Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 163, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

INFINITELY MANY SOLUTIONS FOR CLASS OF NAVIER BOUNDARY (p,q)-BIHARMONIC SYSTEMS

MOHAMMED MASSAR, EL MILOUD HSSINI, NAJIB TSOULI

Abstract. This article shows the existence and multiplicity of weak solutions for the (p,q)-biharmonic type system

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u)=\lambda F_u(x,u,v)\quad \text{in } \Omega,\\ \Delta(|\Delta v|^{q-2}\Delta v)=\lambda F_v(x,u,v)\quad \text{in } \Omega,\\ u=v=\Delta u=\Delta v=0\quad \text{on } \partial\Omega.\end{gathered}$$

Under certain conditions on F, we show the existence of infinitely many weak solutions. Our technical approach is based on Bonanno and Molica Bisci's general critical point theorem.

1. Introduction

In this paper we are concerned with the existence and multiplicity of weak solutions for the (p,q)-biharmonic type system

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u)=\lambda F_u(x,u,v)\quad \text{in } \Omega,\\ \Delta(|\Delta v|^{q-2}\Delta v)=\lambda F_v(x,u,v)\quad \text{in } \Omega,\\ u=v=\Delta u=\Delta v=0\quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N\geq 1)$, with smooth boundary, $\lambda\in(0,\infty)$, $p>\max\{1,\frac{N}{2}\}$, $q>\max\{1,\frac{N}{2}\}$. $F:\overline{\Omega}\times\mathbb{R}^2\to\mathbb{R}$ is a function such that $F(.,s,t)$ is continuous in $\overline{\Omega}$, for all $(s,t)\in\mathbb{R}^2$ and $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for every $x\in\Omega$, and F_u,F_v denote the partial derivatives of F, with respect to u,v respectively.

The investigation of existence and multiplicity of solutions for problems involving biharmonic and p-biharmonic operators has drawn the attention of many authors, see [5, 9, 12, 15] and references therein. Candito and Livrea [5] considered the nonlinear elliptic Navier boundary-value problem

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u)=\lambda f(x,u)\quad \text{i n}\Omega,\\ u=\Delta u=0\quad \text{on } \partial\Omega.\end{gathered} \tag{1.2}$$

There the authors established the existence of infinitely many solutions.

2000 *Mathematics Subject Classification.* 35J40, 35J60.
Key words and phrases. Navier value problem; infinitely many solutions;
Ricceri's variational principle.

Submitted June 4, 2012. Published September 21, 2012.

In the present paper, we look for the existence of infinitely many solutions of system (1.1). More precisely, we will prove the existence of well precise intervals of parameters such that problem (1.1) admits either an unbounded sequence of solutions provided that $F(x,u,v)$ has a suitable behaviour at infinity or a sequence of nontrivial solutions converging to zero if a similar behaviour occurs at zero. Our main tool is a general critical points theorem due to Bonanno and Molica Bisci [2] that is a generalization of a previous result of Ricceri [11].

In the sequel, X will denote the space $\big(W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\big)\times\big(W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)\big)$, which is a reflexive Banach space endowed with the norm

$$\|(u,v)\| = \|u\|_p + \|v\|_q,$$

where

$$\|u\|_p = \Big(\int_\Omega |\Delta u|^p dx\Big)^{1/p} \quad \text{and} \quad \|v\|_q = \Big(\int_\Omega |\Delta v|^q dx\Big)^{1/q}.$$

Let

$$K := \max\Big\{\sup_{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\max_{x\in\Omega}|u(x)|^p}{\|u\|_p^p}, \sup_{v\in W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)\setminus\{0\}} \frac{\max_{x\in\Omega}|v(x)|^q}{\|v\|_q^q}\Big\}. \tag{1.3}$$

Since $p>\max\{1,\frac{N}{2}\}$ and $q>\max\{1,\frac{N}{2}\}$, the Rellich Kondrachov theorem assures that $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\hookrightarrow C(\overline{\Omega})$ and $W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)\hookrightarrow C(\overline{\Omega})$ are compact, and hence $K<\infty$.

Definition 1.1. We say that $(u,v)\in X$ is a weak solution of problem (1.1) if

$$\begin{aligned}&\int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi\,dx + \int_\Omega |\Delta v|^{q-2}\Delta v\Delta\psi\,dx\\ &- \lambda\int_\Omega F_u(x,u,v)\varphi\,dx - \lambda\int_\Omega F_v(x,u,v)\psi\,dx = 0,\end{aligned}$$

for all $(\varphi,\psi)\in X$.

Define the functional $I_\lambda : X\to\mathbb{R}$, given by

$$I_\lambda(u,v) = \Phi(u,v) - \lambda\Psi(u,v),$$

for all $(u,v)\in X$, where

$$\Phi(u,v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q \quad \text{and} \quad \Psi(u,v) = \int_\Omega F(x,u,v)dx.$$

Since X is compactly embedded in $C^0(\overline{\Omega})\times C^0(\overline{\Omega})$, it is well known that Φ and Ψ are well defined Gâteaux differentiable functionals whose Gâteaux derivatives at $(u,v)\in X$ are given by

$$\begin{aligned}\langle\Phi'(u,v),(\varphi,\psi)\rangle &= \int_\Omega |\Delta u|^{p-2}\Delta u\Delta\varphi dx + \int_\Omega |\Delta v|^{q-2}\Delta v\Delta\psi dx,\\ \langle\Psi'(u,v),(\varphi,\psi)\rangle &= \int_\Omega F_u(x,u,v)\varphi dx + \int_\Omega F_v(x,u,v)\psi dx,\end{aligned}$$

for all $(\varphi,\psi)\in X$. Moreover, by the weakly lower semicontinuity of norm, we see that Φ is sequentially weakly lower semi continuous. Since Ψ has compact derivative, it follows that Ψ is sequentially weakly continuous.

In view of (1.3), for every $(u,v) \in X$, we have

$$\sup_{x\in\Omega}|u(x)|^p \le K\|u\|_p^p \quad \text{and} \quad \sup_{x\in\Omega}|v(x)|^q \le K\|v\|_q^q,$$

thus

$$\sup_{x\in\Omega}\Big(\frac{1}{p}|u(x)|^p+\frac{1}{q}|v(x)|^q\Big) \le K\Big(\frac{1}{p}\|u\|_p^p+\frac{1}{q}\|v\|_q^q\Big). \tag{1.4}$$

Hence, for every $r>0$

$$\begin{aligned}
\Phi^{-1}(]-\infty,r[) &:= \big\{(u,v)\in X : \Phi(u,v)<r\big\}\\
&= \big\{(u,v)\in X : \frac{1}{p}\|u\|_p^p+\frac{1}{q}\|v\|_q^q<r\big\}\\
&\subseteq \big\{(u,v)\in X : \frac{1}{p}|u(x)|^p+\frac{1}{q}|v(x)|^q<Kr,\ \forall\, x\in\Omega\big\}.
\end{aligned} \tag{1.5}$$

Let us recall for the reader's convenience a smooth version of a previous result of Ricceri [11].

Theorem 1.2. *Let X be a reflexive real Banach space, let $\Phi,\Psi : X\to\mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r>\inf_X\Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)}\frac{\big(\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Psi(v)\big)-\Psi(u)}{r-\Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty}\varphi(r), \quad \delta := \liminf_{r\to(\inf_X\Phi)^+}\varphi(r).$$

Then, one has

(a) for every $r>\inf_X\Phi$ and every $\lambda\in]0,\frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda=\Phi-\lambda\Psi$ to $\Phi^{-1}(]-\infty,r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma<+\infty$ then, for each $\lambda\in]0,\frac{1}{\gamma}[$, the following alternative holds: either

(b1) *I_λ possesses a global minimum,or*

(b2) *there is a sequence (u_n) of critical points (local minima) of I_λ such that $\lim_{n\to+\infty}\Phi(u_n)=+\infty$.*

(c) If $\delta<+\infty$ then, for each $\lambda\in]0,\frac{1}{\delta}[$, the following alternative holds: either

(c1) *there is a global minimum of Φ which is a local minimum of I_λ,or*

(c2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to global minimum of Φ.*

2. Main results

Fix $x^0\in\Omega$ and pick $R_2>R_1>0$ such that $B(x^0,R_2)\subseteq\Omega$. Set

$$\begin{aligned}
L_p &:= \frac{\Gamma(1+N/2)}{\big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)}\pi^{N/2}}\Big(\frac{R_2^2-R_1^2}{2N}\Big)^p\frac{1}{R_2^N-R_1^N},\\
L_q &:= \frac{\Gamma(1+N/2)}{\big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)}\pi^{N/2}}\Big(\frac{R_2^2-R_1^2}{2N}\Big)^q\frac{1}{R_2^N-R_1^N}
\end{aligned} \tag{2.1}$$

where Γ denotes the Gamma function and K is given by (1.3). Now we are ready to state our main results.

Theorem 2.1. *Assume that*

(i1) $F(x,s,t) \geq 0$ *for every* $(x,s,t) \in \Omega \times [0,+\infty)^2$;

(i2) *There exist* $x^0 \in \Omega$, $0 < R_1 < R_2$ *as considered in* (2.1) *such that, if we put*

$$\alpha := \liminf_{\sigma \to +\infty} \frac{\int_\Omega \sup_{|s|+|t| \leq \sigma} F(x,s,t)dx}{\sigma^{\min(p,q)}}, \quad \beta := \limsup_{s,t \to +\infty} \frac{\int_{B(x^0,R_1)} F(x,s,t)dx}{\frac{s^p}{p} + \frac{t^q}{q}},$$

one has

$$\alpha < L\beta, \tag{2.2}$$

where $L := \min\{L_p, L_q\}$.

Then, for every

$$\lambda \in \Lambda := \frac{1}{\big((Kp)^{1/p} + (Kq)^{1/q}\big)^{\min(p,q)}} \Big] \frac{1}{L\beta}, \frac{1}{\alpha} \Big[$$

problem (1.1) *admits an unbounded sequence of weak solutions.*

Theorem 2.2. *Assume that* (i1) *holds and*

(i3) $F(x,0,0) = 0$ *for every* $x \in \Omega$.

(i4) *There exist* $x^0 \in \Omega$, $0 < R_1 < R_2$ *as considered in* (2.1) *such that, if we put*

$$\alpha^0 := \liminf_{\sigma \to 0^+} \frac{\int_\Omega \sup_{|s|+|t| \leq \sigma} F(x,s,t)dx}{\sigma^{\min(p,q)}}, \quad \beta^0 := \limsup_{s,t \to 0^+} \frac{\int_{B(x^0,R_1)} F(x,s,t)dx}{\frac{s^p}{p} + \frac{t^q}{q}},$$

one has

$$\alpha^0 < L\beta^0. \tag{2.3}$$

where $L := \min\{L_p, L_q\}$.

Then, for every

$$\lambda \in \Lambda := \frac{1}{\big((Kp)^{1/p} + (Kq)^{1/q}\big)^{\min(p,q)}} \Big] \frac{1}{L\beta^0}, \frac{1}{\alpha^0} \Big[,$$

problem (1.1) *admits a sequence* (u_n) *of weak solutions such that* $u_n \rightharpoonup 0$.

3. Proofs of main results

Proof of Theorem 2.1. To apply Theorem 1.2, we set

$$\varphi(r) := \inf_{(u,v) \in \Phi^{-1}(]-\infty,r[)} \frac{\Big(\sup_{(w,z) \in \Phi^{-1}(]-\infty,r[)} \Psi(w,z)\Big) - \Psi(u,v)}{r - \Phi(u,v)}$$

Note that $\Phi(0,0) = 0$, and by (i1), $\Psi(0,0) \geq 0$. Therefore, for every $r > 0$,

$$\begin{aligned} \varphi(r) &= \inf_{(u,v) \in \Phi^{-1}(]-\infty,r[)} \frac{\Big(\sup_{(w,z) \in \Phi^{-1}(]-\infty,r[)} \Psi(w,z)\Big) - \Psi(u,v)}{r - \Phi(u,v)} \\ &\leq \frac{\sup_{\Phi^{-1}(]-\infty,r[)} \Psi}{r} \\ &= \frac{\sup_{\Phi(u,v) < r} \int_\Omega F(x,u,v)dx}{r}. \end{aligned} \tag{3.1}$$

Hence, from (1.5), we have

$$\varphi(r) \le \frac{1}{r} \sup_{\{(u,v)\in X:\, \frac{|u(x)|^p}{p}+\frac{|v(x)|^q}{q}<Kr,\,\forall\, x\in\Omega\}} \int_\Omega F(x,u,v)dx$$

Let (σ_n) a sequence of positive numbers such that $\sigma_n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\le\sigma_n} F(x,s,t)dx}{\sigma_n^{\min(p,q)}} = \alpha < +\infty. \tag{3.2}$$

Put

$$r_n := \Big(\frac{\sigma_n}{(Kp)^{1/p}+(Kq)^{1/q}}\Big)^{\min(p,q)}$$

Let $(u,v)\in\Phi^{-1}(]-\infty,r_n[)$, from (1.5) we have

$$\frac{|u(x)|^p}{p}+\frac{|v(x)|^q}{q} < Kr_n, \quad \text{for all } \ x\in\Omega.$$

Thus

$$|u(x)| \le (Kpr_n)^{1/p} \quad \text{and} \quad |v(x)| \le (Kqr_n)^{1/q},$$

hence, for n large enough ($r_n>1$),

$$\begin{aligned} |u(x)|+|v(x)| &\le (Kpr_n)^{1/p}+(Kqr_n)^{1/q} \\ &\le \Big((Kp)^{1/p}+(Kq)^{1/q}\Big) r_n^{\frac{1}{\min(p,q)}} = \sigma_n. \end{aligned}$$

Therefore,

$$\begin{aligned} \varphi(r_n) &\le \frac{\sup_{\{(u,v)\in X:|u(x)|+|v(x)|<\sigma_n,\,\forall\, x\in\Omega\}} \int_\Omega F(x,u,v)dx}{\big(\frac{\sigma_n}{(Kp)^{1/p}+(Kq)^{1/q}}\big)^{\min(p,q)}} \\ &\le \Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)} \frac{\int_\Omega \sup_{|s|+|t|<\sigma_n} F(x,s,t)dx}{\sigma_n^{\min(p,q)}}. \end{aligned} \tag{3.3}$$

Let

$$\gamma := \liminf_{r\to+\infty} \varphi(r).$$

It follows from (3.2) and (3.3) that

$$\begin{aligned} \gamma &\le \liminf_{n\to+\infty} \varphi(r_n) \\ &\le \Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)} \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|<\sigma_n} F(x,s,t)}{\sigma_n^{\min(p,q)}} \\ &= \alpha\Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)} < +\infty. \end{aligned} \tag{3.4}$$

From (3.4), it is clear that $\Lambda \subseteq]0,\frac{1}{\gamma}[$.

For $\lambda\in\Lambda$, we claim that the functional I_λ is unbounded from below. Indeed, since $\frac{1}{\lambda} < \big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)} L\beta$, we can consider a sequence (τ_n) of positive numbers and $\eta>0$ such that $\tau_n\to+\infty$ and

$$\frac{1}{\lambda} < \eta < L\Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)} \frac{\int_{B(x^0,R_1)} F(x,\tau_n,\tau_n)dx}{\frac{\tau_n^p}{p}+\frac{\tau_n^q}{q}}, \tag{3.5}$$

for n large enough. Define a sequence (u_n) as follows

$$u_n(x) = \begin{cases} 0, & x \in \overline{\Omega} \backslash B(x_0, R_2) \\ \frac{\tau_n}{R_2^2 - R_1^2}[R_2^2 - \big(\sum_{i=1}^N (x^i - x_0^i)^2\big)], & x \in B(x_0, R_2) \backslash B(x_0, R_1) \\ \tau_n, & x \in B(x_0, R_1) \end{cases} \tag{3.6}$$

Then $(u_n, u_n) \in X$ and

$$\|u_n\|_p^p = \frac{\pi^{N/2}}{\Gamma(1+N/2)} \Big(\frac{2N\tau_n}{R_2^2 - R_1^2}\Big)^p (R_2^N - R_1^N),$$
$$\|u_n\|_q^q = \frac{\pi^{N/2}}{\Gamma(1+N/2)} \Big(\frac{2N\tau_n}{R_2^2 - R_1^2}\Big)^q (R_2^N - R_1^N).$$

This and (2.1) imply that

$$\Phi(u_n, u_n) = \frac{1}{\big((Kp)^{1/p} + (Kq)^{1/q}\big)^{\min(p,q)}} \Big(\frac{\tau_n^p}{pL_p} + \frac{\tau_n^q}{qL_q}\Big). \tag{3.7}$$

By (i1), we have

$$\Psi(u_n, u_n) = \int_\Omega F(x, u_n, u_n)dx \geq \int_{B(x^0, R_1)} F(x, \tau_n, \tau_n)dx. \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we obtain

$$\begin{aligned} &I_\lambda(u_n, u_n) \\ &= \Phi(u_n, u_n) - \lambda\Psi(u_n, u_n) \\ &\leq \frac{1}{\big((Kp)^{1/p} + (Kq)^{1/q}\big)^{\min(p,q)}} \Big(\frac{\tau_n^p}{pL_p} + \frac{\tau_n^q}{qL_q}\Big) - \lambda \int_{B(x^0,R_1)} F(x, \tau_n, \tau_n)dx \\ &\leq \frac{1}{L\,\big((Kp)^{1/p} + (Kq)^{1/q}\big)^{\min(p,q)}} \Big(\frac{\tau_n^p}{p} + \frac{\tau_n^q}{q}\Big) - \lambda \int_{B(x^0,R_1)} F(x, \tau_n, \tau_n)dx \\ &< \frac{1 - \lambda\eta}{L\,\big((Kp)^{1/p} + (Kq)^{1/q}\big)^{\min(p,q)}} \Big(\frac{\tau_n^p}{p} + \frac{\tau_n^q}{q}\Big), \end{aligned} \tag{3.9}$$

for n large enough, so

$$\lim_{n\to+\infty} I_\lambda(u_n, u_n) = -\infty,$$

and hence the claim follows.

The alternative of Theorem 1.2 case (b) assures the existence of unbounded sequence (u_n) of critical points of the functional I_λ and the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. First, note that

$$\min_X \Phi = \Phi(0,0) = 0. \tag{3.10}$$

Let (σ_n) be a sequence of positive numbers such that $\sigma_n \to 0^+$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\leq\sigma_n} F(x,s,t)dx}{\sigma_n^{\min(p,q)}} = \alpha^0 < +\infty. \tag{3.11}$$

Put

$$r_n = \Big(\frac{\sigma_n}{(Kp)^{1/p} + (Kq)^{1/q}}\Big)^{\min(p,q)}, \quad \delta := \liminf_{r\to 0^+} \varphi(r).$$

It follows from (3.1) and (3.11) that

$$\begin{aligned}\delta &\le \liminf_{n\to+\infty}\varphi(r_n)\\ &\le \Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)}\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|s|+|t|<\sigma_n}F(x,s,t)}{\sigma_n^{\min(p,q)}}\\ &= \alpha^0\Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)} < +\infty.\end{aligned}\tag{3.12}$$

By (3.12), we see that $\Lambda \subseteq]0,\frac{1}{\delta}[$.

Now, for $\lambda\in\Lambda$, we claim that I_λ has not a local minimum at zero. Indeed, since $\frac{1}{\lambda} < \big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)}L\beta^0$, we can consider a sequence (τ_n) of positive numbers and $\eta>0$ such that $\tau_n\to 0^+$ and

$$\frac{1}{\lambda} < \eta < L\Big((Kp)^{1/p}+(Kq)^{1/q}\Big)^{\min(p,q)}\frac{\int_{B(x^0,R_1)}F(x,\tau_n,\tau_n)dx}{\frac{\tau_n^p}{p}+\frac{\tau_n^q}{q}},\tag{3.13}$$

for n large enough. Let (u_n) be the sequence defined in (3.6). By combining (3.7), (3.8) and (3.13), and taking into account (i_3), we obtain

$$\begin{aligned} &I_\lambda(u_n,u_n)\\ &=\Phi(u_n,u_n)-\lambda\Psi(u_n,u_n)\\ &\le \frac{1}{\big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)}}\Big(\frac{\tau_n^p}{pL_p}+\frac{\tau_n^q}{qL_q}\Big)-\lambda\int_{B(x^0,R_1)}F(x,\tau_n,\tau_n)dx\\ &\le \frac{1}{L\big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)}}\Big(\frac{\tau_n^p}{p}+\frac{\tau_n^q}{q}\Big)-\lambda\int_{B(x^0,R_1)}F(x,\tau_n,\tau_n)dx\\ &< \frac{1-\lambda\eta}{L\big((Kp)^{1/p}+(Kq)^{1/q}\big)^{\min(p,q)}}\Big(\frac{\tau_n^p}{p}+\frac{\tau_n^q}{q}\Big)\\ &< 0 = I_\lambda(0,0)\end{aligned}\tag{3.14}$$

for n large enough. This together with the fact that $\|(u_n,u_n)\|\to 0$ shows that I_λ has not a local minimum at zero, and the claim follows.

The alternative of Theorem 1.2 case (c) ensures the existence of sequence (u_n) of pairwise distinct critical points (local minima) of I_λ which weakly converges to 0. This completes the proof of Theorem 2.2. □

Example. It could be possible to consider the same example given in [14] for the p-Laplacian system. Let $\Omega\subset\mathbb{R}^2, p=3, q=4$ and $F:\mathbb{R}^2\to\mathbb{R}$ be a function defined by

$$F(s,t)=\begin{cases}(a_{n+1})^5 e^{-\frac{1}{1-[(s-a_{n+1})^2+(t-a_{n+1})^2]}} & (s,t)\in\cup_{n\ge1}B((a_{n+1},a_{n+1}),1)\\ 0 & \text{otherwise,}\end{cases}\tag{3.15}$$

for all $x\in\Omega$, where

$$a_1 := 2,\quad a_{n+1} := n!(a_n)^{5/4}+2$$

and $B((a_{n+1},a_{n+1}),1)$ is an open unit ball of center (a_{n+1},a_{n+1}).

We see that F is non-negative and $F\in C^1(\mathbb{R}^2)$. For every $n\in\mathbb{N}$, the restriction of F on $B((a_{n+1},a_{n+1}),1)$ attains its maximum in (a_{n+1},a_{n+1}) and

$$F(a_{n+1},a_{n+1}) = (a_{n+1})^5e^{-1},$$

then

$$\limsup_{n\to+\infty}\frac{F(a_{n+1},a_{n+1})}{\frac{a_{n+1}^3}{3}+\frac{a_{n+1}^4}{4}}=+\infty.$$

So,

$$\begin{aligned}\beta:&=\limsup_{s,t\to+\infty}\frac{\int_{B(x^0,R_1)}F(s,t)dx}{\frac{s^3}{3}+\frac{t^4}{4}}\\ &=|B(x^0,R_1)|\limsup_{s,t\to+\infty}\frac{F(s,t)}{\frac{s^3}{3}+\frac{t^4}{4}}=+\infty.\end{aligned}$$

On the other hand, for every $n\in\mathbb{N}$, we have

$$\sup_{|s|+|t|\le a_{n+1}-1}F(s,t)=a_n^5e^{-1}\quad\text{for all } n\in\mathbb{N}.$$

Then

$$\lim_{n\to+\infty}\frac{\sup_{|s|+|t|\le a_{n+1}-1}F(s,t)}{(a_{n+1}-1)^3}=0,$$

and hence

$$\lim_{\sigma\to+\infty}\frac{\sup_{|s|+|t|\le\sigma}F(s,t)}{\sigma^3}=0.$$

Finally

$$\begin{aligned}\alpha:&=\liminf_{\sigma\to+\infty}\frac{\int_\Omega\sup_{|s|+|t|\le\sigma}F(s,t)dx}{\sigma^3}\\ &=|\Omega|\liminf_{\sigma\to+\infty}\frac{\sup_{|s|+|t|\le\sigma}F(s,t)}{\sigma^3}\\ &=0<L\beta\end{aligned}$$

So, applying Theorem 2.1, we have that for every $\lambda\in]0,+\infty[$ the system

$$\begin{gathered}\Delta(|\Delta u|\Delta u)=\lambda F_u(u,v)\quad\text{in }\Omega,\\ \Delta(|\Delta v|^2\Delta v)=\lambda F_v(u,v)\quad\text{in }\Omega,\\ u=v=\Delta u=\Delta v=0\quad\text{on }\partial\Omega,\end{gathered}\tag{3.16}$$

admits an unbounded sequence of weak solutions.

References

[1] G. A. Afrouzi, S. Heidarkhani; Existence of three solutions for a class of Dirichlet quasilinear elliptic systems involving the $(p_1,\dots,p_n)$-Laplacian, *Nonlinear Anal.* 70, 135-143 (2009).

[2] G. Bonanno, G. Molica Bisci; Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.*, 2009 (2009), 1-20.

[3] G. Bonanno, G. Molica Bisci; Infinitely many solutions for a Dirichlet problem involving the p-Laplacian, *Proc. Royal Soc. Edinburgh*, 140A, 737-752 (2010).

[4] G. Bonanno, G. Molica Bisci, D. O'Regan; Infinitely many weak solutions for a class of quasilinear elliptic systems, *Math. Comput. Model.* 52,152-160 (2010).

[5] P. Candito, R. Livrea; Infintely many solution for a nonlinear Navier boundary value problem involving p-biharmonic, *Studia Univ."Babeş-Bolyai", Mathematica*, 4 (2010).

[6] P. Candito, L. Li, R. Livrea; Infinitely many solutions for a perturbed nonlinear Navier boundary value problem involving the p-biharmonic, *Nonlinear Anal.*, 75 (17) (2012) 6360-6369.

[7] G. D'Agui, A. Sciammetta; Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions, *Nonlinear Anal.*, 75, (14) (2012) 5612-5618.

[8] H. M. Guo D. Geng; Infinitely many solutions for the Dirichlet problem involving the p-biharmonic-like equation, *J. South China Normal Univ. Natur. Sci. Ed.*, 28 (2009), 18-21.

[9] C. Li, C. L. Tang; Three solutions for a Navier boundary value problem involving the p-biharmonic, *Nonlinear Anal.*, 72 (2010), 1339-1347.
[10] C. Li, C. L. Tang; Existence of three solutions for (p,q)-biharmonic systems, *Nonlinear Anal.*, 73 (2010), 796-805.
[11] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl.Math.*, 133 (2000), 401-410.
[12] Y. Shen, J. Zhang, Existence of two solutions for a Navier boundary value Problem involving the p-biharmonic, *Differential Equations and Applications*, 3 (2011), 399-414.
[13] Y. Shen, J. Zhang; Multiplicity of positive solutions for a Navier boundary-value problem involving the p-biharmonic with critical exponent, *Electronic J. of Differential Equations*, 2011, no. 47 (2011) 1-14.
[14] D. M. Shoorabi, G. A. Afrouzi; Infinitely many solutions for class of Neumann quasilinear elliptic systems, *Bound. Value Probl.* 2012, 2012:54 doi:10.1186/1687-2770-2012-54
[15] Y. Wang, Y. Shen; Infinitely many sign-changing solutions for a class of biarmonic equations without symmetry, *Nonlinear Anal.*, 71 (2009), 967-977.
[16] W. Wang, P. Zhao; Nonuniformly nonlinear elliptic equations of p-biharmonic type, *J. Math. Anal. Appl.*, 348 (2008) 730-738.
[17] Z. Yang, D. Geng, H. Yan; Existence of multiple solutions for a semilinear biharmonic equations with critical exponent, *ActaMath. Sci. Ser. A*, 27 (1)(2006), 129-142.

Mohammed Massar
University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco
E-mail address: massarmed@hotmail.com

El Miloud Hssini
University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco
E-mail address: hssini1975@yahoo.fr

Najib Tsouli
University Mohamed I, Faculty of Sciences, Department of Mathematics, Oujda, Morocco
E-mail address: tsouli@hotmail.com