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APPROXIMATE SOLUTIONS TO NEUTRAL TYPE FINITE DIFFERENCE EQUATIONS

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Abstract. In this article, we study the approximate solutions and the dependency of solutions on parameters to a neutral type finite difference equation, under a given initial condition. A fundamental finite difference inequality, with explicit estimate, is used to establish the results.

1. Introduction

Let $\mathbb{R}$ denote the set of real numbers $\mathbb{R}_+ = [0, \infty)$, and $N_0 = \{0, 1, 2, \dots\}$. Let $D(S_1, S_2)$ denote the class of discrete functions from the set S_1 to the set S_2. For any function $z \in D(N_0, \mathbb{R})$ we define the operator Δ by $\Delta z(n) = z(n+1) - z(n)$. We use the conventions that empty sums and products are taken to be 0 and 1 respectively. In the present article we consider the neutral type finite difference equation

$$\Delta x(n) = f(n, x(n), \Delta x(n)), \tag{1.1}$$

with the initial condition

$$x(0) = x_0, \tag{1.2}$$

for $n \in N_0$, where $f \in D(N_0 \times \mathbb{R}^2, \mathbb{R})$ is a given function, x_0 is a real constant and x is an function to be found. The continuous analogue of equation (1.1) is often referred to as a neutral differential equation, see for example [3, p. 155] and [4]. It is easy to observe that the equation (1.1) contains as special case the discrete analogue of the well known Clairaut's differential equation, see [1, pp 117-118] and [8].

In general the solutions to (1.1) cannot be found analytically and thus will need more insight to study the qualitative properties of its solutions. The problem of existence of solutions for (1.1) with (1.2) (IVP (1.1)-(1.2), for short) can be dealt with the method employed in [13, Theorem 1] with suitable modifications, see also [1, 2, 5, 6, 7]. In this article we offer the conditions for the error evaluation of approximate solutions of (1.1)-(1.2) by establishing some new bounds on solutions of approximate problems. We also study the dependency of solutions of (1.1)-(1.2) on parameters. The main tool employed in the analysis is based on the application of a certain basic finite difference inequality with explicit estimate given in [9] (see also [1, Theorem 4.1.1] and [9, Theorem 1.2.3]). Results on two independent

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variables are also given as a generalization of (1.1)-(1.2). A particular feature of our approach is to present in a simple and unified way conditions some of the important qualitative properties of solutions of (1.1)-(1.2).

2. Main Results

Let $x_i(n) \in D(N_0, \mathbb{R})$ $(i = 1, 2)$ be functions such that $\Delta x_i(n)$ exist for $n \in N_0$ and satisfy the inequalities

$$|\Delta x_i(n) - f(n, x_i(n), \Delta x_i(n))| \le \varepsilon_i, \tag{2.1}$$

for given constants $\varepsilon_i \ge 0$ where it is supposed that the initial conditions

$$x_i(0) = x_i, \tag{2.2}$$

are fulfilled. Then we call $x_i(n)$ $(i = 1, 2)$ the ε_i-approximate solutions of (1.1)-(1.2).

We use the following finite difference inequality established in [9] (see also [1, 10]).

Lemma 2.1. *Let $u, a, b, p \in D(N_0, \mathbb{R}_+)$ and*

$$u(n) \le a(n) + p(n) \sum_{s=0}^{n-1} b(s)u(s), \tag{2.3}$$

for $n \in N_0$. Then

$$u(n) \le a(n) + p(n) \sum_{s=0}^{n-1} a(s)b(s) \prod_{\sigma=s+1}^{n-1} [1 + b(\sigma)p(\sigma)], \tag{2.4}$$

for $n \in N_0$.

The following theorem estimates the difference between the two approximate solutions of (1.1)-(1.2).

Theorem 2.2. *Suppose that the function f in (1.1) satisfies the condition*

$$|f(n, x, y) - f(n, \bar{x}, \bar{y})| \le p(n)[|x - \bar{x}| + |y - \bar{y}|], \tag{2.5}$$

where $p \in D(N_0, \mathbb{R}_+)$ and $p(n) < 1$ for $n \in N_0$. Let $x_i(n)$ $(i = 1, 2)$ be ε_i-approximate solutions of equation (1.1) with (2.2) on N_0 such that

$$|x_1 - x_2| \le \delta, \tag{2.6}$$

where $\delta \ge 0$ is a constant. Then

$$\begin{aligned} &|x_1(n) - x_2(n)| + |\Delta x_1(n) - \Delta x_2(n)| \\ &\le A(n) + B(n) \sum_{s=0}^{n-1} A(s)p(s) \prod_{\sigma=s+1}^{n-1} [1 + p(\sigma)B(\sigma)], \end{aligned} \tag{2.7}$$

for $n \in N_0$, where

$$A(n) = \frac{(\varepsilon_1 + \varepsilon_2)(n + 1) + \delta}{1 - p(n)}, \quad B(n) = \frac{1}{1 - p(n)}. \tag{2.8}$$

Proof. Since $x_i(n)$ $(i=1,2)$ for $n \in N_0$ are ε_i-approximate solutions of (1.1) with (2.2), we have (2.1). By taking $n = s$ in (2.1) and summing up both sides over s from 0 to $n-1$, we have

$$\begin{aligned} \varepsilon_i n &\geq \sum_{s=0}^{n-1} |\Delta x_i(s) - f(s, x_i(s), \Delta x_i(s))| \\ &\geq |\sum_{s=0}^{n-1} \{\Delta x_i(s) - f(s, x_i(s), \Delta x_i(s))\}| \\ &= \Big| x_i(n) - x_i(0) - \sum_{s=0}^{n-1} f(s, x_i(s), \Delta x_i(s)) \Big|, \end{aligned} \tag{2.9}$$

for $i = 1, 2$. From (2.9) and using the elementary inequalities

$$|v - z| \leq |v| + |z|, \quad |v| - |z| \leq |v - z|, \tag{2.10}$$

we observe that

$$\begin{aligned} (\varepsilon_1 + \varepsilon_2)n &\geq \Big| x_1(n) - x_1(0) - \sum_{s=0}^{n-1} f(s, x_1(s), \Delta x_1(s)) \Big| \\ &\quad + \Big| x_2(n) - x_2(0) - \sum_{s=0}^{n-1} f(s, x_2(s), \Delta x_2(s)) \Big| \\ &\geq \Big| x_1(n) - x_1(0) - \sum_{s=0}^{n-1} f(s, x_1(s), \Delta x_1(s)) \\ &\quad - \Big(x_2(n) - x_2(0) - \sum_{s=0}^{n-1} f(s, x_2(s), \Delta x_2(s)) \Big) \Big| \\ &\geq |x_1(n) - x_2(n)| - |x_1(0) - x_2(0)| \\ &\quad - \Big| \sum_{s=0}^{n-1} f(s, x_1(s), \Delta x_1(s)) - \sum_{s=0}^{n-1} f(s, x_2(s), \Delta x_2(s)) \Big|. \end{aligned} \tag{2.11}$$

Moreover, from (2.1) and using the elementary inequalities in (2.10), we observe that

$$\begin{aligned} &\varepsilon_1 + \varepsilon_2 \\ &\geq |\Delta x_1(n) - f(n, x_1(n), \Delta x_1(n))| + |\Delta x_2(n) - f(n, x_2(n), \Delta x_2(n))| \\ &\geq |\{\Delta x_1(n) - f(n, x_1(n), \Delta x_1(n))\} - \{\Delta x_2(n) - f(n, x_2(n), \Delta x_2(n))\}| \\ &\geq |\Delta x_1(n) - \Delta x_2(n)| - |f(n, x_1(n), \Delta x_1(n)) - f(n, x_2(n), \Delta x_2(n))|. \end{aligned} \tag{2.12}$$

Let $u(n) = x_1(n) - x_2(n)$ for $n \in N_0$. From (2.11), (2.12) and using the hypotheses, we observe that

$$\begin{aligned} &|u(n)| + |\Delta u(n)| \\ &\leq (\varepsilon_1 + \varepsilon_2)n + |u(0)| + \sum_{s=0}^{n-1} |f(s, x_1(s), \Delta x_1(s)) - f(s, x_2(s), \Delta x_2(s))| \\ &\quad + (\varepsilon_1 + \varepsilon_2) + |f(n, x_1(n), \Delta x_1(n)) - f(n, x_2(n), \Delta x_2(n))| \end{aligned}$$

$$\leq (\varepsilon_1+\varepsilon_2)(n+1)+\delta+\sum_{s=0}^{n-1} p(s)[|u(s)|+|\Delta u(s)|]+p(n)[|u(n)|+|\Delta u(n)|].$$

From this inequality, we obtain

$$|u(n)|+|\Delta u(n)| \leq A(n)+B(n)\sum_{s=0}^{n-1} p(s)[|u(s)|+|\Delta u(s)|], \tag{2.13}$$

for $n \in N_0$, where $A(n), B(n)$ are given as in (2.8). Now an application of Lemma 2.1 to (2.13) yields (2.7). □

Remark 2.3. We note that, if $x_1(n)$ is a solution of (1.1)-(1.2) with $x_1(0)=x_1$, then we have $\varepsilon_1=0$ and from (2.7) we see that $x_1(n)\to x_2(n)$ as $\varepsilon_2\to 0$ and $\delta\to 0$. Moreover, if we put: (i) $\varepsilon_1=\varepsilon_2=0$, $x_1=x_2$ in (2.7), then the uniqueness of solutions of (1.1) is established, and (ii) $\varepsilon_1=\varepsilon_2=0$ in (2.7), then we obtain the bound which shows the dependency of solutions of (1.1) on given initial values. For the estimate on the difference between the two approximate solutions of special version of (1.1) when the function f in (1.1) is independent of $\Delta x(n)$, by using comparison theorem, we refer the interested reader to [13, Theorem 1.5].

Consider (1.1)-(1.2) together with the initial-value problem

$$\Delta y(n) = g(n, y(n), \Delta y(n)), \tag{2.14}$$

$$y(0)=y_0, \tag{2.15}$$

for $n \in N_0$, where $g \in D(N_0\times\mathbb{R}^2, R)$ and y_0 is a real constant.

Next, we shall prove the following theorem concerning the closeness of the solutions of (1.1)-(1.2) and (2.14)-(2.15).

Theorem 2.4. *Suppose that the function f in* (1.1) *satisfies* (2.5)*, and that there exist constants $\bar{\varepsilon}\geq 0$, $\bar{\delta}\geq 0$ such that*

$$|f(n,u,v)-g(n,u,v)| \leq \bar{\varepsilon}, \tag{2.16}$$

$$|x_0-y_0| \leq \bar{\delta}, \tag{2.17}$$

where f, x_0 and g, y_0 are as in (1.1)-(1.2) *and* (2.14)-(2.15)*. Let $x(n)$ and $y(n)$ be respectively, solutions of* (1.1)-(1.2) *and* (2.14)-(2.15) *on N_0. Then*

$$\begin{aligned} &|x(n)-y(n)|+|\Delta x(n)-\Delta y(n)| \\ &\leq \bar{A}(n)+B(n)\sum_{s=0}^{n-1}\bar{A}(s)p(s)\prod_{\sigma=s+1}^{n-1}[1+p(\sigma)B(\sigma)], \end{aligned} \tag{2.18}$$

for $n\in N_0$, where

$$\bar{A}(n)=\frac{\bar{\varepsilon}(n+1)+\bar{\delta}}{1-p(n)}, \tag{2.19}$$

and $B(n)$ is as in (2.8).

Proof. Let $v(n) = x(n) - y(n)$ for $n \in N_0$. Using the facts that $x(n)$ and $y(n)$ are the solutions of (1.1)-(1.2) and (2.14)-(2.15) and hypotheses, we observe that

$$\begin{aligned} |v(n)| + |\Delta v(n)| &\le |x_0 - y_0| + \sum_{s=0}^{n-1} |f(s, x(s), \Delta x(s)) - f(s, y(s), \Delta y(s))| \\ &\quad + \sum_{s=0}^{n-1} |f(s, y(s), \Delta y(s)) - g(s, y(s), \Delta y(s))| \\ &\quad + |f(n, x(n), \Delta x(n)) - f(n, y(n), \Delta y(n))| \\ &\quad + |f(n, y(n), \Delta y(n)) - g(n, y(n), \Delta y(n))| \\ &\le \bar{\delta} + \sum_{s=0}^{n-1} p(s)[|v(s)| + |\Delta v(s)|] + \sum_{s=0}^{n-1} \bar{\varepsilon} \\ &\quad + p(n)[|v(n)| + |\Delta v(n)|] + \bar{\varepsilon}. \end{aligned} \tag{2.20}$$

From (2.20), we obtain

$$|v(n)| + |\Delta v(n)| \le \bar{A}(n) + B(n) \sum_{s=0}^{n-1} p(s)[|v(s)| + |\Delta v(s)|]. \tag{2.21}$$

Now an application of Lemma 2.1 to (2.21) yields (2.18). □

Remark 2.5. We note that the result given in Theorem 2.4 relates the solutions of (1.1)-(1.2) and (2.14)-(2.15) in the sense that if f is close to g and x_0 is close to y_0, then the solutions of (1.1)-(1.2) and (2.14)-(2.15) are also close together. For further results on the qualitative properties of solutions of various types of finite difference equations, see [1, 2, 5, 6, 7, 9, 10, 11, 13].

A slight variant of Theorem 2.4 is embodied in the following theorem.

Theorem 2.6. *Suppose that the functions f and g in (1.1) and (2.14) satisfy the condition*

$$|f(n, u, v) - g(n, \bar{u}, \bar{v})| \le q(n)[|u - \bar{u}| + |v - \bar{v}|], \tag{2.22}$$

where $q \in D(N_0, \mathbb{R}_+)$ and $q(n) < 1$ for $n \in N_0$ and (2.17) holds. Let $x(n)$ and $y(n)$ be respectively, solutions of (1.1)-(1.2) and (2.14)-(2.15) on N_0. Then

$$\begin{aligned} &|x(n) - y(n)| + |\Delta x(n) - \Delta y(n)| \\ &\le A_0(n) + B_0(n) \sum_{s=0}^{n-1} A_0(s) q(s) \prod_{\sigma=s+1}^{n-1} [1 + q(\sigma) B_0(\sigma)], \end{aligned} \tag{2.23}$$

for $n \in N_0$, where

$$A_0(n) = \frac{\bar{\delta}}{1 - q(n)}, \quad B_0(n) = \frac{1}{1 - q(n)}. \tag{2.24}$$

Proof. Let $w(n) = x(n) - y(n)$ for $n \in N_0$. Using the facts that $x(n)$ and $y(n)$ are respectively, solutions of (1.1)-(1.2) and (2.14)-(2.15), and the hypotheses, we observe that

$$
\begin{aligned}
&|w(n)| + |\Delta w(n)| \\
&\le |x_0 - y_0| + \sum_{s=0}^{n-1} |f(s, x(s), \Delta x(s)) - g(s, y(s), \Delta y(s))| \\
&\quad + |f(n, x(n), \Delta x(n)) - g(n, y(n), \Delta y(n))| \\
&\le \bar{\delta} + \sum_{s=0}^{n-1} q(s)[|w(s)| + |\Delta w(s)|] + q(n)[|w(n)| + |\Delta w(n)|].
\end{aligned} \tag{2.25}
$$

From (2.25), we obtain

$$
|w(n)| + |\Delta w(n)| \le A_0(n) + B_0(n) \sum_{s=0}^{n-1} q(s)[|w(s)| + |\Delta w(s)|]. \tag{2.26}
$$

Now an application of Lemma 2.1 to (2.26) yields (2.23). □

We consider the following two neutral type difference equations

$$
\Delta z(n) = h(n, z(n), \Delta z(n), \mu), \tag{2.27}
$$

$$
\Delta z(n) = h(n, z(n), \Delta z(n), \mu_0), \tag{2.28}
$$

with the initial condition

$$
z(0) = z_0, \tag{2.29}
$$

for $n \in N_0$, where $h \in D(N_0 \times \mathbb{R}^3, R)$, z_0 is a real constant and μ, μ_0 are parameters.

The following theorem deals with the dependency of solutions of (2.27)-(2.29) and (2.28)-(2.29) on parameters.

Theorem 2.7. *Suppose that the function h in* (2.27), (2.28) *satisfy the conditions*

$$
|h(n, x, y, \mu) - h(n, \bar{x}, \bar{y}, \mu)| \le a(n)[|x - \bar{x}| + |y - \bar{y}|], \tag{2.30}
$$

$$
|h(n, x, y, \mu) - h(n, x, y, \mu_0)| \le b(n)|\mu - \mu_0|, \tag{2.31}
$$

where $a, b \in D(N_0, \mathbb{R}_+)$ and $a(n) < 1$ for $n \in N_0$. Let $z_1(n)$ and $z_2(n)$ be the solutions of (2.27)-(2.29) *and* (2.28)-(2.29) *respectively. Then*

$$
\begin{aligned}
&|z_1(n) - z_2(n)| + |\Delta z_1(n) - \Delta z_2(n)| \\
&\le C(n) + D(n) \sum_{s=0}^{n-1} C(s) a(s) \prod_{\sigma=s+1}^{n-1} [1 + a(\sigma) D(\sigma)],
\end{aligned} \tag{2.32}
$$

for $n \in N_0$, where

$$
C(n) = \frac{|\mu - \mu_0|}{1 - a(n)} \Big[b(n) + \sum_{s=0}^{n-1} b(s) \Big], \quad D(n) = \frac{1}{1 - a(n)}. \tag{2.33}
$$

Proof. Let $e(n) = z_1(n) - z_2(n)$ for $n \in N_0$. Using the facts that $z_1(n)$ and $z_2(n)$ are the solutions of (2.27)-(2.29) and (2.28)-(2.29) respectively, we observe that

$$\begin{aligned} |e(n)| + |\Delta e(n)| \le \sum_{s=0}^{n-1} &|h(s, z_1(s), \Delta z_1(s), \mu) - h(s, z_2(s), \Delta z_2(s), \mu)| \\ &+ \sum_{s=0}^{n-1} |h(s, z_2(s), \Delta z_2(s), \mu) - h(s, z_2(s), \Delta z_2(s), \mu_0)| \\ &+ |h(n, z_1(n), \Delta z_1(n), \mu) - h(n, z_2(n), \Delta z_2(n), \mu)| \\ &+ |h(n, z_2(n), \Delta z_2(n), \mu) - h(n, z_2(n), \Delta z_2(n), \mu_0)|. \end{aligned} \tag{2.34}$$

The rest of the proof can be completed by closely looking at the proofs of the above theorems and hence we omit it here. □

3. Two independent variable generalization

Our approach here allow us to deal with the following initial boundary value problem (IBVP, for short) for neutral type finite difference equation in two independent variables

$$\Delta_2\Delta_1 u(m, n) = F(m, n, u(m, n), \Delta_2\Delta_1 u(m, n)), \tag{3.1}$$

with the initial boundary conditions

$$u(m, 0) = \alpha(m), \quad u(0, n) = \beta(n), \quad u(0, 0) = 0, \tag{3.2}$$

for $m, n \in N_0$, where $u \in D(N_0^2, \mathbb{R})$, $\alpha, \beta \in D(N_0, \mathbb{R})$, $F \in D(N_0^2 \times \mathbb{R}^2, \mathbb{R})$ and the operators $\Delta_1, \Delta_2, \Delta_2\Delta_1$ are as defined in [10, p. 3]. In this section, we formulate in brief the results analogues to Theorems 2.2 and 2.4 related to the solution of (3.1)-(3.2) only. One can formulate results similar to those in Theorems 2.6 and 2.7 with suitable modifications.

We require the following finite difference inequality presented in [10, Corollary 4.2.1].

Lemma 3.1. *Let $u, a, b, c \in D(N_0^2, \mathbb{R}_+)$. If*

$$u(m, n) \le a(m, n) + b(m, n) \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} c(s, t) u(s, t), \tag{3.3}$$

for $m, n \in N_0$, then

$$u(m, n) \le a(m, n) + b(m, n) \Big(\sum_{s=0}^{m-1} \sum_{t=0}^{n-1} c(s, t) a(s, t) \Big) \prod_{s=0}^{m-1} [1 + \sum_{t=0}^{n-1} c(s, t) b(s, t)], \tag{3.4}$$

for $m, n \in N_0$.

Let $u_i(m, n) \in D(N_0^2, \mathbb{R})$ $(i = 1, 2)$, and $\Delta_2\Delta_1 u_i(m, n)$ $(m, n \in N_0)$ exist and satisfy the inequalities

$$|\Delta_2\Delta_1 u_i(m, n) - F(m, n, u_i(m, n), \Delta_2\Delta_1 u_i(m, n))| \le \varepsilon_i, \tag{3.5}$$

for given constants $\varepsilon_i \ge 0$ $(i = 1, 2)$, where it is supposed that the initial boundary conditions

$$u_i(m, 0) = \alpha_i(m), \quad u_i(0, n) = \beta_i(n), \quad u_i(0, 0) = 0, \tag{3.6}$$

are fulfilled and $\alpha_i, \beta_i \in D(N_0, \mathbb{R})$. Then we call $u_i(m, n)$ the ε_i-approximate solutions with respect to (3.1)-(3.2).

Theorem 3.2. *Suppose that the function F in* (3.1) *satisfies the condition*

$$|F(m,n,u,v) - F(m,n,\bar{u},\bar{v})| \leq p(m,n)[|u-\bar{u}| + |v-\bar{v}|], \tag{3.7}$$

where $p \in D(N_0^2, \mathbb{R}_+)$ and $p(m,n) < 1$ for $m,n \in N_0$. For $i = 1,2$, let $u_i(m,n)$ $(m,n \in N_0)$ be ε_i-approximate solutions of (3.1) *with* (3.6) *such that*

$$|\alpha_1(m) - \alpha_2(m) + \beta_1(n) - \beta_2(n)| \leq \delta, \tag{3.8}$$

where $\delta \geq 0$ is a constant. Then

$$\begin{aligned} &|u_1(m,n) - u_2(m,n)| + |\Delta_2\Delta_1 u_1(m,n) - \Delta_2\Delta_1 u_2(m,n)| \\ &\leq L(m,n) + E(m,n)\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)L(s,t)\Big)\prod_{s=0}^{m-1}\Big[1 + \sum_{t=0}^{n-1} p(s,t)E(s,t)\Big], \end{aligned} \tag{3.9}$$

for $m,n \in N_0$, where

$$L(m,n) = \frac{(\varepsilon_1+\varepsilon_2)(mn+1) + \delta}{1 - p(m,n)}, \quad E(m,n) = \frac{1}{1-p(m,n)}. \tag{3.10}$$

Proof. Since $u_i(m,n)$ $(i = 1,2)$ for $m,n \in N_0$ are respectively ε_i-approximate solutions of (3.1) with (3.6), we have (3.5). Keeping m fixed in (3.5), setting $n = t$ and taking sum on both sides over t from 0 to $n-1$, then keeping n fixed in the resulting inequality and setting $m = s$ and taking sum over s from 0 to $m-1$ and using (3.6), we observe that

$$\begin{aligned} \varepsilon_i mn &\geq \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} |\Delta_2\Delta_1 u_i(s,t) - F(s,t,u_i(s,t),\Delta_2\Delta_1 u_i(s,t))| \\ &\geq \Big|\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\{\Delta_2\Delta_1 u_i(s,t) - F(s,t,u_i(s,t),\Delta_2\Delta_1 u_i(s,t))\}\Big| \\ &= \Big|\Big\{u_i(s,t) - [\alpha_i(m) + \beta_i(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,u_i(s,t),\Delta_2\Delta_1 u_i(s,t))\Big\}\Big|. \end{aligned}$$

From this inequality and using the elementary inequalities in (2.10), we observe that

$$\begin{aligned} &(\varepsilon_1+\varepsilon_2)mn \\ &\geq \Big|\{u_1(s,t) - [\alpha_1(m) + \beta_1(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,u_1(s,t),\Delta_2\Delta_1 u_1(s,t))\}\Big| \\ &+ \Big|\Big\{u_2(s,t) - [\alpha_2(m) + \beta_2(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,u_2(s,t),\Delta_2\Delta_1 u_2(s,t))\Big\}\Big| \\ &\geq \Big|\Big\{u_1(s,t) - [\alpha_1(m) + \beta_1(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,u_1(s,t),\Delta_2\Delta_1 u_1(s,t))\Big\} \\ &\quad - \Big\{u_2(s,t) - [\alpha_2(m) + \beta_2(n)] - \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} F(s,t,u_2(s,t),\Delta_2\Delta_1 u_2(s,t))\Big\}\Big| \\ &\geq \Big|u_1(s,t) - u_2(s,t)| - |[\alpha_1(m) + \beta_1(n)] - [\alpha_2(m) + \beta_2(n)]\Big| \end{aligned}$$

$$
\begin{aligned}
&- \Big| \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} F(s,t,u_1(s,t),\Delta_2\Delta_1 u_1(s,t)) \\
&- \sum_{s=0}^{m-1} \sum_{t=0}^{n-1} F(s,t,u_2(s,t),\Delta_2\Delta_1 u_2(s,t)) \Big|
\end{aligned}
$$

Furthermore, from (3.5) and using the elementary inequalities in (2.10), we observe that

$$
\begin{aligned}
&\varepsilon_1 + \varepsilon_2 \\
&\ge |\Delta_2\Delta_1 u_1(m,n) - F(m,n,u_1(m,n),\Delta_2\Delta_1 u_1(m,n))| \\
&\quad + |\Delta_2\Delta_1 u_2(m,n) - F(m,n,u_2(m,n),\Delta_2\Delta_1 u_2(m,n))| \\
&\ge |\{\Delta_2\Delta_1 u_1(m,n) - F(m,n,u_1(m,n),\Delta_2\Delta_1 u_1(m,n))\} \\
&\quad - \{\Delta_2\Delta_1 u_2(m,n) - F(m,n,u_2(m,n),\Delta_2\Delta_1 u_2(m,n))\}| \\
&\ge |\Delta_2\Delta_1 u_1(m,n) - \Delta_2\Delta_1 u_2(m,n)| \\
&\quad - |F(m,n,u_1(m,n),\Delta_2\Delta_1 u_1(m,n)) - F(m,n,u_2(m,n),\Delta_2\Delta_1 u_2(m,n))|.
\end{aligned}
$$

The remaining proof can be completed by following the proof of Theorem 2.2 with suitable modifications and using Lemma 3.1. We omit the further details. □

Consider (3.1)-(3.2) with the IBVP

$$\Delta_2\Delta_1 v(m,n) = G(m,n,v(m,n),\Delta_2\Delta_1 v(m,n)), \tag{3.11}$$

$$v(m,0) = \bar{\alpha}(m), \quad v(0,n) = \bar{\beta}(n), \quad v(0,0) = 0, \tag{3.12}$$

for $m,n \in N_0$, where $v \in D(N_0^2,\mathbb{R})$, $\bar{\alpha},\bar{\beta} \in D(N_0,\mathbb{R})$, $G \in D(N_0^2 \times \mathbb{R}^2,\mathbb{R})$.

Theorem 3.3. *Suppose that the function F in* (3.1) *satisfies* (3.7) *and that there exist constants $\bar{\varepsilon} \ge 0, \bar{\delta} \ge 0$ such that*

$$|F(m,n,u,v) - G(m,n,u,v)| \le \bar{\varepsilon}, \tag{3.13}$$

$$|\alpha(m) - \bar{\alpha}(m) + \beta(n) - \bar{\beta}(n)| \le \bar{\delta}, \tag{3.14}$$

where F,α,β and $G,\bar{\alpha},\bar{\beta}$ are as in (3.1)-(3.2) *and* (3.11)-(3.12). *Let $u(m,n)$ and $v(m,n)$ be respectively the solutions of* (3.1)-(3.2) *and* (3.11)-(3.12) *for $m,n \in N_0$. Then*

$$
\begin{aligned}
&|u(m,n) - v(m,n)| + |\Delta_2\Delta_1 u(m,n) - \Delta_2\Delta_1 v(m,n)| \\
&\le \bar{L}(m,n) + E(m,n)\Big(\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} p(s,t)\bar{L}(s,t)\Big) \prod_{s=0}^{m-1}\Big[1 + \sum_{t=0}^{n-1} p(s,t)E(s,t)\Big],
\end{aligned} \tag{3.15}
$$

for $m,n \in N_0$, where

$$\bar{L}(m,n) = \frac{\bar{\varepsilon}(mn+1) + \bar{\delta}}{1 - p(m,n)}, \tag{3.16}$$

and $E(m,n)$ is as in (3.10).

The proof follows by the similar argument as in the proof of Theorem 2.4 given above with suitable modifications. Here, we omit the details.

Remark 3.4. We note that the idea used in this paper can be extended very easily to establish similar results as given above for the following finite difference equations

$$\Delta_2\Delta_1 u(m,n) = F(m,n,v(m,n),\Delta_1 u(m,n)), \tag{3.17}$$

$$\Delta_2\Delta_1 u(m,n) = F(m,n,v(m,n),\Delta_2 u(m,n)), \tag{3.18}$$

with the given initial boundary conditions in (3.2) under some suitable conditions and by making use of a suitable variant of the inequality in Lemma 3.1 (see also [11]).

In concluding we note that, in the study of convergence of finite element approximations to solutions of various types of dynamic equations, the dependence of the error bounds on certain derivatives (or differences) of the exact solution will become apparent in the course of analysis (see, for example [14]). Here, it is to be noted that our analysis yields explicit bounds not only on the solutions of the problems but also on the forward differences of the solutions. We hope that our approach here will revel as a model for future investigations.

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