Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 171, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO BOUNDARY-VALUE PROBLEMS GOVERNED BY GENERAL NON-AUTONOMOUS NONLINEAR DIFFERENTIAL OPERATORS

CRISTINA MARCELLI

Abstract. This article concerns the existence and non-existence of solutions to the strongly nonlinear non-autonomous boundary-value problem

$$(a(t,x(t))\Phi(x'(t)))' = f(t,x(t),x'(t)) \quad \text{a.e. } t\in\mathbb{R}$$
$$x(-\infty)=\nu^-, \quad x(+\infty)=\nu^+$$

with $\nu^- < \nu^+$, where $\Phi:\mathbb{R}\to\mathbb{R}$ is a general increasing homeomorphism, with $\Phi(0)=0$, a is a positive, continuous function and f is a Carathéodory nonlinear function. We provide sufficient conditions for the solvability which result to be optimal for a wide class of problems. In particular, we focus on the role played by the behaviors of $f(t,x,\cdot)$ and $\Phi(\cdot)$ as $y\to 0$ related to that of $f(\cdot,x,y)$ and $a(\cdot,x)$ as $|t|\to+\infty$.

1. Introduction

In the previous decade an increasing interest has been devoted to differential equations of the type

$$(\Phi(x'))' = f(t,x,x'),$$

governed by nonlinear differential operators such as the classical p-Laplacian or its generalizations. Various types of differential operators, even singular or non-surjective, have been considered due to many applications in different fields. We quote for the scalar case Bereanu and Mawhin [4]-[5], Garcia-Huidobro, Manásevich and Zanolin [16], Dosla, Marini and Matucci [13, 21], Cabada and Pouso [6, 7], and Papageorgiou and Papalini [23]. Moreover, Manásevich and Mawhin treated systems of equations in [20], where they studied a periodic problem. Finally, in the more general framework of differential inclusions we quote [14] and the paper by Kyritsi, Matzakos and Papageorgiou [17] for systems of differential inclusions involving maximal monotone operators and with various boundary conditions.

Different types of differential operators, depending also on x are involved in reaction-diffusion equations with non-constant diffusivity (see, e.g. [1, 18, 19]), in porous media equations and other models. So, it naturally arises the interest in

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2000 *Mathematics Subject Classification.* 34B40, 34C37, 34B15, 34L30.
Key words and phrases. Boundary value problems; unbounded domains; heteroclinic solutions; nonlinear differential operators; p-Laplacian operator; Φ-Laplacian operator.

Submitted September 3, 2012. Published October 4, 2012.

mixed differential operators, that is strongly nonlinear equations as

$$\big(a(x)\Phi(x')\big)' = f(t,x,x').$$

In [22] a periodic problem for a vectorial differential inclusion involving an operator of the type $(a(x)\|x'\|^{p-2}x')'$ is studied, where $a:\mathbb{R}\to\mathbb{R}$ is a positive, continuous function. Moreover, in [17] the differential operator is even more general, having structure $(A(x,x'))'$, and the existence of solutions is proved for a Dirichlet vector problem. In these last two papers, the boundary-value problem is studied on a compact interval.

Recently, boundary-value problems on the whole real line, of the type

$$\begin{gathered}\big(a(x(t))\Phi(x'(t))\big)' = f(t,x(t),x'(t)) \quad \text{for a.e. } t\in\mathbb{R}\\ x(-\infty)=\nu^-, \quad x(+\infty)=\nu^+\end{gathered}$$

have been studied in [11] where existence and non-existence results have been proved for various types of differential operator Φ, including the classical p-laplacian. It was proved that the existence of heteroclinic solutions depends on the behavior of Φ and $f(t,x,\cdot)$ at 0 and $f(\cdot,x,y)$ at infinity, while the presence of the function a does not influence the existence of solutions.

The aim of this article is to introduce a dependence also on t on the function a appearing in the differential operator; that is, to study the solvability of the boundary-value problem

$$\begin{gathered}\big(a(t,x(t))\Phi(x'(t))\big)' = f(t,x(t),x'(t)) \quad \text{for a.e. } t\in\mathbb{R}\\ x(-\infty)=\nu^-, \quad x(+\infty)=\nu^+\end{gathered} \tag{1.1}$$

where $\nu^- < \nu^+$ are given constants, $\Phi:\mathbb{R}\to\mathbb{R}$ is a general increasing homeomorphism, with $\Phi(0)=0$, and a is a positive, continuous function. We underline that we allow the function a to have null infimum.

Contrary to the autonomous case $a=a(x)$, where the presence of the function a does not influence the existence and non-existence of solutions (see [11]), in the present setting the dependence on t of the function a is instead very relevant. In more detail, the asymptotic behavior of $a(\cdot,x)$ as $|t|\to+\infty$, related to that of $f(\cdot,x,y)$ and compared with the asymptotic behavior of $f(t,x,\cdot)$ and Φ as $y\to 0$, plays a central role in the existence and non-existence results.

We provide sufficient conditions guaranteeing the solvability of problem (1.1), that cannot be improved, in the sense that in a wide range of cases they are both necessary and sufficient for the existence of solutions.

For instance, when a and f have the product structure

$$f(t,x,y)=h(t)g(x)c(y), \quad a(t,x)=\alpha(t)\beta(x)$$

with $h\in L^q_{\rm loc}(\mathbb{R})$, for some $1\le q\le\infty$, satisfying $th(t)g(x)\le 0$ for every (t,x) and $c(y)$ satisfying $c(0)=0$, $0<c(y)<K|y|^{2-\frac{1}{q}}$ for $y\ne 0$, we have (see Corollary 4.8):

$$\begin{gathered}\text{if } \alpha(t)\sim \text{const.}|t|^p \text{ and } |h(t)|\sim\text{const.}|t|^\delta \text{ as } |t|\to+\infty,\\ |c(y)|\sim\text{const.}|y|^\beta, \quad |\Phi(y)|\sim\text{const.}|y|^\mu \text{ as } y\to 0\end{gathered}$$

with $0<\mu\le 1$, $\mu(\delta+1)>p\beta$ and $\mu<\beta\le\mu(2-\frac{1}{q})$, then (1.1) admits solutions if and only if $\mu>\beta+p-\delta-1$.

We underline that in the framework of autonomous functions a treated in [11], only the case when $\delta\ge -1$ can be handled (see also [12]). Here, the dependence on t of the function a allows us to avoid this limitation, provided that $p < 0$; that is, when $a(t,x)$ vanishes as $|t| \to +\infty$.

To the best of our knowledge, the results presented here are new even if for $\Phi(y) \equiv y$; that is, for differential equations of the form

$$\big(a(t,x(t))x'(t)\big)' = f(t,x(t),x'(t)) \quad \text{a.e. } t.$$

Moreover, the operators considered here are quite general and extend the classical p-laplacian. Nevertheless, when dealing just with the p-laplacian the results can be slightly improved, by using the positive homogeneity of the operator, as we will show in a forthcoming paper.

2. Notation and auxiliary results

In the whole paper we will consider a general increasing homeomorphism Φ on $\mathbb{R}$, such that $\Phi(0) = 0$, a positive continuous function $a : \mathbb{R}^2 \to \mathbb{R}$ and a Carathéodory function $f : \mathbb{R}^3 \to \mathbb{R}$. We will deal with the nonlinear differential equation

$$\big((a(t,x(t))\Phi(x'(t))\big)' = f(t,x(t),x'(t)) \quad \text{a.e. } t \tag{2.1}$$

We will adopt the following notation:

$$m(t) := \min_{x\in[\nu^-\nu^+]} a(t,x), \quad M(t) := \max_{x\in[\nu^-\nu^+]} a(t,x),$$
$$m^*(t) := \min_{(s,x)\in[-t,t]\times[\nu^-\nu^+]} a(s,x), \quad M^*(t) := \max_{(s,x)\in[-t,t]\times[\nu^-\nu^+]} a(s,x).$$

Of course, $M^*(t) \ge M(t) \ge m(t) \ge m^*(t) > 0$ for every $t \in \mathbb{R}$, with $\inf_{t\in\mathbb{R}} m(t)$ possibly null.

The approach we adopt to handle the nonlinear problem on the whole real line is based on a sequential technique, considering boundary-value problems in compact intervals exhausting $\mathbb{R}$. The next lemma is just the key result for the convergence of sequences of solutions in compact intervals towards a solution of (2.1) in $\mathbb{R}$. It was proved in the case the operator a is autonomous; that is, $a(t,x) \equiv a(x)$, but the same proof works also with the dependence on t.

Lemma 2.1 ([11, Lemma 2.2]). *For all $n \in \mathbb{N}$ let $I_n := [-n,n]$ and let $u_n \in C^1(I_n)$ be such that the function $t \mapsto a(t,u_n(t))\Phi(u_n'(t))$ belongs to $W^{1,1}(I_n)$, the sequences $(u_n(0))_n$ and $(u_n'(0))_n$ are bounded and*

$$(a(t,u_n(t))\Phi(u_n'(t)))' = f(t,u_n(t),u_n'(t)) \quad \text{for a.e. } t \in I_n.$$

Assume that there exist two functions $H, \gamma \in L^1(\mathbb{R})$ such that

$$|u_n'(t)| \le H(t), \quad |a(t,u_n(t))\Phi(u_n'(t))| \le \gamma(t) \quad \text{a.e. on } I_n, \text{ for all } n \in \mathbb{N}.$$

Then, the sequence $(x_n)_n$ of continuous functions on $\mathbb{R}$ defined by $x_n(t) := u_n(t)$ for $t \in I_n$ and constant outside I_n, admits a subsequence uniformly convergent in $\mathbb{R}$ to a function $x \in C^1(\mathbb{R})$, such that the composition $t \mapsto a(t,x(t))\Phi(x'(t))$ belongs to $W^{1,1}(\mathbb{R})$, and it is a solution of (2.1). Moreover, if $\lim_{n\to+\infty} u_n(-n) = \nu^-$ and $\lim_{n\to+\infty} u_n(n) = \nu^+$, then we have that $\lim_{t\to-\infty} x(t) = \nu^-$ and $\lim_{t\to+\infty} x(t) = \nu^+$.

To achieve the solvability of the boundary-value problem in compact intervals, we will use the following existence result proved in [15], concerning a two-point functional differential problem.

Theorem 2.2. [15, Theorem 1] *Let $I=[a,b]\subset\mathbb{R}$ and let $A: C^1(I)\to C(I)$, $x\mapsto A_x$, and $F: C^1(I)\to L^1(I)$, $x\mapsto F_x$, be two continuous functionals. Suppose that A maps bounded sets of $C^1(I)$ into uniformly continuous sets in $C(I)$. Moreover, assume that*

$$m\le A_x(t)\le M \quad \text{for every } x\in C^1(I), t\in I, \text{ for some } M>m>0; \tag{2.2}$$

and that there exists $\eta\in L^1(I)$ such that

$$|F_x(t)|\le \eta(t), \quad \text{a.e. on } I, \text{ for every } x\in C^1(I). \tag{2.3}$$

Then, there exists a function $u\in C^1(I)$ such that $A_u\cdot(\Phi\circ u')\in W^{1,1}(I)$ and

$$\begin{gathered}(A_u(t)\Phi(u'(t)))'=F_u(t), \quad \text{a.e. on } I\\ u(a)=\nu^-, \quad u(b)=\nu^+.\end{gathered}$$

For recent results on two-point boundary-value problems in different settings see [2, 3, 8, 9, 10].

3. Existence and non-existence theorems

We begin with an existence result for differential operators growing at most linearly at infinity.

Theorem 3.1. *Let Φ be such that*

$$\limsup_{|y|\to+\infty}\frac{|\Phi(y)|}{|y|}<+\infty. \tag{3.1}$$

Assume that

$$f(t,\nu^-,0)\le 0\le f(t,\nu^+,0) \quad \text{for a.e. } t\in\mathbb{R} \tag{3.2}$$

and that there exist constants $L,H>0$, a continuous function $\theta:\mathbb{R}^+\to\mathbb{R}^+$ and a function $\lambda\in L^q([-L,L])$, with $1\le q\le\infty$, such that

$$|f(t,x,y)|\le\lambda(t)\theta(a(t,x)|\Phi(y)|) \quad \text{for a.e. } |t|\le L, \text{ every } x\in[\nu^-,\nu^+], |y|\ge H, \tag{3.3}$$

$$\int^{+\infty}\frac{\tau^{1-\frac1q}}{\theta(\tau)}\,d\tau=+\infty \tag{3.4}$$

(with $\frac1q=0$ if $q=+\infty$). Also assume that there exists a constant $\gamma>1$ such that for every $C>0$ there exist a function $\eta_C\in L^1(\mathbb{R})$ and a function $K_C\in W^{1,1}_{\rm loc}([0,+\infty))$, null in $[0,L]$ and strictly increasing in $[L,+\infty)$, such that

$$N_C(t):=\Phi^{-1}\Big(\frac{1}{m(t)}\Big\{(M^*(L)\Phi(C))^{1-\gamma}+(\gamma-1)\Big|\int_0^t\frac{K_C'(|s|)}{M(s)^\gamma}\,ds\Big|\Big\}^{\frac{1}{1-\gamma}}\Big)\in L^1(\mathbb{R}), \tag{3.5}$$

$$\begin{gathered}f(t,x,y)\le -K_C'(t)\Phi(|y|)^\gamma, \quad f(-t,x,y)\ge K_C'(t)\Phi(|y|)^\gamma\\ \text{for a.e. } t\ge L, \text{ every } x\in[\nu^-,\nu^+], |y|\le N_C(t),\end{gathered} \tag{3.6}$$

$$|f(t,x,y)|\le\eta_C(t) \quad \text{if } x\in[\nu^-,\nu^+], |y|\le N_C(t), \text{ for a.e. } t\in\mathbb{R}. \tag{3.7}$$

Then, there exists a function $x\in C^1(\mathbb{R})$, such that $t\mapsto a(t,x(t))\Phi(x'(t))$ belongs to $W^{1,1}(\mathbb{R})$ and

$$\begin{gathered}\big(a(t,x(t))\Phi(x'(t))\big)'=f(t,x(t),x'(t)) \quad \text{for a.e. } t\in\mathbb{R}\\ \nu^-\le x(t)\le\nu^+ \quad \text{for every } t\in\mathbb{R}\end{gathered}$$

$$x(-\infty) = \nu^-, \quad x(+\infty) = \nu^+.$$

Proof. By (3.1) we have

$$|\Phi(y)| \le K|y| \quad \text{for every } |y| \ge H \tag{3.8}$$

for some constant $K > 0$, and $H > \frac{\nu^+ - \nu^-}{2L}$. Moreover, by (3.4) there exists a constant

$$C > \max\{\Phi^{-1}(\frac{M^*(L)}{m^*(L)}\Phi(H)), -\Phi^{-1}(\frac{M^*(L)}{m^*(L)}\Phi(-H))\} \ge H$$

such that

$$\int_{M^*(L)\Phi(H)}^{m^*(L)\Phi(C)} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)}\, \mathrm{d}\tau > \|\lambda\|_q [KM^*(L)(\nu^+ - \nu^-)]^{1-\frac{1}{q}} \tag{3.9}$$

and

$$\int_{-M^*(L)\Phi(-H)}^{-m^*(L)\Phi(-C)} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)}\, \mathrm{d}\tau > \|\lambda\|_q [KM^*(L)(\nu^+ - \nu^-)]^{1-\frac{1}{q}}. \tag{3.10}$$

Fix $n \in \mathbb{N}$, $n > L$, and put $I_n := [-n, n]$. Consider the truncation operator $T : W^{1,1}(I_n) \to W^{1,1}(I_n)$ defined by

$$T(x) := T_x \quad \text{where } T_x(t) := \max\{\nu^-, \min\{\nu^+, x(t)\}\}. \tag{3.11}$$

Of course, T is well-defined and $T_x'(t) = x'(t)$ for a.e. $t \in I_n$ such that $\nu^- < x(t) < \nu^+$, whereas $T_x'(t) = 0$ for a.e. t such that $x(t) \le \nu^-$ or $x(t) \ge \nu^+$. For every $x \in W^{1,1}_{\rm loc}(\mathbb{R})$, put

$$Q_x(t) := \max\{-N_C(t), \min\{T_x'(t), N_C(t)\}\}. \tag{3.12}$$

Moreover, for every $x \in \mathbb{R}$, put $w(x) := \max\{x - \nu^+, 0\} + \min\{x - \nu^-, 0\}$. Of course, $w(x) = 0$ if $\nu^- \le x \le \nu^+$, $w(x) > 0$ if $x > \nu^+$ and $w(x) < 0$ if $x > \nu^-$.

Let us consider the auxiliary boundary-value problem on the compact interval I_n:

$$\begin{gathered} \big(a(t, T_x(t))\Phi(x'(t))\big)' = f(t, T_x(t), Q_x(t)) + \arctan(w(x(t))), \quad \text{a.e. in } I_n \\ x(-n) = \nu^-, \quad x(n) = \nu^+. \end{gathered} \tag{3.13}$$

Let us now prove that this problem admits solutions for every $n > L$. To this aim, let $A : C^1(I_n) \to C(I_n)$, $x \mapsto A_x$, and $F : C^1(I_n) \to L^1(I_n)$, $x \mapsto F_x$, be the functionals defined by

$$A_x(t) := a(t, T_x(t)), \quad F_x(t) := f(t, T_x(t), Q_x(t)) + \arctan(w(x(t))).$$

As it is easy to check, by (3.7) the functionals A, F are well-defined, continuous and they respectively satisfy assumptions (2.2), (2.3) of Theorem 2.2, taking $m := m^*(n)$ and $M := M^*(n)$. Furthermore, by the uniform continuity of $a(\cdot,\cdot)$ in $[-n, n] \times [\nu^-, \nu^+]$, for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|a(t_1, \xi_1) - a(t_2, \xi_2)| < \epsilon$ whenever $|t_2 - t_1| \le \delta$ and $|\xi_1 - \xi_2| < \delta$. Let D be a bounded subset of $C^1(I_n)$; i.e., there exists $S > 0$ such that $\|x\|_{C^1(I)} \le S$ for every $x \in D$. Put $\rho := \min\{\delta, \frac{\delta}{S}\}$, if $|t_1 - t_2| < \rho$ we have

$$|T_x(t_1) - T_x(t_2)| \le |x(t_1) - x(t_2)| \le \Big| \int_{t_1}^{t_2} |x'(\tau)| d\tau \Big| \le S|t_1 - t_2| < \delta$$

for all $x \in D$ and consequently $|A_x(t_1) - A_x(t_2)| < \epsilon$ for every $x \in D$, whenever $|t_1 - t_2| < \rho$, that is A maps bounded sets of $C^1(I_n)$ into uniformly continuous sets of $C(I_n)$. Therefore, we can apply Theorem 2.2 and obtain the existence of a function $u_n \in C^1(I_n)$ such that $t \mapsto a(t, u_n(t))\Phi(u_n'(t)) \in W^{1,1}(I_n)$, solution of (3.13).

Now we will show that u_n is a solution of (2.1), in order to apply Lemma 2.1. To this aim, split the proof in steps.

Step 1. We have $\nu^- \le u_n(t) \le \nu^+$ for all $t \in I_n$. Indeed, let t_0 be such that $u_n(t_0) = \min_{t \in I_n} u_n(t)$. If $u_n(t_0) < \nu^-$, by the boundary conditions in (3.13), t_0 belongs to a compact interval $[t_1, t_2] \subset I_n$ satisfying $u_n(t_1) = u_n(t_2) = \nu^-$ and $u_n(t) < \nu^-$ for every $t \in (t_1, t_2)$. Hence, by (3.11) we have $T_{u_n}(t) \equiv \nu^-$ and $Q_{u_n}(t) \equiv 0$ in $[t_1, t_2]$, and by (3.2) for a.e. $t \in (t_1, t_2)$ we have

$$\big(a(t, u_n(t))\Phi(u_n'(t))\big)' = f(t, \nu^-, 0) + \arctan(u_n(t) - \nu^-) < 0.$$

Thus, the function $t \mapsto a(t, u_n(t))\Phi(u_n'(t))$ is strictly decreasing in (t_1, t_2) and being $u_n'(t_0) = 0$ we have

$$a(t, u_n(t))\Phi(u_n'(t)) < a(t_0, u_n(t_0))\Phi(u_n'(t_0)) = 0 \quad \text{for every } t \in (t_0, t_2).$$

Hence, $u_n'(t) < 0$ in (t_0, t_2), in contradiction with the definition of t_0. Similarly one can show that $u_n(t) \le \nu^+$ for every $t \in I_n$.

Step 2. The function u_n is increasing in $[-n, -L]$ and in $[L, n]$. Moreover, if $u_n'(t_0) = 0$ for some $|t_0| > L$, then $u_n'(t) = 0$ whenever $|t| > |t_0|$. To prove this claim, first observe that the function $t \mapsto a(t, u_n(t))\Phi(u_n'(t))$ is increasing in $[-n, -L]$ and decreasing in $[L, n]$. In fact, since u_n is a solution of (3.13) and $|Q_{u_n}(t)| \le N_C(t)$, using Step 1 and assumption (3.6) for a.e. $t \ge L$ we have

$$\big(a(t, u_n(t))\Phi(u_n'(t))\big)' = f(t, u_n(t), Q_{u_n}(t)) \le -K_C'(t)\Phi(|Q_{u_n}(t)|)^\gamma \le 0 \tag{3.14}$$

and we obtain the monotonicity in $[L, n]$. Analogously we can proceed for the interval $[-n, -L]$.

Suppose now, by contradiction, that $u_n'(\bar t) < 0$ for some $\bar t \in [L, n)$. Then

$$a(t, u_n(t))\Phi(u_n'(t)) \le a(\bar t, u_n(\bar t))\Phi(u_n'(\bar t)) < 0 \quad \text{for every } t \in [\bar t, n]$$

and so $u_n'(t) < 0$ for every $t \in [\bar t, n]$. This contradicts what proved in Step 1, since $u_n(n) = \nu^+$. Hence u_n is increasing in $[L, n]$. Similarly we can reason in the interval $[-n, -L]$. Finally, if $u_n'(t_0) = 0$ for some $t_0 \in [L, n)$, for every $t \in (t_0, n)$ we have $a(t, u_n(t))\Phi(u_n'(t)) \le a(t_0, u_n(t_0))\Phi(u_n'(t_0)) = 0$, hence $u_n'(t) \le 0$ in $[t_0, n]$. Therefore, since u_n is increasing in the same interval, we deduce that u_n is constant in $[t_0, n]$.

Step 3. We have $|u_n'(t)| < C$ for every $t \in [-L, L]$. As it is easy to check, put $g(t) := a(t, u_n(t))\Phi(u_n'(t))$, the claim will be proved if we show that

$$m^*(L)\Phi(-C) < g(t) < m^*(L)\Phi(C) \quad \text{for every } t \in [-L, L]. \tag{3.15}$$

To this aim, note that by the Lagrange Theorem there exists a point $\tau_0 \in I_n$ such that

$$|u_n'(\tau_0)| = \frac{1}{2L}|u_n(L) - u_n(-L)| \le \frac{\nu^+ - \nu^-}{2L} < H < C,$$

so

$$m^*(L)\Phi(-C) < M^*(L)\Phi(-H) < g(\tau_0) < M^*(L)\Phi(H) < m^*(L)\Phi(C).$$

Assume, by contradiction, the existence of an interval $(\tau_1, \tau_2) \subset [-L, L]$ such that $M^*(L)\Phi(H) < g(t) < m^*(L)\Phi(C)$ in (τ_1, τ_2) and $g(\tau_1) = M^*(L)\Phi(H)$, $g(\tau_2) = m^*(L)\Phi(C)$ or viceversa.

Then we have $H < u_n'(t) < C$ in (τ_1, τ_2) and since $N_C(t) = \Phi^{-1}(\frac{M^*(L)}{m(t)}\Phi(C)) \geq C$ for every $t \in [-L, L]$, we have $|u_n'(t)| < N_C(t)$ for every $t \in (\tau_1, \tau_2)$. Then, by Step 1, the definition of (3.13) and assumption (3.3), for a.e. $t \in (\tau_1, \tau_2)$ we have

$$|g'(t)| = |f(t, u_n(t), u_n'(t))| \leq \lambda(t)\theta(g(t)).$$

Therefore, by the Hölder inequality and (3.8), we obtain

$$\begin{aligned}
\int_{M^*(L)\Phi(H)}^{m^*(L)\Phi(C)} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)}\,\mathrm{d}\tau &\leq \int_{\tau_1}^{\tau_2} \frac{g(t)^{1-\frac{1}{q}}}{\theta(g(t))}|g'(t)|\,\mathrm{d}t \leq \int_{\tau_1}^{\tau_2} \lambda(t) g(t)^{1-\frac{1}{q}}\,\mathrm{d}t \\
&\leq \|\lambda\|_q \Big(M^*(L)\int_{\tau_1}^{\tau_2} |\Phi(u_n'(t))|\,\mathrm{d}t\Big)^{1-\frac{1}{q}} \\
&\leq \|\lambda\|_q \Big(KM^*(L)\int_{\tau_1}^{\tau_2} |u_n'(t)|\,\mathrm{d}t\Big)^{1-\frac{1}{q}} \\
&\leq \|\lambda\|_q [KM^*(L)(\nu^+ - \nu^-)]^{1-\frac{1}{q}}
\end{aligned}$$

in contradiction with (3.9).

Similarly, assuming that $m^*(L)\Phi(-C) < g(t) < M^*(L)\Phi(-H)$ in (τ_1, τ_2) and $g(\tau_1) = m^*(L)\Phi(-C)$, $g(\tau_2) = M^*(L)\Phi(-H)$ or viceversa, reasoning as above we obtain a contradiction to (3.10). Thus, (3.15) holds and the claim is proved.

Step 4. We have $|u_n'(t)| \leq N_C(t)$ for a.e. $t \in I_n$. Observe that by virtue of what we proved in Step 3, for every $t \in [-L, L]$ we have $|u_n'(t)| < C \leq N_C(t)$. Moreover, in force of Step 2, we have $u_n'(t) \geq 0$ for every $t \in I_n \setminus [-L, L]$. Hence, in order to prove the claim, it remains to show that $u_n'(t) \leq N_C(t)$ for every $t \in I_n \setminus [-L, L]$.

To this aim, let $\hat{t} := \sup\{t > L : u_n'(\tau) < N_C(\tau) \text{ in } [L, t]\}$. By Step 3, $\hat{t}$ is well defined. Assume, by contradiction, $\hat{t} < n$. By Step 1 and the definition (3.12) we have

$$\big(a(t, u_n(t))\Phi(u_n'(t))\big)' = f(t, T_{u_n}(t), Q_{u_n}(t)) = f(t, u_n(t), u_n'(t)) \quad \text{a.e. in } [L, \hat{t}].$$

Since $u_n'(t) \geq 0$ in $[L, n)$, by (3.14) we have

$$(a(t, u_n(t))\Phi(u_n'(t)))' \leq -K_C'(t)\Phi(u_n'(t))^\gamma \leq -\frac{K_C'(t)}{M(t)^\gamma}(a(t, u_n(t))\Phi(u_n'(t)))^\gamma$$

for a.e. $t \in [L, \hat{t}]$. Then

$$\begin{aligned}
&\frac{1}{1-\gamma}[(a(t, u_n(t))\Phi(u_n'(t)))^{1-\gamma} - (a(L, u_n(L))\Phi(u_n'(L)))^{1-\gamma}] \\
&= \int_L^t \frac{(a(u_n(s))\Phi(u_n'(s)))'}{(a(u_n(s))\Phi(u_n'(s)))^\gamma}\,\mathrm{d}s \leq -\int_L^t \frac{K_C'(s)}{M(s)^\gamma}\,\mathrm{d}s = -\int_0^t \frac{K_C'(s)}{M(s)^\gamma}\,\mathrm{d}s
\end{aligned}$$

for every $t \in [L, \bar{t}]$. Therefore,

$$\begin{aligned}
(a(t, u_n(t))\Phi(u_n'(t)))^{1-\gamma} &\geq (a(L, u_n(L))\Phi(u_n'(L)))^{1-\gamma} + (\gamma - 1)\int_0^t \frac{K_C'(s)}{M(s)^\gamma}\,\mathrm{d}s \\
&> (M^*(L)\Phi(C))^{1-\gamma} + (\gamma - 1)\int_0^t \frac{K_C'(s)}{M(s)^\gamma}\,\mathrm{d}s
\end{aligned}$$

implying that

$$u_n'(t) < \Phi^{-1}\Big(\frac{1}{m(t)}\big\{(M^*(L)\Phi(C))^{1-\gamma} + (\gamma - 1)\int_0^t \frac{K_C'(s)}{M(s)^\gamma}\,\mathrm{d}s\big\}^{\frac{1}{1-\gamma}}\Big) = N_C(t).$$

for every $t \in [L, \hat{t}]$, a contradiction when $\hat{t} < n$. So, $\hat{t} = n$ and the claim is proved. The same argument works in the interval $[-n, -L]$ too.

Summarizing, taking account of the properties proved in Steps 1-4, we infer that

$$(a(t, u_n(t))\Phi(u_n'(t)))' = f(t, u_n(t), u_n'(t)) \quad \text{a.e. } t \in I_n$$

for every $n \in \mathbb{N}$. Therefore, by (3.7) the sequence of solutions $(u_n)_n$ satisfies all the assumptions of Lemma 2.1, applied with $H(t) = N_C(t)$ and $\gamma(t) = \eta_C(t)$, for $t \in \mathbb{R}$, where C is the constant fixed at the beginning of the proof. So, we obtain the existence of a solution x of equation (2.1), such that $t \mapsto a(t, x(t))\Phi(x'(t))$ belongs to $W^{1,1}(\mathbb{R})$, satisfying $x(-\infty) = \nu^-$, $x(+\infty) = \nu^+$. □

It is also possible to deal with differential operators having superlinear growth at infinity, provided that condition (3.4) is strengthened requiring that the Nagumo function has sublinear growth at infinity, as the following result states.

Theorem 3.2. *Suppose that all the assumptions of Theorem 3.1 are satisfied, with the exception of* (3.1)*, and with* (3.4) *replaced by*

$$\lim_{y \to +\infty} \frac{\theta(y)}{y} = 0. \tag{3.16}$$

Then the assertion of Theorem 3.1 follows.

Proof. The proof is quite similar to that of the previous theorem. Few modifications only regard Step 3, the unique part in which we used assumption (3.4). Indeed, notice that the new assumption (3.16) implies that

$$\lim_{\xi \to +\infty} \frac{1}{\xi^{1-\frac{1}{q}}} \int_{M^*(L)\Phi(H)}^{m^*(L)\xi} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)} \, d\tau = +\infty;$$
$$\lim_{\xi \to -\infty} \frac{1}{|\xi|^{1-\frac{1}{q}}} \int_{-M^*(L)\Phi(-H)}^{-m^*(L)\xi} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)} \, d\tau = +\infty$$

hence we can choose the constant C in such a way that

$$\int_{M^*(L)\Phi(H)}^{m^*(L)\Phi(C)} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)} \, d\tau > \|\lambda\|_q (2LM^*(L)\Phi(C))^{1-\frac{1}{q}},$$
$$\int_{-M^*(L)\Phi(-H)}^{-m^*(L)\Phi(-C)} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)} > \|\lambda\|_q (-2LM^*(L)\Phi(-C))^{1-\frac{1}{q}},$$

which respectively replace conditions (3.9) and (3.10). From now on the proof proceeds as in the previous result, with the exception of the last chain of inequalities of Step 3, which now becomes

$$\begin{aligned} \int_{M^*(L)\Phi(H)}^{m^*(L)\Phi(C)} \frac{\tau^{1-\frac{1}{q}}}{\theta(\tau)} \, d\tau &\le \|\lambda\|_q \Big(M^*(L) \int_{\tau_1}^{\tau_2} |\Phi(u_n'(t))| \, dt \Big)^{1-\frac{1}{q}} \\ &\le \|\lambda\|_q (2LM^*(L)\Phi(C))^{1-\frac{1}{q}}. \end{aligned}$$
□

The key tools in the previous existence Theorems is the summability of function $N_C(t)$ (condition (3.5)) joined with assumption (3.6). Such conditions are not improvable in the sense that if (3.6) is satisfied with the reversed inequality and N_C is not summable, then problem (1.1) does not admit solutions, as the following result states.

Theorem 3.3. *Suppose that there exist two constants $\rho > 0$, $\gamma > 1$ and a positive strictly increasing function $K \in W^{1,1}_{\rm loc}([0,+\infty))$, such that the following pair of conditions hold:*

$$f(t,x,y) \ge -K'(t)\Phi(y)^\gamma \quad \text{for a.e. } t \ge 0, \text{ every } x \in [\nu^-,\nu^+],\ y \in (0,\rho), \tag{3.17}$$

$$f(t,x,y) \le K'(-t)\Phi(y)^\gamma \quad \text{for a.e. } t \le 0, \text{ every } x \in [\nu^-,\nu^+],\ y \in (0,\rho) \tag{3.18}$$

and for every constant C the function

$$N_C(t) := \Phi^{-1}\Big(\frac{1}{M(t)}\Big\{C + (\gamma-1)\Big|\int_0^t \frac{K'(|s|)}{m(s)^\gamma}\,\mathrm{d}s\Big|\Big\}^{\frac{1}{1-\gamma}}\Big) \tag{3.19}$$

does not belong to $L^1(\mathbb{R})$.

Moreover, assume that

$$tf(t,x,y) \le 0 \quad \text{for a.e. } t \in \mathbb{R}, \text{ every } (x,y) \in [\nu^-,\nu^+] \times \mathbb{R} \tag{3.20}$$

and there exist two constants $\mu, H > 0$ such that

$$a(t,x_1) \le Ha(t+\delta,x_2) \quad \text{for every } t \ge 0,\ x_1,x_2 \in [\nu^-,\nu^+],\ 0 < \delta < \mu, \tag{3.21}$$

$$a(t+\delta,x_1) \le Ha(t,x_2) \quad \text{for every } t \le 0,\ x_1,x_2 \in [\nu^-,\nu^+],\ 0 < \delta < \mu. \tag{3.22}$$

Then, (1.1) *does not admit solutions such that $\nu^- \le x(t) \le \nu^+$; that is, no function $x \in C^1(\mathbb{R})$, with $t \mapsto a(t,x(t))\Phi(x'(t))$ almost everywhere differentiable and $\nu^- \le x(t) \le \nu^+$, exists solving problem* (1.1).

Proof. Let $x \in C^1(\mathbb{R})$, with $a(t,x(t))\Phi(x'(t))$ almost everywhere differentiable and $\nu^- \le x(t) \le \nu^+$ (not necessarily belonging to $W^{1,1}(\mathbb{R})$), be a solution of (1.1).

First of all let us prove that the function x is monotone increasing. Indeed, notice that by assumption (3.20) we deduce that the function $t \mapsto a(t,x(t))\Phi(x'(t))$ is decreasing in $[0,+\infty)$ and increasing in $(-\infty,0]$. Then, if $x'(t_0) = 0$ for some $t_0 \ge 0$, we have $a(t,x(t))\Phi(x'(t)) \le a(t_0,x(t_0))\Phi(x'(t_0)) = 0$ for every $t > t_0$; hence, $x'(t) \le 0$ for every $t \ge t_0$. Since $\nu^- \le x(t) \le \nu^+$ and $x(+\infty) = \nu^+$, this implies that $x(t) \equiv \nu^+$ in $[t_0,+\infty)$. Therefore, for every $t \ge 0$ we have $x'(t) \ge 0$ and $x'(t) > 0$ whenever $x(t) < \nu^+$. Similarly, if $x'(t_0) = 0$ for some $t_0 \le 0$, we have $a(t,x(t))\Phi(x'(t)) \le a(t_0,x(t_0))\Phi(x'(t_0)) = 0$ for every $t < t_0$; hence, $x'(t) \le 0$ for every $t \le t_0$, implying $x'(t) = 0$ in $(-\infty,t_0]$. Therefore, we have $x'(t) \ge 0$ for every $t \in \mathbb{R}$ and $x'(t) > 0$ whenever $\nu^- < x(t) < \nu^+$.

Let us now prove that $\lim_{t\to\pm\infty} x'(t) = 0$. Since x is increasing, it suffices to prove that $\ell : \limsup_{t\to\pm\infty} x'(t) = 0$. Assume, by contradiction, that $\ell > 0$. Then there exists an interval $[t_1,t_2] \subset [0,+\infty)$ such that $|t_1 - t_2| < \mu$, $0 < x'(t) < \rho$ in $[t_1,t_2]$ and $\Phi(x'(t_2)) > H\Phi(x'(t_1))$, where μ and H are the constants appearing in assumption (3.21). Hence,

$$\Phi(x'(t_2)) > H\Phi(x'(t_1)) \ge \frac{a(t_1,x(t_1))}{a(t_2,x(t_2))}\Phi(x'(t_1)$$

so

$$a(t_2,x(t_2))\Phi(x'(t_2)) > a(t_1,x(t_1))\Phi(x'(t_1))$$

a contradiction, since the function $t \mapsto a(t,x(t))\Phi(x'(t))$ is decreasing in $[0,+\infty)$. Similarly, by using (3.22) we obtain $\limsup_{t\to-\infty} x'(t) = 0$. Then, $\lim_{t\to\pm\infty} x'(t) = 0$.

Let us now define $t^* := \inf\{t \geq 0 : x'(t) < \rho \text{ in } [t, +\infty)\}$ and assume by contradiction that $x'(t^*) > 0$. Put $T := \sup\{t : x(t) < \nu^+\}$, so that $0 < x'(t) < \rho$ in (t^*, T). By (3.17) for every $t \in (t^*, T)$ we obtain

$$\begin{aligned}&\frac{1}{1-\gamma}[(a(t,x(t))\Phi(x'(t)))^{1-\gamma} - (a(t^*,x(t^*))\Phi(x'(t^*)))^{1-\gamma}]\\&= \int_{t^*}^{t} \frac{(a(s,x(s))\Phi(x'(s)))'}{(a(s,x(s))\Phi(x'(s)))^{\gamma}}\,\mathrm{d}s \geq \int_{t^*}^{t} -\frac{K'(s)}{m(s)^{\gamma}}\,\mathrm{d}s\end{aligned}$$

therefore,

$$a(t,x(t))\Phi(x'(t)) \geq \Big((a(t^*,x(t^*))\Phi(x'(t^*)))^{1-\gamma} + (\gamma-1)\int_{t^*}^{t}\frac{K'(s)}{m(s)^{\gamma}}\,\mathrm{d}s\Big)^{\frac{1}{1-\gamma}}$$

and finally

$$x'(t) \geq \Phi^{-1}\Big(\frac{1}{M(t)}\Big\{(a(t^*,x(t^*))\Phi(x'(t^*)))^{1-\gamma} + (\gamma-1)\int_{t^*}^{t}\frac{K'(s)}{m(s)^{\gamma}}\,\mathrm{d}s\Big\}^{\frac{1}{1-\gamma}}\Big).$$

Then, if $T < +\infty$, necessarily we have $x'(T) = 0$, in contradiction with the above inequality. Therefore, $T = +\infty$ and again by the above inequality we deduce $x(+\infty) = +\infty$ since the function on the right side in not summable by assumption (3.19). Therefore, $x'(t^*) = 0$, implying $t^* = 0$ and $x(0) = \nu^+$. Similarly one can show that $x(0) = \nu^-$, a contradiction, by using (3.18). □

Remark 3.4. Assumptions (3.21), (3.22) in the previous non-existence theorem have been introduced just to deal with non-autonomous differential operators. Notice that when dealing with autonomous operators, that is for $a(t,x) = a(x)$, they are trivially satisfied. However, also in the non-autonomous case they hold in many relevant situations. For instance, when $a(t,x)$ has the product structure $a(t,x) = \alpha(t)\beta(x)$, then it is easy to check that assumptions (3.21) and (3.22) hold if one the following conditions is satisfied:

- $\alpha(t)$ is decreasing in $(-\infty, 0)$ and increasing in $(0, +\infty)$;
- α is uniformly continuous in $\mathbb{R}$ and $\inf_{t\in\mathbb{R}}\alpha(t) > 0$;
- $\alpha(t) \sim |t|^{-p}$ as $|t| \to +\infty$ for some $p > 0$.

4. Some asymptotic criteria

In this section we present some operative criteria applicable for operators and right-hand side having the product structure

$$a(t,x) = \alpha(t)\beta(x) \quad \text{and} \quad f(t,x,y) = b(t,x)c(x,y).$$

We will focus on the link between the local behaviors of $c(x,\cdot)$ at $y = 0$ and of $b(\cdot,x)$, $\alpha(\cdot)$ at infinity, which play a key role for the existence or non-existence of solutions.

In what follows we assume that α, β are continuous positive functions, b is a Carathéodory function and c is a continuous function satisfying

$$c(x,y) > 0 \quad \text{for every } y \neq 0 \text{ and } x \in [\nu^-, \nu^+]; \quad c(\nu^-, 0) = c(\nu^+, 0) = 0.$$

In this framework, put $\tilde{m} := \min_{x\in[\nu^-,\nu^+]}\beta(x)$ and $\tilde{M} := \max_{x\in[\nu^-,\nu^+]}\beta(x)$, we have

$$m(t) = \tilde{m}\alpha(t), \quad M(t) = \tilde{M}\alpha(t), \quad \text{for every } t \in \mathbb{R}$$

where recall that $m(t) := \min_{x\in[\nu^-,\nu^+]} a(t,x)$ and $M(t) := \max_{x\in[\nu^-,\nu^+]} a(t,x)$. We put

$$m_\infty := \inf_{t\in\mathbb{R}} \alpha(t) \geq 0. \tag{4.1}$$

4.1. **Case of Φ growing at most linearly.** In this subsection we deal with differential operators Φ satisfying condition (3.1); that is, such that $|\Phi(y)| \leq \Lambda|y|$ whenever every $|y| > H$, for some $H, \Lambda > 0$. With this class of operators we cover differential equations of the type

$$(a(t,x(t))x'(t))' = f(t,x(t),x'(t)).$$

The first two existence theorems are applications of Theorem 3.1.

Proposition 4.1. *Suppose that*

$$t\cdot b(t,x) < 0 \quad \text{for a.e. } t \text{ such that } |t| \geq L, \text{ every } x\in[\nu^-,\nu^+] \tag{4.2}$$

for some $L > 0$ and there exists a function $\lambda \in L^q_{\rm loc}(\mathbb{R})$, $1 \leq q \leq +\infty$, such that

$$|b(t,x)| \leq \lambda(t) \quad \text{for a.e. } t\in\mathbb{R}, \text{ every } x\in[\nu^-,\nu^+]. \tag{4.3}$$

Moreover, assume that there exist real constants $\gamma > 1$, p, δ, with $p < \delta + 1$, satisfying

$$\delta + 1 > p\gamma \tag{4.4}$$

such that for every $x \in [\nu^-,\nu^+]$ we have

$$h_1|t|^p \leq \alpha(t) \leq h_2|t|^p, \quad \text{a.e. } |t| > L, \tag{4.5}$$

$$h_1|t|^\delta \leq |b(t,x)| \leq h_2|t|^\delta, \quad \text{a.e. } |t| > L. \tag{4.6}$$

$$c(x,y) \geq k_1\Phi(|y|)^\gamma \quad \text{for every } y\in\mathbb{R}, \tag{4.7}$$

$$c(x,y) \leq k_2\Phi(|y|)^\gamma, \quad \text{whenever } |y| < \rho, \tag{4.8}$$

$$c(x,y) \leq k_2|\Phi(y)|^{2-\frac{1}{q}} \quad \text{whenever } |y| > H \tag{4.9}$$

for certain positive constants $h_1, h_2, k_1, k_2, \rho, H$. Let (3.1) be satisfied and assume that

$$\limsup_{y\to 0^+} \frac{\Phi(y)}{y^\mu} > 0 \tag{4.10}$$

for some positive constant μ satisfying

$$\mu < \frac{\delta + 1 - p}{\gamma - 1}. \tag{4.11}$$

Then, problem (1.1) admits solutions.

Proof. Without loss of generality we can assume $H > \max\{L, \frac{\nu^+-\nu^-}{2L}\}$. Put $\theta(r) := k_2(\frac{r}{m^*(L)})^{2-\frac{1}{q}}$ for $r > 0$, from (4.3) and (4.9) it is immediate to verify the validity of conditions (3.3) and (3.4). Put

$$K(t) := \begin{cases} k_1\int_L^t \min\{\min_{x\in[\nu^-,\nu^+]} b(-\tau,x), -\max_{x\in[\nu^-,\nu^+]} b(\tau,x)\}\,d\tau, & t\geq L\\ 0, & 0\leq t\leq L. \end{cases}$$

By condition (4.3) we have $K \in W^{1,1}_{\rm loc}([0,+\infty))$ and by (4.2) we have that K is strictly increasing for $t \geq L$. Observe that by (4.7) it follows that

$$f(t,x,y) = b(t,x)c(x,y) \leq k_1 b(t,x)\Phi(|y|)^\gamma \leq -K'(t)\Phi(|y|)^\gamma$$

and

$$f(-t,x,y) = b(-t,x)c(x,y) \geq k_1\, b(-t,x)\Phi(|y|)^\gamma \geq K'(t)\Phi(|y|)^\gamma$$

for a.e. $t \geq L$, every $x \in [\nu^-,\nu^+]$ and every $y \in \mathbb{R}$. Then, condition (3.6) of Theorem 3.1 holds.

Now, from (4.6) it follows that $h_1 k_1 t^\delta \leq K'(t)$ for a.e. $t \geq L$ and by (4.5) we deduce that

$$\int_L^t \frac{K'(\tau)}{\alpha(\tau)^\gamma}\, \mathrm{d}\tau \geq \frac{h_1 k_1}{h_2^\gamma} \int_L^t \tau^{\delta - p\gamma}\, \mathrm{d}\tau \quad \text{for every } t > L.$$

Hence, by (4.4) we obtain $\int_L^t \frac{K'(\tau)}{\alpha(\tau)^\gamma}\, \mathrm{d}\tau \to +\infty$ as $t \to +\infty$ and so by condition (4.5) we deduce that for every fixed $C \in \mathbb{R}$ the function $N_C(t)$ defined in (3.5) satisfies

$$\Phi(N_C(t)) \leq \text{const. } t^{\frac{\delta+1-p\gamma}{1-\gamma} - p} \quad \text{for } t \text{ large enough.} \tag{4.12}$$

Since $p < \delta + 1$, we obtain $\frac{\delta+1-p\gamma}{1-\gamma} - p < 0$, so $N_C(t) \to 0$ as $t \to +\infty$. Therefore, by (4.10) and (4.12) we deduce

$$N_C(t) \leq \text{const. } t^{\frac{\delta+1-p\gamma}{\mu(1-\gamma)} - \frac{p}{\mu}} \quad \text{for } t \text{ large enough.}$$

implying that $N_C(t) \in L^1(\mathbb{R})$ by (4.11). Then also (3.5) holds.

Since $\lim_{|t|\to+\infty} N_C(t) = 0$ a constant $L_C^* > L$ exists such that $N_C(t) \leq \rho$ for every $|t| \geq L_C^*$. Let us define $\hat{C} := \max_{|t| \leq L_C^*} N_C(t)$ and

$$\eta_C(t) := \begin{cases} \max_{x\in[\nu^-,\nu^+]} |b(t,x)| \cdot \max_{(x,y)\in[\nu^-,\nu^+]\times[-\hat{C},\hat{C}]} c(x,y) & \text{if } |t| \leq L_C^* \\ h_2 k_2 |t|^\delta \Phi(N_C(t))^\gamma & \text{if } |t| > L_C^*. \end{cases}$$

By (4.6) and (4.8), for a.e. $t \in \mathbb{R}$, for every $x \in [\nu^-,\nu^+]$ and every $y \in \mathbb{R}$ such that $|y| \leq N_C(t)$ we have

$$|f(t,x,y)| = |b(t,x)| c(x,y) \leq \eta_C(t),$$

so it remains to prove that $\eta_C \in L^1(\mathbb{R})$.

By (4.3) and the continuity of the function c we have $\eta_C \in L^1([-L_C^*, L_C^*])$. Moreover, when $|t| > L_C^*$, by (4.12) we have

$$\eta_c(t) \leq \text{const. } |t|^{\delta + \gamma\frac{\delta+1-p\gamma}{1-\gamma} - p\gamma} = \text{const. } |t|^{\frac{\delta+\gamma-p\gamma}{1-\gamma}}$$

implying that $\eta_c(t) \in L^1(\mathbb{R} \setminus [-L_C^*, L_C^*])$ by condition (4.4). Therefore, Theorem 3.1 applies and guarantees the assertion of the present result. $\square$

Remark 4.2. Notice that $\gamma \leq 2 - \frac{1}{q} \leq 2$ is a necessary compatibility condition in order to have both (4.7) and (4.9) for large $|y|$. But when $m_\infty > 0$ (see (4.1)), then condition (4.7) can be weakened, requiring that it holds only for $|y|$ small enough, as the following result states.

Proposition 4.3. *Let all the assumptions of Proposition 4.1 be satisfied, with the exception of* (4.7)*, replaced by*

$$c(x,y) \geq k_1 \Phi(|y|)^\gamma \quad \textit{whenever } |y| < \rho. \tag{4.13}$$

Moreover, assume that $m_\infty > 0$*. Then, problem* (1.1) *admits solutions.*

Proof. For every fixed $C > 0$ let

$$\Gamma_C := \max\big\{\rho, \Phi^{-1}\big(\frac{M^*(L)}{m_\infty}\Phi(C)\big)\big\}, \quad \hat{m}_C := \min_{(x,y)\in[\nu^-\nu^+]\times[\rho,\Gamma_C]} c(x,y),$$
$$h_C := \min\{k_1, \frac{\hat{m}_C}{\Phi(\Gamma_C)^\gamma}\}.$$

Note that

$$c(x,y) \geq h_C\Phi(|y|)^\gamma \quad \text{for every } x \in [\nu^-,\nu^+], \text{ whenever } |y| \leq \Gamma_C.$$

So, put

$$K_C(t) := h_C \int_L^t \min\{\min_{x\in[\nu^-,\nu^+]} b(-\tau,x), -\max_{x\in[\nu^-,\nu^+]} b(\tau,x)\}\,\mathrm{d}\tau$$

for $t \geq L$ (and $K_C(t) = 0$ for $t \in [0,L]$), we deduce that (3.6) holds since $N_C(t) \leq \Gamma_C$ for every $t \geq L$. From now on, the proof proceeds as that of Proposition 4.1. □

In view of the proof of Proposition 4.1, condition (4.11) guarantees the summability of the function $N_C(t)$, in the case when (4.4) holds. The following results cover cases when the reversed inequality holds.

Proposition 4.4. *Let all the assumptions of Proposition 4.1 hold, with the exception of* (4.4) *and* (4.11)*, replaced by*

$$\delta + 1 < p\gamma; \tag{4.14}$$
$$p > \mu. \tag{4.15}$$

Then problem (1.1) *admits solutions.*

Proof. If K is the function defined in the proof of Proposition 4.1, by (4.14) we have

$$\int_L^t \frac{K'(\tau)}{\alpha(\tau)^\gamma}\,\mathrm{d}\tau \leq \text{const.} \int_L^t \tau^{\delta-p\gamma}\,\mathrm{d}\tau \leq \text{const.}$$

Therefore, $\Phi(N_C(T)) \sim \frac{\text{const.}}{\alpha(t)}$ as $t \to +\infty$ (see (3.5)), hence $N_C(t) \leq \text{const.}t^{-p/\mu}$ implying that N_C is summable by virtue of (4.15).

Moreover, if η_C is defined as in the proof of Proposition 4.1, then

$$\eta_C(t) \leq \text{const. } |t|^\delta \frac{1}{\alpha(t)^\gamma} \leq \text{const. } t^{\delta-p\gamma} \quad \text{for } t \text{ large enough}$$

and we conclude that η_C is summable by condition (4.14). Then, the proof proceeds as that of Theorem 4.1 □

By the same proof of the previous Proposition one can prove also the following result, applicable when $m_\infty > 0$.

Proposition 4.5. *Let all the assumptions of Proposition 4.3 hold, with the exception of* (4.4) *and* (4.11)*, replaced by* (4.14) *and* (4.15)*. Then problem* (1.1) *admits solutions.*

We state two non-existence results, obtained by applying Theorem 3.3.

Proposition 4.6. *Suppose that*

$$t \cdot b(t,x) \le 0 \quad \text{for a.e. } t \in \mathbb{R} \text{ and every } x \in [\nu^-, \nu^+]$$

and let there exist real constants δ, $\gamma > 1$, $\Lambda > 0$ *and a positive function* $\ell(t) \in L^1([0,\Lambda])$ *such that*

$$|b(t,x)| \le \lambda_1 |t|^\delta, \quad \text{for every } x \in [\nu^-, \nu^+],\ \text{a.e. } |t| > \Lambda, \tag{4.16}$$

$$|b(t,x)| \le \ell(|t|) \quad \text{for a.e. } |t| \le \Lambda, x \in [\nu^-, \nu^+], \tag{4.17}$$

$$c(x,y) \le \lambda_2 \Phi(y)^\gamma, \quad \text{for every } x \in [\nu^-, \nu^+],\ 0 < y < \rho \tag{4.18}$$

for some positive constants $\lambda_1, \lambda_2, \rho$. *Moreover, assume that* (4.5) *holds for some constants* h_1, h_2, p *such that*

$$\delta + 1 > p\gamma. \tag{4.19}$$

Furthermore, suppose that

$$\limsup_{y \to 0} \frac{\Phi(y)}{y^\mu} < +\infty \tag{4.20}$$

for some positive constant μ *satisfying*

$$\mu \ge \frac{\delta + 1 - p}{\gamma - 1}. \tag{4.21}$$

Also suppose that there exist two constants $\epsilon, H > 0$ *such that*

$$\alpha(t) \le H\alpha(t+r) \quad \text{for every } t > 0 \text{ and } 0 < r < \epsilon, \tag{4.22}$$

$$\alpha(t+r) \le H\alpha(t) \quad \text{for every } t < 0 \text{ and } 0 < r < \epsilon. \tag{4.23}$$

Then, problem (1.1) *does not admit solutions.*

Proof. Put

$$K(t) := \begin{cases} \lambda_2 \int_0^t \ell(\tau)\,\mathrm{d}\tau & \text{for } t \in [0,\Lambda] \\ \lambda_2 \int_0^\Lambda \ell(\tau)\,\mathrm{d}\tau + \lambda_1\lambda_2 \int_\Lambda^t \tau^\delta\,\mathrm{d}\tau & \text{for } t > \Lambda \end{cases}$$

we have that K is a strictly increasing function belonging to $W^{1,1}_{\rm loc}([0,+\infty)$ and one can easily verify that conditions (4.16), (4.17) and (4.18) guarantee the validity of (3.17) and (3.18). Moreover, by (4.5) we obtain

$$\int_L^t \frac{K'(\tau)}{\alpha(\tau)^\gamma}\,\mathrm{d}\tau \ge \text{const. } t^{\delta - p\gamma + 1} \quad \text{for } t \text{ large enough,}$$

hence by (4.19) we have $\int_L^t \frac{K'(\tau)}{\alpha(\tau)^\gamma}\,\mathrm{d}\tau \to +\infty$ as $t \to +\infty$. Therefore, by (4.5) and (4.16), if $N_C(t)$ is the function defined in (3.19) we have

$$\Phi(N_C(t)) \ge \text{const. } t^{\frac{\delta - p\gamma + 1}{1-\gamma} - p} \quad \text{for } t \text{ large enough}$$

implying that

$$N_C(t) \ge \text{const. } t^{\frac{\delta - p\gamma + 1}{\mu(1-\gamma)} - \frac{p}{\mu}} \quad \text{for } t \text{ large enough}$$

by virtue of (4.20). Finally, assumption (4.21) implies that $N_C(t)$ is not summable in $[L, +\infty)$ and the assertion follows as an immediate application of Theorem 3.3. $\square$

When condition (4.19) does not hold, we can use the following non-existence result.

Proposition 4.7. *Let all the assumption of Proposition 4.6 be satisfied with the exception of* (4.19) *and* (4.21)*, which are replaced by*

$$\delta + 1 \le p\gamma, \tag{4.24}$$

$$p \le \mu. \tag{4.25}$$

Then, problem (1.1) *does not admit solutions.*

Proof. With the same notation of the proof of Proposition 4.6, notice that under condition (4.24) we have $\limsup_{t\to+\infty} \int_L^t \frac{K'(\tau)}{\alpha(\tau)^\gamma}\,d\tau < +\infty$, hence $\Phi(N_C(t)) \ge$ const. t^{-p}, implying that $N_C(t) \ge$ const. $t^{-p/\mu}$. Therefore N_C is not summable at infinity owing to assumption (4.25) and the assertion follows from Theorem 3.3. $\square$

For sufficient conditions ensuring the validity of assumptions (4.22) and (4.23), see Remark 3.4. As an immediate application of the previous results, the following operative criteria hold.

Corollary 4.8. *Let* (3.1) *be satisfied. Let* $f(t,x,y) = h(t)g(x)c(y)$*, where* $h \in L^q_{\rm loc}(\mathbb{R})$*, for some* $1 \le q \le +\infty$*,* c *is continuous in* $\mathbb{R}$ *and* g *is continuous and positive in* $[\nu^-, \nu^+]$*. Assume that* $c(y) > 0$ *for* $y \ne 0$*;* $t \cdot h(t) \le 0$ *for every* t *and suppose that there exist constants* $C_1, \dots, C_4 > 0$ *such that*

$$\alpha(t) \sim C_1|t|^p \text{ as } |t| \to +\infty, \quad \text{for some } p \in \mathbb{R}, \tag{4.26}$$

$$|h(t)| \sim C_2|t|^\delta \text{ as } |t| \to +\infty, \quad \text{for some } \delta \in \mathbb{R}, \tag{4.27}$$

$$\Phi(y) \sim C_3|y|^\mu \text{ as } y \to 0, \quad \text{for some } \mu > 0, \tag{4.28}$$

$$c(y) \sim C_4|y|^\beta \text{ as } y \to 0, \quad \text{for some } \beta > \mu, \tag{4.29}$$

with

$$\delta + 1 > \frac{p\beta}{\mu}. \tag{4.30}$$

Then, if conditions (4.22)*,* (4.23) *hold and* $\mu \le \beta + p - \delta - 1$*, Problem* (1.1) *has no solution.*

Viceversa, if $p < \delta + 1$*,* $\mu > \beta + p - \delta - 1$ *and we further assume that*

$$\limsup_{|y|\to+\infty} c(y)|\Phi(y)|^{\frac{1}{q}-2} < +\infty, \tag{4.31}$$

$$c(y) \ge k_1\Phi(|y|)^{\frac{\beta}{\mu}} \quad \text{for every } y \in \mathbb{R} \tag{4.32}$$

for some $k_1 > 0$*, then* (1.1) *admits solutions.*

The assertion of the above corollary is an immediate consequence of Propositions 4.1 and 4.6 taking $\gamma = \beta/\mu$.

As observed in Remark 4.2, $\frac{\beta}{\mu} \le 2 - \frac{1}{q} \le 2$ is a necessary compatibility condition to have both (4.31) and (4.32), but when $m_\infty > 0$ it can be removed, as we state in the following result, application of Proposition 4.3.

Corollary 4.9. *Let all the assumption of Corollary 4.8 hold, with the exception of* (4.32)*. Then if* $m_\infty > 0$*, problem* (1.1) *admits solutions.*

When assumption (4.30) is not satisfied, we can use the following result, consequence of Propositions 4.4 and 4.7.

Corollary 4.10. *Let all the assumptions of Corollary 4.8 be satisfied, with the exception of* (4.30)*, which is replaced by*
$$\delta + 1 < \frac{p\beta}{\mu}. \tag{4.33}$$
Then, if conditions (4.22)*,* (4.23) *hold and $p \le \mu$, Problem* (1.1) *has no solution.*

Viceversa, if $\mu < p$ and we further assume (4.31) *and* (4.32)*, then* (1.1) *admits solutions.*

Finally, a result analogous to Corollary 4.10 holds when condition (4.32) is removed, provided that $m_\infty > 0$, as in Corollary 4.9.

We provide now an application of the previous results.

Example 4.11. Let $\Phi(y) := y$, $\alpha(t) := |t|^p$, $f(t,x,y) = -t|t|^s|y|^\beta$, for some constants p, s, β (we avoid to introduce a dependence on x since we have showed that it does not influence the existence or non-existence of solutions). In this case we have $\mu = 1$. Assume $s + 1 \ge 0$ (so we can take $q = +\infty$) and $1 < \beta \le 2$ (so that (4.31) holds).

Then, if $s + 2 > p\beta$, $s + 2 > p$, problem (1.1) admits solutions (whatever ν^-, ν^+ may be), if and only if $p < s + 3 - \beta$, as a consequence of Corollary 4.8. Otherwise, if $s + 2 < p\beta$, problem (1.1) admits solutions if and only if $p > 1$, as a consequence of Corollary 4.10.

4.2. **Case of Φ having superlinear growth.** We now deal with operators Φ having possibly superlinear growth at infinity, that is we now remove condition (3.1). The non-existence Propositions 4.6 and 4.7 hold also in this case, since they do not require condition (3.1). As for the existence results, we now use Theorem 3.2 instead of Theorem 3.1, by assuming (3.16).

As it will be clear later, condition (3.16) is not compatible with (4.7) so from now on we will assume $m_\infty > 0$. However, in the special case of the $p-$laplacian, this condition can be removed, as we will show in a forthcoming paper.

Proposition 4.12. *Let all the assumptions of Proposition 4.3 (or Proposition 4.5) hold, with the exception of* (4.9) *which is replaced by*
$$\lim_{|y|\to+\infty} \frac{\max_{x\in[\nu^-,\nu^+]} c(x,y)}{|\Phi(y)|} = 0 \tag{4.34}$$
Then, problem (1.1) *admits solutions.*

Proof. Put
$$\theta(r) := \max_{(t,x)\in[-L,L]\times[\nu^-,\nu^+]} \Big(\max\Big\{ c(x, \Phi^{-1}(\frac{r}{a(t,x)})),\ c(x, \Phi^{-1}(-\frac{r}{a(t,x)})\Big\}\Big),$$
it is immediate to check that θ is a continuous function on $[0, +\infty)$, such that
$$\theta(a(t,x)|\Phi(y)|) \ge c(x,y) \quad \text{for every } t \in [-L, L], x \in [\nu^-, \nu^+], y \in \mathbb{R},$$
hence (3.3) holds. Moreover, by (4.34), for every $\epsilon > 0$ there exists a real c_ϵ such that
$$c(x,y) \le \epsilon|\Phi(y)| \quad \text{for every } x \in [\nu^-, \nu^+], |y| \ge c_\epsilon.$$
Hence, for every $s \ge M^*(L)\max\{\Phi(c_\epsilon), -\Phi(-c_\epsilon)\}$ we have $\theta(s) \le \frac{\epsilon}{m^*(L)} s$; that is,
$$\lim_{s\to+\infty} \frac{\theta(s)}{s} = 0.$$

Then (3.16) holds and the proof proceeds as that of Proposition 4.3 [or Proposition 4.5], applying Theorem 3.2 instead of Theorem 3.1. □

Note that condition (4.34) is not compatible with (4.7), since $\gamma > 1$. As applications of the previous result, the following operative criteria hold.

Corollary 4.13. *Let $f(t,x,y) = h(t)g(x)c(y)$, where $h \in L^q_{\rm loc}(\mathbb{R})$, for some $1 \le q \le +\infty$, c is continuous in $\mathbb{R}$ and g is continuous and positive in $[\nu^-, \nu^+]$. Assume that $c(y) > 0$ for $y \neq 0$; $t \cdot h(t) \le 0$ for every t and suppose that there exist constants $C_1, \dots, C_4 > 0$ such that (4.26), (4.27), (4.28), (4.29), (4.30) hold with $p < \delta + 1$. Moreover, assume that $\mu > \beta + p - \delta - 1$, $m_\infty > 0$ and*

$$\lim_{|y| \to +\infty} \frac{c(y)}{|\Phi(y)|} = 0.$$

Then problem (1.1) admits solutions.

Corollary 4.14. *Let all the assumptions of Corollary 4.13 be satisfied, with the exception of (4.30) which is replaced by (4.33). Then if $p \ge \mu$, problem (1.1) admits solutions.*

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Cristina Marcelli

Dipartimento di Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 60131 Ancona, Italy

E-mail address: `marcelli@dipmat.univpm.it`