and integrate in space to obtain

$$\frac{d}{dt}\Big(\|\nabla v_1\|_{\mathbf{L}^2}^2 + \|\nabla v_2\|_{\mathbf{L}^2}^2\Big) = -\frac{4\mu}{p+1}\frac{d}{dt}\|\frac{\nabla}{|\nabla|}(v_1+v_2)\|_{\mathbf{L}^{p+1}}^{p+1} \tag{1.13}$$

for $\mu \in \mathbb{R}$. This identity is equivalent to

$$\frac{d}{dt}\Big(\|\partial_t v\|_{\mathbf{L}^2}^2 + \frac{1}{4}\|\Delta v\|_{\mathbf{L}^2}^2 + \frac{2\mu}{p+1}\|\nabla v\|_{\mathbf{L}^{p+1}}^{p+1}\Big) = 0.$$

If we assume that $\mu \geq 0$, then (1.13) yields a-priori estimates for $\|\nabla v_1\|_{\mathbf{L}^2}^2, \|\nabla v_2\|_{\mathbf{L}^2}^2$, and $\|\frac{\nabla}{|\nabla|}(v_1 + v_2)\|_{\mathbf{L}^{p+1}}$. Applying the Sobolev imbedding theorem to (1.12) we obtain

$$\begin{aligned}
\frac{d}{dt}(\|v_1\|_{\mathbf{L}^2}^2 + \|v_2\|_{\mathbf{L}^2}^2) &\leq C\|\frac{\nabla}{|\nabla|}(v_1 - v_2)\|_{\mathbf{L}^{p+1}} \\
&\leq C\|\frac{\nabla}{|\nabla|}(v_1 - v_2)\|_{\mathbf{L}^2}^{1-\frac{n}{2}(\frac{p-1}{p+1})}\|\nabla(v_1 - v_2)\|_{\mathbf{L}^2}^{\frac{n}{2}(\frac{p-1}{p+1})} \\
&\leq C\|v_1\|_{\mathbf{L}^2}^{1-\frac{n}{2}(\frac{p-1}{p+1})} + C\|v_2\|_{\mathbf{L}^2}^{1-\frac{n}{2}(\frac{p-1}{p+1})}.
\end{aligned}$$

Therefore we have the estimate

$$(\|v_1\|_{\mathbf{L}^2} + \|v_2\|_{\mathbf{L}^2})^{1+\frac{n}{2}(\frac{p-1}{p+1})} \leq C + Ct. \tag{1.14}$$

Thus by the method in [9], Equation (1.9) has a unique global solution for $\mu \geq 0$, $1 < p < 1 + \frac{4}{n}$ such that

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \Big((\mathbf{C} \cap \mathbf{L}^\infty)(\mathbf{R}; \mathbf{L}^2) \cap \mathbf{L}^\beta_{\rm loc}(\mathbb{R}; \mathbf{L}^{p+1})\Big)^2,$$

where $\beta = \frac{4}{n}\frac{p+1}{p-1}$ (see Appendix 5). However, as far as we know, the large time asymptotic behavior of such solutions is not well established.

We denote the weighted Sobolev space by

$$\mathbf{H}^{m,s} = \Big\{f = (f_1, f_2) \in \mathbf{L}^2 \times \mathbf{L}^2; \|f\|_{\mathbf{H}^{m,s}} = \sum_{j=1}^{2}\|f_j\|_{\mathbf{H}^{m,s}} < \infty\Big\},$$

where

$$\|f\|_{\mathbf{H}^{m,s}} = \|(1-\Delta)^{m/2}(1+|x|^2)^{s/2}f\|_{\mathbf{L}^2}.$$

We write $\mathbf{H}^m = \mathbf{H}^{m,0}$ for simplicity. As usual, let the Fourier transform be defined by

$$\mathcal{F}\phi = \hat{\phi}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(x\cdot\xi)}\phi(x)dx$$

and the inverse Fourier transform be given by

$$\mathcal{F}^{-1}\phi = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(x\cdot\xi)}\phi(\xi)d\xi.$$

Denote by $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-\frac{it}{2}|\xi|^2}\mathcal{F}$ the free Schrödinger evolution group.

In what follows, we assume that (1.7) or (1.9) has a unique global solution. To solve the usual scattering problem we need to find a solution of (1.7) or (1.9) in a neighborhood of a free solution $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ of the linear equation

$$\mathbf{L}\mathbf{w} = 0 \tag{1.15}$$

with some initial data $\mathbf{w}(0) = \mathbf{w}_0 = \begin{pmatrix} w_{1,0} \\ w_{2,0} \end{pmatrix}$.

The purpose of this paper is to show that it is impossible to find a solution to (1.7) or (1.9) in any neighborhood of any free solution $\mathbf{w}$. We use the functional used in [1] for the Schrödinger equations which is a version of the one used for the nonlinear Klein-Gordon equation in [3] and [6].

Our main results are the following.

Theorem 1.1. *Let* $\mathrm{Im}\lambda \neq 0$, $\beta = \frac{4}{n}\frac{p+1}{p-1}$, *and let*

$$\mathbf{u} \in \Big((\mathbf{C} \cap \mathbf{L}^\infty)(\mathbf{R}; \mathbf{L}^2) \cap \mathbf{L}^\beta_{\rm loc}(\mathbf{R}; \mathbf{L}^{p+1})\Big)^2$$

be a solution of (1.7) *with* $1 < p \leq 1 + \frac{2}{n}$ *for* $n \geq 2$, $1 < p \leq 2$ *for* $n = 1$. *Then, there does not exist any free solution* $\mathbf{w}$ *of* (1.15) *with the initial data* $\mathbf{w}_0 \neq 0$ *such that* $\mathbf{w}_0 \in (\mathbf{H}^{0,1} \cap \mathbf{L}^1)^2$ *and*

$$\lim_{t\to\infty} \|\mathbf{u}(t) - \mathbf{w}(t)\|_{\mathbf{L}^2} = 0.$$

For the case of the system (1.9) we have

Theorem 1.2. *Let* $\mathrm{Re}\mu \neq 0, \beta = \frac{4}{n}\frac{p+1}{p-1}$ *and let*

$$\mathbf{v} \in \Big((\mathbf{C} \cap \mathbf{L}^\infty)(\mathbf{R}; \mathbf{L}^2) \cap \mathbf{L}^\beta_{\rm loc}(\mathbf{R}; \mathbf{L}^{p+1})\Big)^2$$

be a solution of (1.9) *with* $1 < p \leq 1 + \frac{2}{n}$ *for* $n \geq 2$, $1 < p \leq 2$ *for* $n = 1$. *Then, there does not exist any free solution* $\mathbf{w}$ *of* (1.15) *with the initial data* $\mathbf{w}_0 \neq 0$ *such that* $\mathbf{w}_0 \in (\mathbf{H}^{0,1} \cap \mathbf{L}^1)^2$ *and*

$$\lim_{t\to\infty} \|\mathbf{v}(t) - \mathbf{w}(t)\|_{\mathbf{L}^2} = 0.$$

For (1.1) and (1.2), we define free solution to be a solution $u_+(t)$ to the linear dispersive equation

$$\Big(\partial_t^2 + \frac{1}{4}\Delta^2\Big)u_+ = 0 \tag{1.16}$$

with initial data $u_+(0) = u_{1+}$, $\partial_t u_+(0) = u_{2+}$. As a consequence of the above results we have the corollaries for (1.1) and (1.2).

Corollary 1.3. *Let* $\mathrm{Im}\,\lambda \neq 0$, $\beta = \frac{4}{n}\frac{p+1}{p-1}$ *and let* u *be a solution of* (1.1) *with* $1 < p \leq 1 + \frac{2}{n}$ *for* $n \geq 2$, $1 < p \leq 2$ *for* $n = 1$ *such that*

$$\Delta u, \partial_t u \in (\mathbf{C} \cap \mathbf{L}^\infty)(\mathbf{R}; \mathbf{L}^2) \cap \mathbf{L}^\beta_{\rm loc}(\mathbf{R}; \mathbf{L}^{p+1}).$$

Then, there does not exist any free solution $u_+(t)$ *of* (1.16) *with the initial data* $(u_{1+}, u_{2+}) \neq 0$ *such that* $\Delta u_{1+}, u_{2+} \in \mathbf{H}^{0,1} \cap \mathbf{L}^1$ *and*

$$\lim_{t\to\infty} (\|\partial_t(u(t) - u_+(t))\|_{\mathbf{L}^2} + \|\Delta(u(t) - u_+(t))\|_{\mathbf{L}^2}) = 0.$$

Corollary 1.4. *Let* $\mathrm{Re}\,\mu \neq 0$, $\beta = \frac{4}{n}\frac{p+1}{p-1}$ *and let* v *be a solution of* (1.2) *with* $1 < p \leq 1 + \frac{2}{n}$ *for* $n \geq 2$, $1 < p \leq 2$ *for* $n = 1$ *such that*

$$|\nabla| v,\, |\nabla|^{-1}\partial_t v \in (\mathbf{C} \cap \mathbf{L}^\infty)(\mathbf{R}; \mathbf{L}^2) \cap \mathbf{L}^\beta_{\rm loc}(\mathbb{R}; \mathbf{L}^{p+1}).$$

Then, there does not exist any free solution $u_+(t)$ *of* (1.16) *with the initial data* $(u_{1+}, u_{2+}) \neq 0$ *such that* $|\nabla| u_{1+}$, $|\nabla|^{-1}u_{2+} \in \mathbf{H}^{0,1} \cap \mathbf{L}^1$ *and*

$$\lim_{t\to\infty} \Big(\||\nabla|^{-1}\partial_t(v(t) - u_+(t))\|_{\mathbf{L}^2} + \|\nabla(v(t) - u_+(t))\|_{\mathbf{L}^2}\Big) = 0.$$

To prove our results using the methods in [3] and [6], we need a-priori lower bounds for the solutions to the linear problem. Our main point in this paper is to prove the lower bound of time decay estimates of solutions to the linear problem which is a main tool on the proof of [3] and [6]. To get these estimates in the case of nonlinear Klein-Gordon equations, the finite propagation property of solutions was used in [3] and [6]. However, the equations considered in this paper do not have this property, so instead we study the large time asymptotic behavior for a linear combination of two types of free Schrödinger evolution groups to get the lower bound of solutions. We note that the lower bound of time decay estimates of solutions was shown in [1] for a single free Schrödinger evolution group.

2. A-priori estimates of solutions to the linear problem from below

In this section we prove the estimates $\|\mathbf{a}\cdot\mathbf{w}\|_{\mathbf{L}^{p+1}} \geq Ct^{-\frac{n}{2}\frac{p-1}{p+1}}$ and $\|\mathbf{b}\cdot\mathbf{w}\|_{\mathbf{L}^{p+1}} \geq Ct^{-\frac{n}{2}\frac{p-1}{p+1}}$ for the solution $\mathbf{w}(t) = \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$ of the linear problem

$$\mathbf{L}\mathbf{w} = 0$$

with the initial data

$$\mathbf{w}(0) = \mathbf{w}_0 = \begin{pmatrix} w_{1,0} \\ w_{2,0} \end{pmatrix}.$$

Note that

$$\mathbf{w}(t) = \begin{pmatrix} \mathcal{U}(-t)w_{1,0} \\ \mathcal{U}(t)w_{2,0} \end{pmatrix}.$$

Since $\mathbf{a}\cdot\mathbf{w} = w_1 + w_2$ and $\mathbf{b}\cdot\mathbf{w} = w_1 - w_2$ do not have the finite propagation property which was used in papers [3] and [6], we need to find the large time asymptotic behavior for $\mathcal{U}(-t)\phi \pm \mathcal{U}(t)\psi$. Denote $M(t) = e^{\frac{i}{2t}|x|^2}$ and $D(t)\phi = \frac{1}{(it)^{n/2}}\phi(\frac{x}{t})$. We only consider the case $t \geq 0$, since the case $t \leq 0$ can be treated in the same way.

Lemma 2.1. *Let $0 \leq \gamma \leq 1$. For any $\phi \in \mathbf{H}^{2\gamma}$*

$$\|\mathcal{U}(t)\phi - M(t)D(t)\mathcal{F}\phi\|_{\mathbf{L}^2} \leq Ct^{-\gamma}\|\widehat{\phi}\|_{\mathbf{H}^{2\gamma}}$$

for $t > 0$.

Proof. By the identity $\mathcal{U}(t) = M(t)D(t)\mathcal{F}M(t)$ we find

$$\mathcal{U}(t)\phi = M(t)D(t)\mathcal{F}M(t)\phi = M(t)D(t)\mathcal{F}\phi + M(t)D(t)\mathcal{F}(M(t)-1)\phi.$$

The $\mathbf{L}^2$-norm of the last term in the right-hand side of the above identity is estimated by

$$\begin{aligned} \|M(t)D(t)\mathcal{F}(M(t)-1)\phi\|_{\mathbf{L}^2} &= \|(M(t)-1)\phi\|_{\mathbf{L}^2} \\ &\leq Ct^{-\gamma}\|\phi\|_{\mathbf{H}^{0,2\gamma}} = Ct^{-\gamma}\|\widehat{\phi}\|_{\mathbf{H}^{2\gamma}}. \end{aligned}$$

This proves the lemma. □

In the next lemma we find a lower bound of the norm $\|\mathcal{U}(-t)\phi \pm \mathcal{U}(t)\psi\|_{\mathbf{L}^{p+1}}$.

Lemma 2.2. *For $\phi, \psi \in \mathbf{H}^{0,1} \cap \mathbf{L}^1$,*

$$\begin{aligned} &\|\mathcal{U}(-t)\phi \pm \mathcal{U}(t)\psi\|_{\mathbf{L}^r} \\ &\geq \frac{1}{2}(2kt)^{\frac{n}{2}(\frac{2}{r}-1)}\Big(\|\widehat{\phi}\|_{\mathbf{L}^2(|\xi|\leq k)} + \|\widehat{\psi}\|_{\mathbf{L}^2(|\xi|\leq k)}\Big) \end{aligned}$$

$$- C(k)(kt)^{\frac{n}{2}(\frac{2}{r}-1)}t^{-\alpha/2}\Big(\|\widehat{\phi}\|_{\mathbf{H}^1}+\|\widehat{\psi}\|_{\mathbf{H}^1}+\|\widehat{\phi}\|_{\mathbf{L}^\infty}+\|\widehat{\psi}\|_{\mathbf{L}^\infty}\Big)$$

for all $t>0$ and $k>0$, where $2\le r\le\infty,\alpha<\frac{1}{2}$ for $n=1$ and $\alpha=\frac{1}{2}$ for $n\ge 2$, and $C(k)$ is a positive constant depending on k.

Proof. By Hölder's inequality

$$\begin{aligned}
&\|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi\|_{\mathbf{L}^2(|x|\le kt)}\\
&\le \|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi\|_{\mathbf{L}^r(|x|\le kt)}\Big(\int_{|x|\le kt}dx\Big)^{\frac{r-2}{2r}}\\
&= (2kt)^{\frac{n}{2}(1-\frac{2}{r})}\|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi\|_{\mathbf{L}^r(|x|\le kt)}\\
&\le (2kt)^{\frac{n}{2}(1-\frac{2}{r})}\|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi\|_{\mathbf{L}^r}.
\end{aligned}\tag{2.1}$$

Hence in order to get the desired estimate from below we need to find a lower bound for the norm $\|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi\|_{\mathbf{L}^2(|x|\le kt)}$. By Lemma 2.1 with $\gamma=\frac{1}{2}$ we find

$$\begin{aligned}
&\|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi-(M(-t)D(-t)\widehat{\phi}\pm M(t)D(t)\widehat{\psi})\|_{\mathbf{L}^2(|x|\le kt)}\\
&\le \|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi-(M(-t)D(-t)\widehat{\phi}\pm M(t)D(t)\widehat{\psi})\|_{\mathbf{L}^2}\\
&\le 2|t|^{-1/2}(\|\widehat{\phi}\|_{\mathbf{H}^1}+\|\widehat{\psi}\|_{\mathbf{H}^1}).
\end{aligned}$$

Therefore changing the variable of integration by $\xi=\frac{x}{t}$, we obtain

$$\begin{aligned}
&\|\mathcal{U}(-t)\phi\pm\mathcal{U}(t)\psi\|_{\mathbf{L}^2(|x|\le kt)}\\
&\ge \|M(-t)D(-t)\widehat{\phi}\pm M(t)D(t)\widehat{\psi}\|_{\mathbf{L}^2(|x|\le kt)}-Ct^{-1/2}(\|\widehat{\phi}\|_{\mathbf{H}^1}+\|\widehat{\psi}\|_{\mathbf{H}^1})\\
&\ge \|e^{-\frac{it}{2}|\xi|^2}(-i)^n\widehat{\phi}(-\xi)\pm e^{\frac{it}{2}|\xi|^2}\widehat{\psi}(\xi)\|_{\mathbf{L}^2(|\xi|\le k)}-Ct^{-1/2}(\|\widehat{\phi}\|_{\mathbf{H}^1}+\|\widehat{\psi}\|_{\mathbf{H}^1}).
\end{aligned}\tag{2.2}$$

By a direct computation we have

$$\begin{aligned}
&\|e^{-\frac{it}{2}|\xi|^2}(-i)^n\widehat{\phi}(-\xi)\pm e^{\frac{it}{2}|\xi|^2}\widehat{\psi}(\xi)\|^2_{\mathbf{L}^2(|\xi|\le k)}\\
&= \|\widehat{\phi}\|^2_{\mathbf{L}^2(|\xi|\le k)}+\|\widehat{\psi}\|^2_{\mathbf{L}^2(|\xi|\le k)}\pm 2\,\mathrm{Re}\int_{|\xi|\le k}(-i)^n\widehat{\phi}(-\xi)\overline{\widehat{\psi}(\xi)}e^{-it|\xi|^2}d\xi.
\end{aligned}\tag{2.3}$$

Integration by parts and using the identity

$$e^{-it|\xi|^2}=\frac{1}{n-2it|\xi|^2}\nabla\cdot\xi e^{-it|\xi|^2}$$

yields

$$\begin{aligned}
\int_{|\xi|\le k}F(\xi)e^{-it|\xi|^2}d\xi &= \int_{|\xi|\le k}F(\xi)\frac{1}{n-2it|\xi|^2}\nabla\cdot\xi e^{-it|\xi|^2}d\xi\\
&= \int_{|\xi|\le k}\nabla\cdot\Big(\frac{\xi F(\xi)}{n-2it|\xi|^2}e^{-it|\xi|^2}\Big)d\xi\\
&\quad -\int_{|\xi|\le k}e^{-it|\xi|^2}\xi\cdot\nabla\frac{F(\xi)}{n-2it|\xi|^2}d\xi,
\end{aligned}$$

for any $F\in\mathbf{L}^\infty$ with $\nabla F\in\mathbf{L}^1$. Therefore,

$$\Big|\int_{|\xi|\le k}\nabla\cdot\Big(\frac{\xi F(\xi)}{n-2it|\xi|^2}e^{-it|\xi|^2}\Big)d\xi\Big|\le C(k)t^{-1/2}\|F\|_{\mathbf{L}^\infty}$$

and by a direct calculation

$$\xi \cdot \nabla \frac{F(\xi)}{n-2it|\xi|^2} = \frac{4it|\xi|^2 F(\xi)}{(n-2it|\xi|^2)^2} + \frac{\xi \cdot \nabla F(\xi)}{n-2it|\xi|^2}.$$

Hence

$$\begin{aligned}
&|\int_{|\xi|\le k} F(\xi) e^{-it|\xi|^2} d\xi| \\
&\le \int_{|\xi|\le k} \frac{2|F(\xi)| + |\xi\cdot\nabla F(\xi)|}{n+2t|\xi|^2} d\xi + C(k)t^{-\frac{1}{2}}\|F\|_{\mathbf{L}^\infty} \\
&\le Ct^{-\alpha}\int_{|\xi|\le k} (\frac{|F(\xi)|}{|\xi|^{2\alpha}} + |\nabla F(\xi)|) d\xi + C(k)t^{-\frac{1}{2}}\|F\|_{\mathbf{L}^\infty} \\
&\le Ct^{-\alpha}\|F\|_{\mathbf{L}^\infty}\int_{|\xi|\le k} |\xi|^{-2\alpha} d\xi + Ct^{-\alpha}\|\nabla F\|_{\mathbf{L}^1} + C(k)t^{-1/2}\|F\|_{\mathbf{L}^\infty} \\
&\le Ct^{-\alpha}(C(k)\|F\|_{\mathbf{L}^\infty} + \|\nabla F\|_{\mathbf{L}^1}),
\end{aligned}$$

where $\alpha < 1/2$ for $n = 1$ and $\alpha = 1/2$ for $n \ge 2$. Therefore taking $F(\xi) = \widehat{\phi}(-\xi)\overline{\widehat{\psi}(\xi)}$ in the above estimate, we obtain

$$\begin{aligned}
&|\int_{|\xi|\le k} \widehat{\phi}(-\xi)\overline{\widehat{\psi}(\xi)} e^{-it|\xi|^2} d\xi| \\
&\le Ct^{-\alpha}\Big((k^{n-2\alpha}+1)\|\widehat{\phi}(-\xi)\overline{\widehat{\psi}(\xi)}\|_{\mathbf{L}^\infty} + \|\nabla(\widehat{\phi}(-\xi)\overline{\widehat{\psi}(\xi)})\|_{\mathbf{L}^1}\Big) \\
&\le C(k)t^{-\alpha}\Big(\|\widehat{\phi}\|_{\mathbf{H}^1} + \|\widehat{\phi}\|_{\mathbf{L}^\infty}\Big)\Big(\|\widehat{\psi}\|_{\mathbf{H}^1} + \|\widehat{\psi}\|_{\mathbf{L}^\infty}\Big).
\end{aligned} \tag{2.4}$$

We apply (2.4) to (2.3) and use (2.2) to obtain

$$\begin{aligned}
&\|\mathcal{U}(-t)\phi \pm \mathcal{U}(t)\psi\|_{\mathbf{L}^2(|x|\le kt)} \\
&\ge \Big(\|\widehat{\phi}\|^2_{\mathbf{L}^2(|\xi|\le k)} + \|\widehat{\psi}\|^2_{\mathbf{L}^2(|\xi|\le k)}\Big)^{1/2} - \Big(2|\int_{|\xi|\le k} \widehat{\phi}(-\xi)\overline{\widehat{\psi}(\xi)} e^{-it|\xi|^2} d\xi|\Big)^{1/2} \\
&\quad - 2|t|^{-\frac{1}{2}}(\|\widehat{\phi}\|_{\mathbf{H}^1} + \|\widehat{\psi}\|_{\mathbf{H}^1}) \\
&\ge \frac{1}{2}\|\widehat{\phi}\|_{\mathbf{L}^2(|\xi|\le k)} + \frac{1}{2}\|\widehat{\psi}\|_{\mathbf{L}^2(|\xi|\le k)} \\
&\quad - C(k)t^{-\alpha/2}\Big(\|\widehat{\phi}\|_{\mathbf{H}^1} + \|\widehat{\psi}\|_{\mathbf{H}^1} + \|\widehat{\phi}\|_{\mathbf{L}^\infty} + \|\widehat{\psi}\|_{\mathbf{L}^\infty}\Big).
\end{aligned}$$

Finally by (2.1) we obtain

$$\begin{aligned}
&\|\mathcal{U}(-t)\phi \pm \mathcal{U}(t)\psi\|_{\mathbf{L}^r} \ge (2kt)^{\frac{n}{2}(\frac{2}{r}-1)}\|\mathcal{U}(-t)\phi \pm \mathcal{U}(t)\psi\|_{\mathbf{L}^2(|x|\le kt)} \\
&\ge \frac{1}{2}(2kt)^{\frac{n}{2}(\frac{2}{r}-1)}(\|\widehat{\phi}\|_{\mathbf{L}^2(|\xi|\le k)} + \|\widehat{\psi}\|_{\mathbf{L}^2(|\xi|\le k)}) \\
&\quad - C(k)(kt)^{\frac{n}{2}(\frac{2}{r}-1)}t^{-\alpha/2}(\|\widehat{\phi}\|_{\mathbf{H}^1} + \|\widehat{\psi}\|_{\mathbf{H}^1} + \|\widehat{\phi}\|_{\mathbf{L}^\infty} + \|\widehat{\psi}\|_{\mathbf{L}^\infty}),
\end{aligned}$$

which proves Lemma 2.2. □

3. Proof of Theorem 1.1

By the contradiction, suppose that there exists a free solution $\mathbf{w}$ of (1.15) defined by the initial data such that $\mathbf{w}_0 \neq 0$: $\mathbf{w}_0 \in (\mathbf{H}^{0,1} \cap \mathbf{L}^1)^2$ satisfying

$$\lim_{t\to\infty} \|\mathbf{u}(t) - \mathbf{w}(t)\|_{\mathbf{L}^2} = 0. \tag{3.1}$$

Define the functional

$$\mathbf{H}_u(t) = \operatorname{Re}\int_{\mathbb{R}^n} i\mathbf{w}\cdot\overline{\mathbf{u}}dx = \operatorname{Re}\sum_{j=1}^{2}\int_{\mathbb{R}^n} iw_j\overline{u}_j dx$$

as in [1]. By (1.15) and (1.6) we have

$$\begin{aligned}
\frac{d}{dt}\mathbf{H}_u(t) &= \operatorname{Re}\int_{\mathbb{R}^n}\Big(\begin{pmatrix} i\partial_t & 0\\ 0 & i\partial_t\end{pmatrix}\mathbf{w}\cdot\overline{\mathbf{u}} + \mathbf{w}\cdot\overline{\begin{pmatrix} -i\partial_t & 0\\ 0 & -i\partial_t\end{pmatrix}\mathbf{u}}\Big)dx\\
&= \operatorname{Re}\int_{\mathbb{R}^n}\Big(\begin{pmatrix} \frac12\Delta & 0\\ 0 & -\frac12\Delta\end{pmatrix}\mathbf{w}\cdot\overline{\mathbf{u}} + \mathbf{w}\cdot\overline{\begin{pmatrix} -i\partial_t & 0\\ 0 & -i\partial_t\end{pmatrix}\mathbf{u}}\Big)dx\\
&= \operatorname{Re}\int_{\mathbb{R}^n}\mathbf{w}\cdot\overline{\begin{pmatrix} -i\partial_t+\frac12\Delta & 0\\ 0 & -(i\partial_t+\frac12\Delta)\end{pmatrix}\mathbf{u}}dx\\
&= \operatorname{Re}\int_{\mathbb{R}^n}\mathbf{w}\cdot\overline{\begin{pmatrix} -2^{-p}i\lambda|u_1-u_2|^{p-1}(u_1-u_2)\\ 2^{-p}i\lambda|u_1-u_2|^{p-1}(u_1-u_2)\end{pmatrix}}dx.
\end{aligned}$$

Letting

$$W = w_1 - w_2,\quad U = u_1 - u_2,$$

from the above identity we have

$$\begin{aligned}
&\frac{d}{dt}\mathbf{H}_u(t)\\
&= 2^{-p}\operatorname{Re}\Big(i\overline{\lambda}\int_{\mathbb{R}^n}|U|^{p-1}\overline{U}W\,dx\Big)\\
&= 2^{-p}\operatorname{Re}\Big(i\overline{\lambda}\int_{\mathbb{R}^n}(|U|^{p-1}\overline{U}W-|W|^{p+1})dx\Big) + 2^{-p}(\operatorname{Im}\lambda)\int_{\mathbb{R}^n}|W|^{p+1}dx.
\end{aligned}\tag{3.2}$$

Due to the inequality

$$\begin{aligned}
\big||a|^{p-1}a-|b|^{p-1}b\big| &\le C(|a|^{p-1}+|b|^{p-1})|a-b|\\
&\le C(|a-b|^{p-1}+|b|^{p-1})|a-b|,
\end{aligned}$$

where $a,b\in\mathbb{C}$ and the Hölder inequality we obtain

$$\begin{aligned}
&\Big|2^{-p}\operatorname{Re}\Big(i\overline{\lambda}\int_{\mathbb{R}^n}(|U|^{p-1}\overline{U}W-|W|^{p+1})dx\Big)\Big|\\
&= \Big|2^{-p}\operatorname{Re}\Big(i\overline{\lambda}\int_{\mathbb{R}^n}(|U|^{p-1}\overline{U}-|W|^{p-1}\overline{W})W\,dx\Big)\Big|\\
&\le C\int_{\mathbb{R}^n}(|U-W|^p|W|+|U-W||W|^p)dx\\
&\le C\|U-W\|_{\mathbf{L}^2}^p\|W\|_{\mathbf{L}^{\frac{2}{2-p}}} + C\|U-W\|_{\mathbf{L}^2}\|W\|_{\mathbf{L}^{2p}}^p\\
&\le C(\delta)|t|^{\frac n2(1-p)}\|U-W\|_{\mathbf{L}^2}(1+\|U-W\|_{\mathbf{L}^2}^{p-1}),
\end{aligned}\tag{3.3}$$

since $1<p\le 1+\frac2n\le 2$ for $n\ge 2$, $1\le p\le 2$ for $n=1$, and $\|W\|_{\mathbf{L}^r}\le C|t|^{\frac n2(\frac2r-1)}\|\mathbf{w}_0\|_{\mathbf{L}^{\frac{r}{r-1}}}$ for $r\ge 2$, where $C(\delta)$ is a constant depends on δ=$\|\mathbf{w}_0\|_{\mathbf{L}^1}+\|\mathbf{w}_0\|_{\mathbf{H}^{0,1}}$.

Since $\mathbf{w}_0\ne 0$, there exists a $k>0$ such that $\|\widehat{w_{1,0}}\|_{\mathbf{L}^2(|\xi|\le k)}+\|\widehat{w_{2,0}}\|_{\mathbf{L}^2(|\xi|\le k)}>0$. We apply Lemma 2.2 with $r=p+1$ to the difference $w_1(t)-w_2(t)=\mathcal{U}(-t)w_{1,0}-$

$\mathcal{U}(t)w_{2,0}$ to find

$$\begin{aligned}
&\|w_1(t)-w_2(t)\|_{\mathbf{L}^{p+1}}^{p+1}\\
&\geq (\frac{1}{2})^{p+1}(2kt)^{\frac{n}{2}(1-p)}\Big(\|\widehat{w_{1,0}}\|_{\mathbf{L}^2(|\xi|\leq k)}+\|\widehat{w_{2,0}}\|_{\mathbf{L}^2(|\xi|\leq k)}\Big)^{p+1}\\
&\quad - C(k)(kt)^{\frac{n}{2}(1-p)}t^{-\alpha\frac{p+1}{2}}\Big(\|\mathbf{w}_0\|_{\mathbf{H}^{0,1}}^{p+1}+\|\mathbf{w}_0\|_{\mathbf{L}^1}^{p+1}\Big)\\
&\geq C(k,\gamma)t^{\frac{n}{2}(1-p)}-C(k,\delta)t^{\frac{n}{2}(1-p)-\alpha\frac{p+1}{2}}
\end{aligned}\tag{3.4}$$

for all $t>0$, where $C(k,\gamma)$ is a constant depending on k and $\gamma=\|\widehat{w_{1,0}}\|_{\mathbf{L}^2(|\xi|\leq k)}+\|\widehat{w_{2,0}}\|_{\mathbf{L}^2(|\xi|\leq k)}$ and $C(k,\delta)$ is a constant depending on k and δ. Integrating (3.2) in time, and using (3.3) and (3.4), we obtain

$$\begin{aligned}
|\mathbf{H}_u(2T)-\mathbf{H}_u(T)| &\geq 2^{-p}|\operatorname{Im}\lambda|\Big|\int_T^{2T}\int_{\mathbb{R}^n}|W(t,x)|^{p+1}dxdt\Big|\\
&\quad - C\int_T^{2T}|t|^{\frac{n}{2}(1-p)}\|U-W\|_{\mathbf{L}^2}^{p-1}(1+\|U-W\|_{\mathbf{L}^2})dt\\
&\geq 2^{-p}|\operatorname{Im}\lambda|\int_T^{2T}\|w_1(t)-w_2(t)\|_{\mathbf{L}^{p+1}}^{p+1}dt\\
&\quad - C(\delta)\int_T^{2T}|t|^{\frac{n}{2}(1-p)}\|\mathbf{u}(t)-\mathbf{w}(t)\|_{\mathbf{L}^2}^{p-1}dt\\
&\geq 2^{-p}|\operatorname{Im}\lambda|\int_T^{2T}(C(k,\gamma)t^{\frac{n}{2}(1-p)}-C(k,\delta)t^{\frac{n}{2}(1-p)-\alpha\frac{p+1}{2}})dt\\
&\quad - C(\delta)\int_T^{2T}t^{n(1-p)/2}\|\mathbf{u}(t)-\mathbf{w}(t)\|_{\mathbf{L}^2}^{p-1}dt.
\end{aligned}$$

By (3.1), it follows that for any ε satisfying $0<\varepsilon\leq 2^{-p-2}|\operatorname{Im}\lambda|C(k,\gamma)/C(\delta)$, there exists a $T_1>0$ such that

$$\|\mathbf{u}(t)-\mathbf{w}(t)\|_{\mathbf{L}^2}<\varepsilon^{\frac{1}{p-1}}$$

for $t\geq T_1$. Let $T_2>0$ be such that

$$C(k,\gamma)t^{\frac{n}{2}(1-p)}-C(k,\delta)t^{\frac{n}{2}(1-p)-\alpha\frac{p+1}{2}}\geq\frac{1}{2}C(k,\gamma)t^{\frac{n}{2}(1-p)}$$

for $t\geq T_2$. Hence

$$\begin{aligned}
|\mathbf{H}_u(2T)-\mathbf{H}_u(T)| &\geq (2^{-p-1}|\operatorname{Im}\lambda|C(k,\gamma)-C(\delta)\varepsilon)\int_T^{2T}t^{\frac{n}{2}(1-p)}dt\\
&\geq (2^{-p-1}|\operatorname{Im}\lambda|C(k,\gamma)-C(\delta)\varepsilon)\int_T^{2T}t^{-1}dt\\
&\geq 2^{-p-2}|\operatorname{Im}\lambda|C(k,\gamma)\log 2>0
\end{aligned}\tag{3.5}$$

for $T\geq\max\{T_1,T_2\}>0$. On the other hand, by the definition of $\mathbf{H}_u(t)$ and (3.1) we find

$$\begin{aligned}
|\mathbf{H}_u(t)| &= |\operatorname{Re}\int_{\mathbb{R}^n}(i\mathbf{w}\cdot(\overline{\mathbf{u}}-\overline{\mathbf{w}}))dx|\leq C\|\mathbf{w}(t)\|_{\mathbf{L}^2}\|\mathbf{u}(t)-\mathbf{w}(t)\|_{\mathbf{L}^2}\\
&\leq C\|\mathbf{w}_0\|_{\mathbf{L}^2}\|\mathbf{u}(t)-\mathbf{w}(t)\|_{\mathbf{L}^2}\to 0
\end{aligned}\tag{3.6}$$

for $t\to\infty$. From (3.6) and (3.5) we have the desired contradiction. This completes the proof.

4. Proof of Theorem 1.2

As in the proof of Theorem 1.1, suppose that there exists a free solution $\mathbf{w}$ of (1.15) defined by the initial data such that $\mathbf{w}_0 \neq 0$, $\mathbf{w}_0 \in (\mathbf{H}^{0,1} \cap \mathbf{L}^1)^2$ satisfying

$$\lim_{t\to\infty} \|\mathbf{v}(t) - \mathbf{w}(t)\|_{\mathbf{L}^2} = 0. \tag{4.1}$$

Define the functional

$$\mathbf{G}_v(t) = \operatorname{Re} \int_{\mathbb{R}^n} (iw_1\overline{v}_1 - iw_2\overline{v}_2)dx$$

and denote

$$\Omega = \frac{\nabla}{|\nabla|}(w_1 + w_2), \quad V = \frac{\nabla}{|\nabla|}(v_1 + v_2).$$

Then by (1.15) and (1.8) we obtain

$$\begin{aligned} \frac{d}{dt}\mathbf{G}_v(t) &= \operatorname{Re} \int_{\mathbb{R}^n} \mathbf{w}\cdot\overline{\mathbf{L}\mathbf{v}}dx \\ &= -\operatorname{Re}\Big(\overline{\mu}\int_{\mathbb{R}^n} (\mathbf{a}\cdot\mathbf{w})\frac{\nabla}{|\nabla|}|V|^{p-1}\overline{V}dx\Big) = \operatorname{Re}\Big(\overline{\mu}\int_{\mathbb{R}^n} \Omega|V|^{p-1}\overline{V}dx\Big) \\ &= \operatorname{Re}\Big(\overline{\mu}\int_{\mathbb{R}^n} \Omega(|V|^{p-1}\overline{V} - |\Omega|^{p-1}\overline{\Omega})dx\Big) + (\operatorname{Re}\overline{\mu})\int_{\mathbb{R}^n} |\Omega|^{p+1}dx. \end{aligned} \tag{4.2}$$

As in (3.3) we find

$$\begin{aligned} &\big|\operatorname{Re}\Big(\overline{\mu}\int_{\mathbb{R}^n} \Omega(|V|^{p-1}\overline{V} - |\Omega|^{p-1}\overline{\Omega})dx\Big)\big| \\ &\leq C\|V-\Omega\|^p_{\mathbf{L}^2}\|\Omega\|_{\mathbf{L}^{\frac{2}{2-p}}} + C\|V-\Omega\|^{p-1}_{\mathbf{L}^2}\|\Omega\|^2_{\mathbf{L}^{\frac{4}{3-p}}} \\ &\leq C(\delta)|t|^{\frac{n}{2}(1-p)}\|V-\Omega\|^{p-1}_{\mathbf{L}^2}(1 + \|V-\Omega\|_{\mathbf{L}^2}). \end{aligned} \tag{4.3}$$

Applying Lemma 2.2 to $\Omega = \frac{\nabla}{|\nabla|}(w_1 + w_2)$, we obtain

$$\|\Omega(t)\|^{p+1}_{\mathbf{L}^{p+1}} \geq C(k,\gamma)t^{\frac{n}{2}(1-p)} - C(k,\delta)t^{\frac{n}{2}(1-p)-\alpha\frac{p+1}{2}} \tag{4.4}$$

for all $t > 0$, since the norm $\|\frac{\nabla}{|\nabla|}\cdot\|_{\mathbf{L}^{p+1}}$ is equivalent to $\|\cdot\|_{\mathbf{L}^{p+1}}$ (see [8]). Integrating (4.2) in time, and using (4.3) and (4.4), we obtain

$$\begin{aligned} &|\mathbf{G}_v(2T) - \mathbf{G}_v(T)| \\ &\geq |\operatorname{Re}\mu|\int_T^{2T} \|\Omega(t)\|^{p+1}_{\mathbf{L}^{p+1}}dt - C(\delta)\int_T^{2T} |t|^{\frac{n}{2}(1-p)}\|\mathbf{v}(t)-\mathbf{w}(t)\|^{p-1}_{\mathbf{L}^2}dt \\ &\geq |\operatorname{Re}\mu|\int_T^{2T} \Big(C(k,\gamma)t^{\frac{n}{2}(1-p)} - C(k,\delta)t^{\frac{n}{2}(1-p)-\alpha\frac{p+1}{2}}\Big)dt \\ &\quad - C(\delta)\int_T^{2T} t^{\frac{n}{2}(1-p)}\|\mathbf{v}(t)-\mathbf{w}(t)\|^{p-1}_{\mathbf{L}^2}dt. \end{aligned}$$

By (4.1), it follows that for any ε satisfying $0 < \varepsilon \leq 2^{-2}|\operatorname{Re}\mu|C(k,\gamma)/C(\delta)$, there exists a $T_1 > 0$ such that

$$\|\mathbf{u}(t)-\mathbf{w}(t)\|_{\mathbf{L}^2} < \varepsilon^{\frac{1}{p-1}}$$

for $t \geq T_1$. Again, let $T_2 > 0$ such that

$$C(k,\gamma)t^{\frac{n}{2}(1-p)} - C(k,\delta)t^{\frac{n}{2}(1-p)-\alpha\frac{p+1}{2}} \geq \frac{1}{2}C(k,\gamma)t^{\frac{n}{2}(1-p)}$$

for $t \ge T_2$. Therefore,

$$\begin{aligned} |\mathbf{G}_v(2T) - \mathbf{G}_v(T)| &\ge (2^{-1}|\operatorname{Re}\mu|C(k,\gamma) - C(\delta)\varepsilon) \int_T^{2T} t^{\frac{n}{2}(1-p)} dt \\ &\ge (2^{-1}|\operatorname{Re}\mu|C(k,\gamma) - C(\delta)\varepsilon) \int_T^{2T} t^{-1} dt \\ &\ge 2^{-2}|\operatorname{Re}\mu|C(k,\gamma)\log 2 > 0 \end{aligned} \tag{4.5}$$

for $T \ge \max\{T_1, T_2\} > 0$. On the other hand, by the definition of $\mathbf{G}_v(t)$ and (4.1) we find

$$\begin{aligned} |\mathbf{G}_v(t)| &= \big|\operatorname{Re}\int_{\mathbb{R}^n} (iw_1(\overline{v}_1 - \overline{w}_1) - iw_2(\overline{v}_2 - \overline{w}_2))dx\big| \\ &\le C\|\mathbf{w}(t)\|_{\mathbf{L}^2}\|\mathbf{v}(t) - \mathbf{w}(t)\|_{\mathbf{L}^2} \\ &\le C\|\mathbf{v}(t) - \mathbf{w}(t)\|_{\mathbf{L}^2} \to 0 \end{aligned} \tag{4.6}$$

as $t \to \infty$. Therefore we have the desired contradiction by (4.5) and (4.6). This completes the proof.

5. Appendix

In this section we prove the existence of global solutions to the systems (1.7)) and (1.9). We introduce the following space-time norm

$$\|\phi\|_{\mathbf{L}^q(\mathbf{I};\mathbf{L}^r)} = \|\|\phi(t,x)\|_{\mathbf{L}^r_x}\|_{\mathbf{L}^q_t(\mathbf{I})},$$

where $\mathbf{I}$ is a bounded or unbounded time interval.

To prove the local existence of $\mathbf{L}^2$-solutions, we write (1.7) as a system of integral equations

$$\begin{aligned} u_1(t) &= \overline{\mathcal{U}(t)}\phi_1 + i\int_0^t \overline{\mathcal{U}(t-\tau)} F(u_1(\tau), u_2(\tau))d\tau, \\ u_2(t) &= \mathcal{U}(t)\phi_2 - i\int_0^t \mathcal{U}(t-\tau) F(u_1(\tau), u_2(\tau))d\tau, \end{aligned} \tag{5.1}$$

where $\mathcal{U}(t)$ is the free Schrödinger evolution group. As in [9], we treat the problem in $\mathbf{L}^2$ space by applying the results for a single nonlinear Schrödinger equation with power nonlinearities.

Define the space

$$\mathbf{X}(\mathbf{I}) = (\mathbf{C} \cap \mathbf{L}^\infty)(\mathbf{I};\mathbf{L}^2) \cap \mathbf{L}^\beta(\mathbf{I};\mathbf{L}^{p+1})$$

with the norm

$$\|u\|_{\mathbf{X}(\mathbf{I})} = \sum_{j=1}^2 \Big(\|u_j\|_{\mathbf{L}^\infty(\mathbf{I};\mathbf{L}^2)} + \|u_j\|_{\mathbf{L}^\beta(\mathbf{I};\mathbf{L}^{p+1})}\Big),$$

on a time-interval $\mathbf{I} = [-T, T]$, where $\beta = \frac{4}{n}\frac{p+1}{p-1}$, $1 < p < 1 + \frac{4}{n}$. We now prove the following result.

Theorem 5.1. *For any $\rho > 0$ there exists a $T(\rho) > 0$ such that for any initial data $\phi = (\phi_1, \phi_2) \in \mathbf{L}^2$ with the norm $\|\phi\|_{\mathbf{L}^2} \le \rho$, the Cauchy problem for (1.7) has a unique solution $\mathbf{u} = (u_1, u_2) \in \mathbf{X}(\mathbf{I})$ with $\mathbf{I} = [-T(\rho), T(\rho)]$.*

Proof. We denote the right-hand sides of (5.1) by $\Phi_j(\mathbf{u})$ for $j = 1, 2$. Applying the Strichartz inequality

$$\| \int_0^t \mathcal{U}(t-\tau)g(\tau)d\tau\|_{\mathbf{L}_t^r(\mathbf{I};\mathbf{L}_x^q)} \le C\|g\|_{\mathbf{L}_t^s(\mathbf{I};\mathbf{L}_x^l)}$$

for $2 \le r \le \infty$, $1 \le s \le 2$, $\frac{1}{q} = \frac{1}{2} - \frac{2}{nr}$, $\frac{1}{l} = \frac{1}{2} + \frac{2}{n}(1 - \frac{1}{s})$ (see [2]), we estimate $\Phi_j(\mathbf{u})$ via the Hölder inequality in space and in time. We choose $r = \beta$, $q = p + 1$, $s = \frac{\beta}{\beta-1}$, $l = \frac{p+1}{p}$ and $\beta = \frac{4}{n}\frac{p+1}{p-1}$, then

$$\begin{aligned}
\|\Phi_j(\mathbf{u})\|_{\mathbf{X}(\mathbf{I})} &\le C\|\phi\|_{\mathbf{L}^2} + C\|\mathbf{F}(\mathbf{u})\|_{\mathbf{L}^s(\mathbf{I};\mathbf{L}^{\frac{p+1}{p}})} \\
&\le C\|\phi\|_{\mathbf{L}^2} + C\Big(\int_I \|\mathbf{u}\|_{\mathbf{L}^{p+1}}^{ps} dt\Big)^{1/s} \\
&\le C\|\phi\|_{\mathbf{L}^2} + CT^{\frac{1}{s}-\frac{p}{\beta}}\|\mathbf{u}\|_{\mathbf{X}(\mathbf{I})}^p.
\end{aligned}$$

Note that $\beta - ps > 0$ since $p < 1 + \frac{4}{n}$. Similarly, we find the estimate for the difference

$$\|\Phi_j(\mathbf{u}) - \Phi_j(\mathbf{u}')\|_{\mathbf{X}(\mathbf{I})} \le CT^{\frac{1}{s}-\frac{p}{\beta}}(\|\mathbf{u}\|_{\mathbf{X}(\mathbf{I})}^{p-1} + \|\mathbf{u}'\|_{\mathbf{X}(\mathbf{I})}^{p-1})\|\mathbf{u} - \mathbf{u}'\|_{\mathbf{X}(\mathbf{I})}.$$

Therefore the conclusion of the theorem follows from the contraction mapping principle if we take $T > 0$ sufficiently small which depends only on the size ρ of the initial data. □

The existence of global solutions for (1.7) follows from Theorem 5.1 and a-priori estimates (1.11). Similarly, a-priori estimates (1.14) and Theorem 5.1 ensure the existence of global solutions to (1.9).

Acknowledgements. The work of P.I.N. is partially supported by CONACYT and PAPIIT. The authors would like to thank an anonymous referee for the useful suggestions and comments.

References

[1] J. E. Barab; *Nonexistence of asymptotically free solutions for a nonlinear Schrödinger equation,* J. Math. Phys., **25** (1984), pp. 3270-3273.
[2] T. Cazenave; *Semilinear Schrödinger equations*, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323.
[3] R. T. Glassey; *On the asymptotic behavior of nonlinear wave equations,* Trans. Amer. Math. Soc., **182** (1973), pp. 187-200.
[4] N. Hayashi, P. I. Naumkin; *Asymptotic properties of solutions to dispersive equation of Schrödinger type,* J. Math. Soc. Japan, **60** (2008), pp. 631-652.
[5] N. Hayashi, S. Kobayashi, P. I. Naumkin; *Global existence of solutions to nonlinear dispersive wave equations,* Differential and Integral Equations, **25** (2012), pp. 685-698.
[6] A. Matsumura; *On the asymptotic behavior of solutions of semi-linear wave equations,* Publ. Res. Inst. Math. Sci., **12** (1976/77), pp. 169-189.
[7] M. Nakamura; *Remarks on Keel-Smith-Sogge estimates and some applications to nonlinear higher order wave equations*, Differential and Integral Equations, **24** (2011), pp. 519-540.
[8] E.M. Stein; *Singular Integrals and Differentiability Properties of Functions*, **30**, Princeton Univ. Press, Princeton, NJ, 1970.
[9] Y. Tsutsumi; $\mathbf{L}^2$*-solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcialaj Ekvacioj, **30** (1987), pp. 115-125.
[10] S. Yoshikawa; *Weak solutions for the Falk model system of shape memory alloys in energy class,* Math. Meth. Appl. Sci., **28** (2005), pp. 1423-1443.

Nakao Hayashi
Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka, 560-0043, Japan
E-mail address: nhayashi@math.sci.osaka-u.ac.jp

Chunhua Li
Department of Mathematics, Graduate School of Science, Yanbian University, Yanji City, Jilin Province, 133002, China
E-mail address: sxlch@ybu.edu.cn

Pavel I. Naumkin
Centro de Ciencias Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico
E-mail address: pavelni@matmor.unam.mx