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PICONE-TYPE IDENTITY FOR PSEUDO P-LAPLACIAN WITH VARIABLE POWER

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Abstract. A Picone type identity is established for homogeneous differential operators involving the pseudo p-Laplacian with variable exponent $p = p(x)$. Using this identity, it is shown that the classical Sturmian theory extends to the associated partial differential equations.

1. Introduction

A Picone type identity for some specific class of equations usually indicates that one can elaborate a "reasonable" Sturmian oscillation and comparison theory for this class of equations. The classical form of this identity concerns the second order linear differential equation

$$(r(t)x')' + c(t)x = 0$$

and its origin goes back to the beginning of the previous century [20]. Since that time, Picone type identity has been established for various types of equations and operators. We refer at least to the papers [1, 6, 10, 11, 12, 13, 16, 19, 25] for more details.

Our research follows this line and it is motivated by the recent papers of Yoshida [26, 27, 28]. These papers concern the partial differential operators involving the p-Laplacian $\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u)$ with the variable power $p(x)$. Operators of this form find applications in various fields as shown e.g. in [21]. The case of the constant power $p(x) \equiv p \in (1,\infty)$ is relatively deeply developed and the Sturmian theory for these operators was elaborated in the papers [2, 3, 8, 17, 29]. More precisely, in these papers, the equation of the form

$$\operatorname{div}(a(x)\|u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0, \tag{1.1}$$

is investigated and an important role was played by the fact that the solution space of this equation is *homogeneous* in the sense that if u is a solution of (1.1) and k is a real constant, then ku is a solution of (1.1) as well.

However, if the power p in (1.1) is not a constant, as observed in the above papers [26, 27, 28], the solution space of (1.1) is no longer homogeneous. This "drawback" was removed by introducing an extra term in (1.1) and after this modification

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the $p(x)$-Laplacian operator became homogeneous and this enabled to introduce a Riccati type equation and to establish Picone's identity and Sturmian theory.

Another motivation are the results in [4, 7] where the equation with the so-called pseudo p-Laplacian

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(a_i(x)\Phi\big(\frac{\partial u}{\partial x_i}\big)\Big) + c(x)\Phi(u) = 0, \quad \Phi(u) = |u|^{p-2}u, \tag{1.2}$$

was investigated and it was shown that similarly to the "classical" p-Laplacian equation (1.1), concepts like Riccati type equation, Picone's identity, and, in turn, the Sturmian theory, can be established also for this equation.

In this paper we combine ideas of the above mentioned papers. We allow the exponent p in the pseudo p-Laplacian operator to be a differentiable function of x. After that, this operator is no longer homogeneous and one needs again to introduce an extra term which cares about homogeneity of the resulting operator.

To describe our idea in more detail, let $\Omega \in \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ and $p : \mathbb{R}^n \to (1, \infty)$ be a differentiable function. Further, let $u \in C^1(\Omega)$ and let us denote

$$\Psi(\nabla u) := \Big(\Phi\big(\frac{\partial u}{\partial x_i}\big), \dots, \Phi\big(\frac{\partial u}{\partial x_n}\big)\Big),$$

where now and later on $\Phi(u) := |u|^{p(x)-2}u$. Our aim is to establish Picone type identity for the pair of operators q and Q defined by

$$\begin{aligned} q[u] := \operatorname{div}\big(a(x)\Psi(\nabla u)\big) - a(x)\log|u|\langle\nabla p(x), \Psi(\nabla u)\rangle + \langle b(x), \Psi(\nabla u)\rangle \\ + c(x)\Phi(u) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} Q[u] := \operatorname{div}\big(A(x)\Psi(\nabla u)\big) - A(x)\log|u|\langle\nabla p(x), \Psi(\nabla u)\rangle + \langle B(x), \Psi(\nabla u)\rangle \\ + C(x)\Phi(u), \end{aligned} \tag{1.4}$$

where a, A, b, B, c, C are continuous functions and $a(x) > 0$, $A(x) > 0$ in $\overline{\Omega}$. Applying Picone's identity we derive then Leighton type Sturmian comparison theorem for Q and q.

Note that we are not concerned with the existence of a solution of equations $q(u) = 0$ and $Q(u) = 0$ in our paper. We refer the reader to [5, 9, 22], where the variational methods of solvability of nonlinear problems are demonstrated. These methods are based on the functional analytic background established in [14].

2. Homogeneity and Riccati type equation

First we show that the operator q is homogeneous in the sense that if u is a solution of $q[u] = 0$ then ku is also a solution of this equation for any constant $k \neq 0$. So, let u be any solution to $q[u] = 0$ and $k \neq 0$ be any constant. One gets

$$\begin{aligned} q[ku] &= \operatorname{div}\big(a\Psi(k\nabla u)\big) - a\log|ku|\langle\Psi(k\nabla u), \nabla p\rangle + \langle b, \Psi(k\nabla u) + c\Phi(ku) \\ &= \nabla\Phi(k)a\Psi(\nabla u) + \Phi(k)\operatorname{div}\big(a\Psi(\nabla u)\big) - a\Phi(k)\log|k|\langle\Psi(\nabla u), \nabla p\rangle \\ &\quad - a\Phi(k)\log|u|\langle\Psi(\nabla u), \nabla p\rangle + \Phi(k)\langle b, \Psi(\nabla u)\rangle + \Phi(k)c\Phi(u). \end{aligned}$$

Since

$$\nabla\Phi(k) = \nabla\big(|k|^{p(x)-2}k\big) = |k|^{p(x)-2}k\log|k|\nabla p(x)$$

we see that the terms containing $\log|k|$ cancel, hence with (1.3) we obtain that

$$q[ku] = \Phi(k)q[u] = 0$$

for any constant $k \neq 0$. Therefore, q is really homogeneous. The same property is valid for Q.

Observe, that a (sufficiently smooth) function u for which $u(\bar{x}) = 0$ at some point $\bar{x} \in \Omega$ *is not* in the domain of the operators q and Q because the logarithmic term is not defined for such a function. However, in Picone-type identities presented later the operator q (and also Q) appears in the form $uq[u]$ only. In this case we define $u \log|u| = 0$ for $u = 0$ and hence a function having a zero point in Ω is in the domain of the operator $\widetilde{q}[u] := uq[u]$ (and also of $\widetilde{Q}[u] := uQ[u]$). With this convention concern in $u = 0$ the operators $\widetilde{q}$ and $\widetilde{Q}$ are homogeneous also for $k = 0$; i.e. if u is a solution of $\widetilde{q}[u] = 0$ then $\widetilde{q}[ku] = 0$ for any $k \in \mathbb{R}$, the same holds for $\widetilde{Q}$.

Once a homogeneity of a differential operator is established, a natural idea is to look for an associated Riccati type differential equation. We denote the "$p(x)$-norm" in $\mathbb{R}^n$ by $\|x\|_{p(x)} := \big(\sum_{i=1}^n |x_i|^{p(x)}\big)^{1/p(x)}$, and the "$q(x)$-norm" in $\mathbb{R}^n$ by $\|x\|_{q(x)} = \big(\sum_{i=1}^n |x_i|^{q(x)}\big)^{1/q(x)}$, where $q(x) := \frac{p(x)}{p(x)-1}$ is the conjugate exponent of $p(x)$.

Introducing the Riccati variable $w := \frac{a\Psi(\nabla u)}{\Phi(u)}$, we have

$$\begin{aligned}
\operatorname{div} w &= \frac{1}{\Phi(u)} \operatorname{div}\big(a\Psi(\nabla u)\big) + \big\langle \nabla\big(\frac{1}{\Phi(u)}\big), a\Psi(\nabla u)\big\rangle \\
&= \frac{1}{\Phi(u)} \operatorname{div}\big(a\Psi(\nabla u)\big) - \frac{\log|u|}{\Phi(u)}\langle \nabla p, a\Psi(\nabla u)\rangle - (p-1)\frac{\langle \nabla u, a\Psi(\nabla u)\rangle}{|u|^p} \\
&= \frac{1}{\Phi(u)} \operatorname{div}\big(a\Psi(\nabla u)\big) - \frac{\log|u|}{\Phi(u)}\langle \nabla p(x), a\Psi(\nabla u)\rangle - (p-1)a^{1-q}\|w\|_{q(x)}^{q(x)}.
\end{aligned}$$

Together with (1.3) we have the identity

$$|u|^p\Big[\operatorname{div} w + c(x) + \langle \frac{b(x)}{a(x)}, w\rangle + (p(x)-1)(a(x))^{1-q(x)}\|w\|_{q(x)}^{q(x)}\Big] = uq[u]. \tag{2.1}$$

Consequently, if u is a solution of $q[u] = 0$ then from (2.1) one gets the Riccati type equation

$$\operatorname{div} w + c(x) + \langle \frac{b(x)}{a(x)}, w\rangle + (p(x)-1)(a(x))^{1-q(x)}\|w\|_{q(x)}^{q(x)} = 0.$$

3. Picone's identity

First we establish a Picone type identity in its simpler form, for one operator only. Before formulating it, we recall the modified Young inequality as given e.g. in [7].

Lemma 3.1. *For any $\alpha, \beta \in \mathbb{R}^n$ and $p > 1$, $q = \frac{p}{p-1}$ we have the inequality*

$$G(\alpha, \beta) := \frac{\|\alpha\|_p^p}{p} - \langle \alpha, \beta\rangle + \frac{\|\beta\|_q^q}{q} \geq 0 \tag{3.1}$$

with equality if and only if $\Psi(\alpha) = \beta$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\Psi(\alpha) = \big(\Phi(\alpha_1), \dots, \Phi(\alpha_n)\big)$.

Theorem 3.2. *Let $u \in C^1(\Omega)$ be such that $a\Psi(\nabla u) \in C^1(\Omega)$ and $u(x) \neq 0$ in Ω. Then for any $y \in C^1(\Omega)$ we have the identity (suppressing the argument x)*

$$\begin{aligned} &\operatorname{div}\Big(\frac{|y|^p a\Psi(\nabla u)}{\Phi(u)}\Big) \\ &= -c|y|^p + a\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap}b\|_p^p + \frac{|y|^p q[u]}{\Phi(u)} \\ &\quad - pa^{1-q}G(a^{q-1}(\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap}b), \frac{\Phi(y)a\Psi(\nabla u)}{\Phi(u)}), \end{aligned} \tag{3.2}$$

where the function G is given in (3.1).

Proof. Using (2.1) and the fact that

$$\nabla(|y|^{p(x)}) = |y|^{p(x)}\Big(\log|y|\nabla p(x) + p(x)\frac{\nabla y}{y}\Big),$$

we have

$$\begin{aligned} &\operatorname{div}\Big(\frac{|y|^p a\Psi(\nabla u)}{\Phi(u)}\Big) \\ &= |y|^p\Big[-c - \langle\frac{b}{a}, \frac{a\Psi(\nabla u)}{\Phi(u)}\rangle - (p-1)a^{1-q}\|\frac{a\Psi(\nabla u)}{\Phi(u)}\|_{q(x)}^{q(x)} + \frac{q[u]}{\Phi(u)}\Big] \\ &\quad + |y|^p\Big\langle\big(\log|y|\nabla p + p\frac{\nabla y}{y}\big), \frac{a\Psi(\nabla u)}{\Phi(u)}\Big\rangle \\ &= \frac{|y|^p}{\Phi(u)}q[u] - |y|^p c - (p-1)|y|^p a^{1-q}\|\frac{a\Psi(\nabla u)}{\Phi(u)}\|_q^q \\ &\quad + p\Big\langle\big(\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap}b\big), \frac{a\Phi(y)\Psi(\nabla u)}{\Phi(u)}\Big\rangle. \end{aligned}$$

In the last two terms we factor out pa^{1-q} and the remaining terms we take as two terms in Young's inequality with

$$\alpha = a^{q-1}\Big(\nabla y + \frac{y\log|y|}{p} - \frac{y}{ap}b\Big), \quad \beta = \frac{a\Phi(y)\Psi(\nabla u)}{\Phi(u)}.$$

so we add and subtract the term $\frac{\|\alpha\|_p^p}{p}$. The resulting identity is

$$\begin{aligned} &\operatorname{div}\Big(\frac{|y|^p a\Psi(\nabla u)}{\Phi(u)}\Big) \\ &= -c|y|^p + a\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap}b\|_p^p + \frac{|y|^p q[u]}{\Phi(u)} \\ &\quad - pa^{1-q}\Big[\frac{a^q}{p}\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap}b\|_p^p \\ &\quad - \Big\langle a^{q-1}\Big(\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap}b\Big), \frac{a\Phi(y)\Psi(\nabla u)}{\Phi(u)}\Big\rangle + \frac{1}{q}\|\frac{a\Phi(y)\Psi(\nabla u)}{\Phi(u)}\|_q^q\Big] \end{aligned}$$

which is what we need to prove. □

Next we establish a Picone type identity for the pair of operators q and Q. Similarly to the case of $p(x)$-Laplacian, we require an extra assumption on the function y appearing in this identity. For the sake of later comparison, let us recall

Picone's identity as established in [4] for $p(x) \equiv p$ and $b(x) = 0 = B(x)$ in (1.3) and (1.4).

Proposition 3.3. *For sufficiently smooth functions y, u with $u(x) \neq 0$ we have the identity*

$$\begin{aligned}
&\operatorname{div}[ya(x)\Psi(\nabla y) - |y|^p \frac{A(x)\Psi(\nabla u)}{\Phi(u)}] \\
&= (a(x) - A(x))\|\nabla y\|_p^p + (C(x) - c(x))|y|^p + yq[y] - \frac{|y|^p}{\Phi(u)} Q[u] - pA^{1-q}(x) \\
&\quad \times \Big[\frac{A^q(x)}{p}\|\nabla y\|_p^p - \langle A^{q-1}(x)\nabla y, \Phi(y)\frac{A(x)\Psi(\nabla u)}{\Phi(u)}\rangle + \frac{|y|^p}{q}\|\frac{A(x)\Psi(\nabla u)}{\Phi(u)}\|_q^q\Big].
\end{aligned}$$

Our computations in the proof of the next theorem follow the general idea of [26], but the technical realization is different because of the difference between p-Laplacian and pseudo p-Laplacian.

Theorem 3.4. *Let y be a $C^1(\Omega)$ function which has no zero in Ω and the following hypothesis holds:*

(H1) *There exists a function $f \in C^1(\overline{\Omega})$ such that*

$$\nabla f(x) = \frac{\log|y(x)|}{p(x)}\nabla p(x) - \frac{1}{a(x)p(x)} b(x) \quad \text{in } \Omega.$$

Then for any $u \in C^1(\Omega)$ which has no zero in Ω and $y \in C^1(\Omega)$ we have the Picone's identity of the form

$$\begin{aligned}
&\operatorname{div}[e^{(1-p(x))f} ya(x)\Psi\big(\nabla(e^f y)\big) - \frac{|y|^{p(x)}}{\Phi(u)}A(x)\Psi(\nabla u)] \\
&= a(x)\|\nabla y + \frac{y\log|y|}{p(x)}\nabla p(x) - \frac{y}{a(x)p(x)} b(x)\|_{p(x)}^{p(x)} \\
&\quad - A(x)\|\nabla y + \frac{y\log|y|}{p(x)}\nabla p(x) - \frac{y}{A(x)p(x)} B(x)\|_{p(x)}^{p(x)} + [C(x) - c(x)]|y|^{p(x)} \\
&\quad - p(x)(A(x))^{1-q(x)} G(\alpha(x), \beta(x)) + e^{(1-p(x))f} yq[e^f y] - \frac{|y|^{p(x)}Q[u]}{\Phi(u)},
\end{aligned} \tag{3.3}$$

where the function G is given in Lemma 3.1 and

$$\begin{gathered}
\alpha(x) = (A(x))^{q(x)-1}\Big(\nabla y + \frac{y\log|y|}{p(x)}\nabla p(x) - \frac{y}{A(x)p(x)} B(x)\Big), \\
\beta(x) = \frac{A(x)\Phi(y)\Psi(\nabla u)}{\Phi(u)}.
\end{gathered} \tag{3.4}$$

Proof. The divergence of the second term in the formula (3.3) (with a replaced by A and q by Q) is computed in the previous theorem. As for the first term, using the assumption (H1) and the form of q, we have

$$\begin{aligned}
&\operatorname{div}[e^{-pf}\big(e^f y\big)a\Psi\big(\nabla(e^f y)\big)] \\
&= \langle \nabla\big[e^{-pf}\big(e^f y\big)\big], a\Psi\big(\nabla(e^f y)\big)\rangle + e^{(1-p)f} y \operatorname{div}\big(a\Psi\big(\nabla(e^f y)\big)\big) \\
&= \langle e^{-pf}[-\nabla p f - p\nabla f](e^f y) + e^{-pf}\nabla(e^f y), a\Psi\big(\nabla(e^f y)\big)\rangle \\
&\quad + e^{(1-p)f} y \operatorname{div}\big(a\Psi\big(\nabla(e^f y)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big\langle e^{(1-p)f} y[-f\nabla p - \log|y|\nabla p + \frac{1}{a}b], a\Psi\big(\nabla(e^f y)\big)\Big\rangle \\
&\quad + e^{-pf} a \sum_{i=1}^n \big|\frac{\partial(e^f y)}{\partial x_i}\big|^p + e^{(1-p)f} y \operatorname{div}\big(a\Psi\big(e^f y)\big)\big) \\
&= e^{(1-p)f} y \langle [-\nabla p \log|e^f y| + \frac{1}{a} b], a\Psi\big(\nabla(e^f y)\big)\rangle \\
&\quad + a \sum_{i=1}^n \big|e^{-f}\frac{\partial(e^f y)}{\partial x_i}\big|^p + e^{(1-p)f} y \operatorname{div}\big(a\Psi\big(\nabla(e^f y)\big)\big) \\
&= e^{(1-p)f} y \log|e^f y| \langle -\nabla p, a\Psi\big(\nabla(e^f y)\big)\rangle + \langle b, \Psi\big(\nabla(e^f y)\big)\rangle \\
&\quad + a\|y\nabla f + \nabla y\|_p^p + e^{(1-p)f} y \operatorname{div}\big(a\Psi\big(e^f y\big)\big) \\
&= e^{(1-p)f} y q\big[e^f y\big] - c|y|^p + a\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap} b\|_p^p.
\end{aligned}
$$

Therefore,

$$
\operatorname{div}[e^{(1-p)f} y a\Psi\big(\nabla(e^f y)\big)] = a\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap} b\|_p^p + e^{(1-p)f} q[e^f y] - c|y|^p.
$$

Altogether, we obtain

$$
\begin{aligned}
&\operatorname{div}\Big[e^{(1-p)f} y a\Psi\big(\nabla(e^f y)\big) - \frac{A|y|^p\Psi(\nabla u)}{\Phi(u)}\Big] \\
&= a\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{ap} b\|_p^p - A\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{Ap} B\|_p^p \\
&\quad + [C(x) - c(x)]|y|^p + A\|\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{Ap} B(x)\|_p^p \\
&\quad - p\big\langle\big(\nabla y + \frac{y\log|y|}{p}\nabla p - \frac{y}{Ap} B\big), \frac{A\Phi(y)\Psi(\nabla u)}{\Phi(u)}\big\rangle \\
&\quad + (p-1)A^{1-q}\|\frac{A\Phi(y)\Psi(\nabla u)}{\Phi(u)}\|_q^q + e^{(1-p)f} y q\big[e^f y\big] - \frac{|y|^p Q[u]}{\Phi(u)}
\end{aligned}
$$

which is the required identity. □

4. Comparison theorem

Using the previous statement we can now prove the following Leighton type comparison theorem. For its original version we refer to [18] and to [15, 19, 23, 24] for its extension to elliptic PDE's.

Theorem 4.1. *Suppose that there exists a function $y \in C^1(\Omega)$ with $y = 0$ on the boundary $\partial\Omega$ such that* (H1) *holds and $yq\big[e^f y\big] \ge 0$ in Ω. Moreover, assume that*

(H2) *there exists a function $F \in C^1(\overline{\Omega})$ such that*

$$
\nabla F(x) = \frac{\log|y(x)|}{p(x)}\nabla p(x) - \frac{1}{A(x)p(x)} B(x) \quad \text{in} \quad \Omega,
$$

If

$$
\begin{aligned}
&\mathcal{F}(y) \\
&:= \int_\Omega \Big\{a(x)\|\nabla y + \frac{y\log|y|}{p}\nabla p(x) - \frac{y}{a(x)p(x)} b(x)\|_{p(x)}^{p(x)}
\end{aligned}
$$

$$- A(x)\|\nabla y + \frac{y\log|y|}{p(x)}\nabla p(x) - \frac{y}{A(x)p(x)}B(x)\|_{p(x)}^{p(x)} + [C(x)-c(x)]|y|^p\Big\}\,dx \ge 0,$$

then every solution of the equation $Q[u]=0$ has a zero in $\overline{\Omega}$.

Proof. By contradiction, suppose that there exists a solution u of $Q[u]=0$ such that $u(x)\neq 0$ in $\overline{\Omega}$. By the divergence theorem, using the fact that $y|_{\partial\Omega}=0$, we have

$$\int_\Omega \operatorname{div}\Big[e^{(1-p(x))f}a(x)\Psi\big(\nabla(e^f y)\big) - \frac{|y|^p}{\Phi(u)}A(x)\Psi(\nabla u)\Big]\,dx = 0.$$

Hence

$$0 \ge \mathcal{F}(y) + \int_\Omega p(x)(A(x))^{1-q(x)}G(\alpha(x),\beta(x))\,dx,$$

where α,β are given by (3.4) with a,b replaced by A,B. Consequently, $G(\alpha,\beta)=0$; i.e., $\alpha=\Psi^{-1}(\beta):=(\Phi^{-1}(\beta_1),\dots,\Phi^{-1}(\beta_n))$, which means that

$$\nabla y + \frac{y\log|y|}{p(x)}\nabla p(x) - \frac{y}{A(x)p(x)}B(x) = \frac{y}{u}\nabla u \quad \text{in } \Omega.$$

Using (H2), $\nabla y + y\nabla F = \frac{y}{u}\nabla u$ which implies

$$e^{-F(x)}u(x)\nabla\Big[\frac{e^{F(x)}y(x)}{u(x)}\Big] = 0, \quad x\in\Omega.$$

This, together with $y|_{\partial\Omega}=0$ implies that $y\equiv 0$, which contradicts the assumption that y has no zero in Ω. □

As a consequence of the previous theorem we have the following Sturmian comparison result.

Corollary 4.2. *Suppose that $a(x)\ge A(x)$, $C(x)\ge c(x)$, and $\frac{b(x)}{a(x)}=\frac{B(x)}{A(x)}$ in Ω. Further, suppose that there exists a differentiable function y such that $y(x)\neq 0$ in Ω, $y|_{\partial\Omega}=0$, the hypothesis* (H1) *holds, and $yq[e^f y]\ge 0$ in Ω. Then any solution u of the equation $Q[u]=0$ has a zero in Ω.*

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