Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 181, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OPTIMAL DESIGN OF A BAR WITH AN ATTACHED MASS FOR MAXIMIZING THE HEAT TRANSFER

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Abstract. We maximize, with respect to the cross sectional area, the rate of heat transfer through a bar of given mass. The bar serves as an extended surface to enhance the heat transfer surface of a larger heated known mass to which the bar is attached. In this paper we neglect heat transfer from the sides of the bar and consider only conduction through its length. The rate of cooling is defined by the first eigenvalue of the corresponding Sturm-Liouville problem. We establish existence of an optimal design via rearrangement techniques. The necessary conditions of optimality admit a unique optimal design. We compare the rate of heat transfer for that bar with the rate for the bar of the same mass but of a constant cross-section area.

1. Introduction

Materials are often cooled by convection to a surrounding ambient medium such as the atmosphere. For example, the heat generated in an automobile engine is transferred first to the cooling water that circulates through the engine and then to the atmosphere through the radiator. Convective heat transfer is described by the equation

$$\dot{Q} = hA_s(T - T_\infty) \tag{1.1}$$

where $\dot{Q}$ is the heat transfer rate, h is an empirical heat transfer coefficient, A_s is the surface area, T is the temperature of the surface and T_∞ is the temperature of the surrounding medium. We want to maximize the surface area since the convective heat transfer rate is proportional to this area. On the other hand, we want to minimize the volume of the heat transfer region, in order to keep its weight and hence material cost as low as possible. Thus we seek to maximize the surface to volume ratio of the heat transfer surface.

Extended surfaces attached to a given base mass, M_0, are frequently used in commercial applications to increase the heat transfer surface area without significantly increasing the associated mass and hence the material cost of the device. The additional surface might be in the form of thin donuts around a central pipe, parallel plates attached to the surface as in small engines or automobile radiators, or fins extending outward like hairs from a surface. The mass of the added extended

2000 *Mathematics Subject Classification.* 74A15, 74P10, 49K15, 34B24.
Key words and phrases. Optimal design; heat transfer; heat equation; least eigenvalue; Sturm-Liouville problem; Helly's principle; calculus of variations.

Submitted August 15, 2012. Published October 19, 2012.

surface is small compared to the base mass M_0. For this reason a high surface-to-volume ratio for the extended surface is sought. Such mass additions to enhance heat transfer are referred to as fins.

Nature also "designs" according to this criterion. The ears of an elephant have large surface area compared to their volume which allows the blood passing through them to be efficiently cooled. Likewise members of species like deer that live near the equator must be able to dissipate heat efficiently. They achieve this desirable ratio by being smaller than their counterparts that live towards the poles (think of a sphere where the surface to volume is inversely proportional to r).

Engineering heat transfer texts sometimes consider fins of variable cross-section [13, pp. 124-126] but the cross-section is assumed to be for a regular shape, e.g. a cylinder with variable radius. Or the optimization of a fin with a given cross-section (e.g. a rectangle) is optimized with respect to the length and thickness [22, pp. 74]. However, a general variation of shape to maximize the heat transfer from the fin is not considered.

Heat transfer within the surface is by conduction and the rate is given by the equation

$$\dot{Q} = -\frac{kA\Delta T}{l} \tag{1.2}$$

where k is the thermal conductivity of the material, A is the cross-sectional area, and l is the length of the material. Here ΔT is the temperature difference between the end points of the heat transfer. The equations above, along with the corresponding physical background, may be found in [13, pp. 110-114].

If an energy balance is performed for the region of the bar between x and $x+\Delta x$, energy enters by conduction at x and leaves by conduction at $x+\Delta x$ and also from the side by convection (see Equation (1.1)). The difference is the rate of change of the energy content of that region of the bar

$$\text{rate of change of energy} = \text{energy in} - \text{energy out} \quad \text{or}$$

$$\rho c A \Delta x \frac{\Delta T}{\Delta t} = -kA\frac{\Delta T}{\Delta x}\Big|_x + kA\frac{\Delta T}{\Delta x}\Big|_{x+\Delta x} - hA_s(T - T_\infty). \tag{1.3}$$

Here $T(x,t)$ is the temperature distribution. The surface area is $A_s = P\Delta x$, where $P(x)$ is the perimeter of the cross-section at the point x. The ratio $\frac{\Delta T}{\Delta t}$ is the rate of change of temperature with time and $\frac{\Delta T}{\Delta x}$ is the local temperature gradient. The following bar material parameters are introduced, the density ρ, the specific heat capacity c, the thermal conductivity k, and the convective heat transfer coefficient h. It is assumed that ρ, c, k, h are positive constants.

Dividing by Δx, and taking the limit as Δx and Δt approach zero, yields the partial differential equation

$$A\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial}{\partial x}\Big(A\frac{\partial T}{\partial x}\Big) - \frac{hP}{\rho c}(T - T_\infty), (x,t) \in (0,l) \times (0,\infty). \tag{1.4}$$

We discuss a particular case of this general equation when convective heat transfer from the side of the bar is neglected, i.e., the limiting case $h \to 0$ is considered. It is the purpose of this paper to find the optimal distribution of the cross-section area, A, of a surface of revolution of a given mass such that the heat transfer rate is a maximum. This will produce a maximum cooling per unit mass and may be considered the optimum.

Detailed discussion of techniques and results in structural optimization can be found in [9, 27] and the references therein. We mention in particular the maximization of a column's buckling load [24, 25], the minimization of the mass of an oscillating bar [23, 26] or a rotating rod [1, 2], the maximization of a column's height [14] and the minimization of the moment of inertia of an oscillating turbine [4]. If the design variable tapers too rapidly, the eigenvalue being optimized is not isolated from the remainder of the spectrum [14, 8, 1]. In these cases optimality conditions can be derived using non-smooth analysis in conjunction with the more classical Calculus of Variations techniques [16, 17].

The complexity of the Sturm-Liouville problem also increases if the boundary conditions contain an eigenparameter. This is due to the fact the the Sturm-Liouville operator is not self-adjoint with respect to the usual $L^2(0,l)$ inner product. The spectral properties of the Sturm-Liouville problems that arise from diverse mechanical models and contain the spectral parameter in the boundary condition(s) have been studied in [28, 10, 12, 5, 3]. Numerical schemes for the inverse problem were developed by [18]. Design problems of this type have been considered by [26, 4]. In the latter, existence of an optimal design was treated seriously, as it will be here.

In the design problem considered here, we encounter a spectral parameter in a boundary condition. In Section 2 we give the mathematical description of the model, apply separation of variables and formulate the spectral properties of the corresponding Sturm-Liouville problem. We also give the solution for an elementary case of the problem when the cross-section area is constant, which we need later for comparison with the solution in case of variable cross-section area. In Section 3 we derive the necessary conditions of optimality and hence find an optimal form of the bar. In Section 4 we use a rearrangement technique to prove that the optimal design is increasing and maximizes the first eigenvalue of the Sturm-Liouville problem. In Section 5 we give the numerical comparison of cooling properties for the bar of optimal shape and a bar having the same mass but with the constant cross-section area. In the Appendix, we prove that the rate of cooling for the bar with the optimal cross-section area is greater than for a bar of the same mass but with constant cross-section area.

2. Heat transfer of a bar of a variable cross-section area: separation of variables and the Sturm-Liouville problem

We consider the heat transfer in a bar $\{0 < x < l\}$ with a base mass M_0 attached at the end point $x = 0$. The temperature distribution $T : [0,l] \times [0,\infty) \to R$ satisfies the transient one-dimensional conduction equation

$$A\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial}{\partial x}\Big(A\frac{\partial T}{\partial x}\Big), \quad (x,t) \in (0,l) \times (0,\infty) \tag{2.1}$$

that is the limiting case of Equation (1.4) as $h \to 0$. Here the cross-section area $A(x) : [0,l] \to R_+$ is a continuous differentiable positive function. As was mentioned in the Introduction, parameters k, ρ, c are positive constants. The end point $x = l$ is kept at the (constant) temperature of the surrounding medium,

$$T(l,t) = T_\infty, t \in [0,\infty). \tag{2.2}$$

The rate of change of the energy content of the base is given by the difference between the energy flow into and out of it, as in the derivation of (1.3). For energy flow only by conduction outward at $x = 0$ this becomes

$$cM_0 \frac{\Delta T}{\Delta t} = kA \frac{\Delta T}{\Delta x}\Big|_{x=0}.$$

In the limit as Δt and $\Delta x \to 0$ this becomes

$$cM_0 \frac{\partial T}{\partial t}(0,t) = kA(0) \frac{\partial T}{\partial x}(0,t) \quad t \in [0, \infty). \tag{2.3}$$

The initial distribution $T_0 : [0, l] \to R$ of the temperature is given,

$$T(x, 0) = T_0(x). \tag{2.4}$$

It is well known that the initial boundary value problem (2.1)-(2.4) has a unique solution [21, 15]. It is convenient to extract the term T_∞ from the solution,

$$\tau(x, t) \equiv T(x, t) - T_\infty. \tag{2.5}$$

The new unknown function $\tau : [0, l] \times [0, \infty) \to R$ is the unique solution of the initial boundary value problem

$$A \frac{\partial \tau}{\partial t} = \frac{k}{\rho c} \frac{\partial}{\partial x}\Big(A \frac{\partial \tau}{\partial x}\Big), \quad (x, t) \in (0, l) \times (0, \infty), \tag{2.6}$$

$$\tau(l, t) = 0, \quad t \in [0, \infty), \tag{2.7}$$

$$cM_0 \frac{\partial \tau}{\partial t}(0, t) = kA(0) \frac{\partial \tau}{\partial x}(0, t), \quad t \in [0, \infty), \tag{2.8}$$

$$\tau(x, 0) = \tau_0(x), \quad \in [0, l]; \text{ where } \tau_0(x) \equiv T_0(x) - T_\infty. \tag{2.9}$$

If we use the standard procedure of separation of variables

$$\tau(x, t) \equiv e^{-\sigma t} u(x) \tag{2.10}$$

and introduce the notation

$$\frac{\rho c}{k} \sigma \equiv \lambda \text{ so that } \frac{c\sigma}{k} = \frac{\lambda}{\rho}, \tag{2.11}$$

then the function $u : [0, l] \to R$ satisfies the following Sturm-Liouville problem

$$\begin{gathered} (Au')' + \lambda A u = 0, \quad x \in (0, l); \\ u(l) = 0, \quad A(0)u'(0) + \frac{M_0}{\rho} \lambda u(0) = 0. \end{gathered} \tag{2.12}$$

As we see, the spectral parameter λ appears in the second boundary condition. The general theory for Sturm-Liouville problems of this type developed in [28, 10, 12, 5, 3] may be used. It can be verified that the conditions of the corresponding theorems are satisfied. In particular, by Walter [28, Theorem 1], we know that the eigenparameter dependent Sturm-Liouville problem (2.12) has a pure discrete positive real spectrum with the only point of accumulation at $+\infty$. The set of eigenfunctions satisfies the orthogonality relation

$$\int_0^l A u_n u_j dx + \frac{M_0}{\rho} u_n(0) u_j(0) = 0 \quad \text{if } n \neq j \tag{2.13}$$

which allows us to define an inner product over which our Sturm-Liouville problem is self-adjoint. The Rayleigh quotient

$$\lambda_n = \frac{\rho \int_0^l A u_n'^2 dx}{\rho \int_0^l A u_n^2 dx + M_0 u_n^2(0)} \tag{2.14}$$

immediately follows.

Existence and uniqueness of the solution of the initial boundary value problem (2.6)-(2.9) follow from known techniques, [15]. Its series representation is given by

$$\tau(x,t) \equiv \sum_{n\geq 1} c_n e^{-\sigma_n t} u_n(x) \tag{2.15}$$

where

$$\sigma_n \equiv \frac{k}{\rho c}\lambda_n$$

and

$$c_n = \frac{\rho \int_0^l A\tau_0 u_n dx + M_0 \tau_0(0) u_n(0)}{\rho \int_0^l A u_n^2 dx + M_0 u_n^2(0)}. \tag{2.16}$$

We note that if the cross-section area is constant, $A(x) = A$, the mass of the bar is $M = \rho A l$, and the exact solution of the problem (2.6)-(2.9) is given by

$$\tau(x,t) \equiv \sum_{n\geq 1} c_n e^{-\sigma_n t} \sin\sqrt{\lambda_n}(x-l).$$

Here λ_n are the positive solutions of the transcendental equation

$$\tan(\sqrt{\lambda_n} l) = \frac{M}{M_0\sqrt{\lambda_n} l}, \quad n = 1, 2, \dots \tag{2.17}$$

and

$$c_n = \frac{\rho A \int_0^l \tau_0(x) \sin\sqrt{\lambda_n}(x-l) dx - M_0 \tau_0(0) \sin\sqrt{\lambda_n} l}{\rho A \int_0^l \sin^2\sqrt{\lambda_n}(x-l) dx + M_0 \sin^2\sqrt{\lambda_n} l}. \tag{2.18}$$

3. Heat transfer of a bar of a variable cross-section area: optimality conditions

The representation (2.15)-(2.16), and (2.5) for the solution shows that the temperature $T(x,t)$ approaches the level T_∞ exponentially fast, and the rate of approach is determined by the first eigenvalue λ_1. We now formulate the problem of optimal design and consider the variational problem:

Find the form of the cross–section $A(x) \in (0,\infty)$ that yields the maximum to the functional

$$\lambda_1 = \min_{u\in H^1[0,l]} \frac{\int_0^l A(x)(u'(x))^2 dx}{\int_0^l A(x)(u(x))^2 dx + \frac{M_0}{\rho} u^2(0)} \tag{3.1}$$

given

$$\int_0^l A(x) = \frac{M}{\rho}. \tag{3.2}$$

We note here that for any $\hat{u} \in H^1[0,l]$

$$\lambda_1(A) \leq \frac{\int_0^l A(x)(\hat{u}'(x))^2 dx}{\int_0^l A(x)(\hat{u}(x))^2 dx + \frac{M_0}{\rho}\hat{u}^2(0)} \tag{3.3}$$

In particular, if we choose $u(x) = l - x$, it follows that $\lambda_1(A)$ is bounded because

$$\lambda_1(A) \leq \frac{\int_0^l A(x) dx}{\int_0^l A(x)(l-x)^2 dx + \frac{M_0}{\rho} l^2} \leq \frac{\rho M}{M_0 l^2}. \tag{3.4}$$

It is well-known that the variational problem of the minimization of the ratio above is equivalent to the following problem: *Find the function $u \in H^1[0,l]$ that yields the minimum to the functional*

$$L = \int_0^l A(x)(u'(x))^2 dx \tag{3.5}$$

subject to the constraint

$$\int_0^l A(x)(u(x))^2 dx + \frac{M_0}{\rho} u^2(0) = 1. \tag{3.6}$$

Hence, we come to the following variational problem:

Find the form of the cross–section $A(x) \in (0,\infty)$ that yields the maximum to the functional

$$L_1 = \min_{u \in H^1[0,l]} \int_0^l A(x)(u'(x))^2 dx \tag{3.7}$$

subject to the constraints

$$\int_0^l A(x)(u(x))^2 dx + \frac{M_0}{\rho} u^2(0) = 1, \int_0^l A(x) = \frac{M}{\rho}. \tag{3.8}$$

Using the Lagrange method, we introduce the new functional

$$\begin{aligned} F[A;u] = \int_0^l A(x)(u'(x))^2 dx - \mu_1 \Big(\int_0^l A(x) u^2(x) dx + \frac{M_0}{\rho} u^2(0) \Big) \\ - \mu_2 \Big(\int_0^l A(x) dx - \frac{M}{\rho} \Big) \end{aligned} \tag{3.9}$$

where μ_1, μ_2 are Lagrange multipliers. The necessary condition of the extremum in terms of the first variation of the functional $F[A;u]$ has the form $\delta F[A;u] = 0$. Using the boundary condition

$$u(l) = 0, \tag{3.10}$$

this can be written as

$$\begin{aligned} &\int_0^l (u')^2 \delta A dx + \int_0^l 2Au'\delta u' dx - \mu_1 \int_0^l u^2 \delta A dx \\ &- \mu_1 \int_0^l 2Au\delta u dx - \mu_1 \frac{M_0}{\rho} 2u(0)\delta u(0) - \mu_2 \int_0^l \delta A dx = 0. \end{aligned}$$

The variations $\delta u(0), \delta u(x), \delta A(x)$ are independent. Equating the corresponding parts of the variation of $F[A;u]$ to zero yields a boundary condition and two differential equations

$$(Au')(0) + \mu_1 \frac{M_0}{\rho} u(0) = 0, \tag{3.11}$$

$$(Au')' + \mu_1 Au = 0, 0 < x < l, \tag{3.12}$$

$$(u')^2 - \mu_1 u^2 - \mu_2 = 0. \tag{3.13}$$

We observe that the differential equation (3.12) subject to the boundary conditions (3.10), (3.11) yields $u(x)$ to be the first eigenfunction and μ_1 to be the first eigenvalue, $\mu_1 = \lambda_1$, of the original Sturm-Liouville problem (2.12). The optimality conditions represented by the nonlinear differential equation (3.13) subject to the boundary condition (3.10) may be solved explicitly,

$$u(x) = \frac{\sqrt{\mu_2}}{\lambda_1} \sinh \sqrt{\lambda_1}(l - x). \tag{3.14}$$

Substituting $u(x)$ from (3.14) into the Sturm-Liouville equation (3.12) yields the following differential equation for $A(x)$

$$\big(A \cosh \sqrt{\lambda_1}(l - x)\big)' - \lambda_1 A \frac{1}{\sqrt{\lambda_1}} \sinh \sqrt{\lambda_1}(l - x) = 0 \tag{3.15}$$

which also may be solved explicitly

$$A(x) = \frac{C}{\cosh^2 \sqrt{\lambda_1}(x - l)}. \tag{3.16}$$

The boundary condition (3.11) yields

$$C = \frac{M_0 \sqrt{\lambda_1}}{2\rho} \sinh(2\sqrt{\lambda_1} l). \tag{3.17}$$

Using Conditions (3.2) yields finally

$$M_0 \sinh^2(\sqrt{\lambda_1} l) = M. \tag{3.18}$$

Solving (3.18) for λ_1 finally yields the optimal rate of cooling for the bar with the given mass

$$\lambda_1 = \Big(\frac{1}{l} \ln\Big(\sqrt{\frac{M}{M_0}} + \sqrt{\frac{M}{M_0} + 1}\Big)\Big)^2. \tag{3.19}$$

We introduce the dimensionless parameter

$$z^{\text{opt}} \equiv \sqrt{\lambda_1} l = \ln\Big(\sqrt{\frac{M}{M_0}} + \sqrt{\frac{M}{M_0} + 1}\Big). \tag{3.20}$$

It is of interest to compare it with the similar parameter z for the constant cross-section. From the transcendental equation (2.17), our dimensionless parameter satisfies

$$z \tan z = \frac{M}{M_0}. \tag{3.21}$$

In the Appendix, we show that the inequality

$$z^{\text{opt}} > z \tag{3.22}$$

holds for any positive ratio M/M_0. This is confirmed by our numerical results in Section 5.

4. The Optimal Design is Increasing

We prove here that the heat transfer rate $\lambda_1(A)$ can be increased through the use of increasing rearrangements of the cross-sectional area $A(x)$. This is achieved through the use of an alternative characterization of $\lambda_1(A)$. We begin by defining decreasing and increasing rearrangements, and stating some of their relevant properties.

Definition 4.1. The decreasing rearrangement of a nonnegative function, f, on (a, b) is simply
$$f^*(x) \equiv \sup\{t > 0 : \mu_f(t) > x\},$$
where μ_f is the distribution function of f,
$$\mu_f(t) = |\{x \in (a, b) : f(x) > t\}| \quad t \geq 0.$$
The increasing rearrangement of f is $f_*(x) \equiv f^*(b - x)$.

If g and h are nonnegative functions on (a, b), with g increasing and h decreasing, then
$$\int_a^b f dx = \int_a^b f^* dx = \int_a^b f_* dx, \tag{4.1}$$
By (4.1), if we replace a particular design $A \in ad$ by either its increasing or decreasing rearrangements A_* or A^* then the new design has the same integral. Furthermore,
$$\int_a^b f^* g dx \leq \int_a^b f g dx, \qquad \int_a^b f_* h dx \leq \int_a^b f h dx. \tag{4.2}$$
These results are a special case of those established in [19, pp. 153].

Theorem 4.2. *For any cross-sectional area A satisfying*
$$0 < A(x) < \infty, x \in [0, l], \int_0^l A(x) dx = \frac{M}{\rho},$$
its increasing rearrangement A_* satisfies
$$\lambda_1(A) \leq \lambda_1(A_*).$$

Proof. Using variation of parameters, as in [12, 5] we find that if $u(x)$ is a solution of (2.12) corresponding to $A(x)$, then $v(x) = \sqrt{A(x)} u(x)$ satisfies
$$v(x) = \lambda[\phi_A(x) + (G_A v)(x)]$$
for $0 < x < l$, where
$$\phi_A(x) = \sqrt{A(x)} \frac{M_0}{\rho} \Big(\int_x^l \frac{dx}{A(x)} \Big), \tag{4.3}$$
$$(G_A v)(x) = \int_0^l g_A(x, t) v(t) dt, \tag{4.4}$$
$$g_A(x, s) = \sqrt{A(x)} \sqrt{A(s)} \int_{x \wedge s}^l \frac{dy}{A(y)} \tag{4.5}$$
and $x \wedge s = \max\{x, s\}$. If $\langle u, v\rangle$ denotes the $L^2(0, l)$ inner product and $\|\cdot\|$ denotes its associated norm, then this can be written as
$$\|v\|^2 = \lambda[\langle \phi_A, v\rangle + \langle G_A v, v\rangle].$$
Thus, our second characterization is a variational characterization similar to that of Porter and Stirling, [20, Lemma 5.1],
$$\frac{1}{\lambda_1(A)} = \max_{\|v\|=1} [\langle \phi_A, v\rangle + \langle G_A v, v\rangle]. \tag{4.6}$$
The maximum is attained at $v_1(x) = \sqrt{A(x)} u_1(x)$ where $u_1(x)$ is the first eigenfunction of the Sturm-Liouville Problem (2.12) associated with $A(x)$ for which $\int_0^l u_1^2(x) A(x) dx = 1$.

Let u_1 be the first eigenfunction of the Sturm-Liouville Problem (2.12) associated with A_*. Using $v(x) = \sqrt{A(x)}u(x)$ in (4.6) and integrating by parts, we find

$$\frac{1}{\lambda_1(A)} = \max_{\|u\|_A=1} \Big[\frac{M_0}{\rho}\int_0^l \Big(\int_0^x A(y)u(y)dy\Big)\frac{dx}{A(x)} + \int_0^l \Big(\int_0^x A(y)u(y)dy\Big)^2 \frac{dx}{A(x)}\Big]$$

where $\|\cdot\|_A$ denotes the norm associated with the $L^2(0,l;A(x))$ inner product. Integrating (2.12) with u_1 and A_* between 0 and x, and using the boundary condition at $x=0$ yields

$$u_1'(x) = \frac{-\lambda_1(A_*)}{A_*(x)}\Big[\int_0^x A_* u_1 dr + \frac{M_0}{\rho}u_1(0)\Big].$$

Binding, Browne and Seddighi established oscillation results for Sturm-Liouville problems with eigenparameter dependent boundary conditions in [5]. In particular, their Corollary 5.2 implies that our first eigenfunction has no interior zeros. We assume without loss of generality that $u_1 > 0$. Since $\lambda_1(A_*), M_0, \rho, u_1(x), A(x) > 0$, this implies that u_1 is decreasing. The second part of (4.2) implies that $\int_0^x Au_1 dy \geq \int_0^x A_* u_1 dy$ and so

$$\frac{1}{\lambda_1(A)} \geq \frac{M_0}{\rho}\int_0^l \left(\int_0^x A_*(y)u_1(y)dy\right)\frac{dx}{A(x)} + \int_0^l \left(\int_0^x A_*(y)u_1(y)dy\right)^2 \frac{dx}{A(x)}.$$

Clearly, the functions $(\int_0^x A_* u_1 dy)$ and $(\int_0^x A_* u_1 dy)^2$ are nonnegative increasing functions of x. Once again (4.2) yields

$$\begin{aligned} \frac{1}{\lambda_1(A)} &\geq \frac{M_0}{\rho}\int_0^l \Big(\int_0^x A_*(y)u_1(y)dy\Big)\Big(\frac{1}{A(x)}\Big)^* dx \\ &\quad + \int_0^l \Big(\int_0^x A_*(y)u_1(y)dy\Big)^2 \Big(\frac{1}{A(x)}\Big)^* dx. \end{aligned}$$

If f is decreasing on the range of g then the composition $(f \circ g)^* = f \circ g_*$, see [7], which implies that

$$\begin{aligned} \frac{1}{\lambda_1(A)} &\geq \frac{M_0}{\rho}\int_0^l \Big(\int_0^x A_*(y)u_1(y)dy\Big)\frac{dx}{A_*(x)} + \int_0^l \Big(\int_0^x A_*(y)u_1(y)dy\Big)^2 \frac{dx}{A_*(x)} \\ &= \frac{1}{\lambda_1(A_*)}. \end{aligned}$$ □

Theorem 4.3. *The design satisfying the first order optimality conditions* (3.10)-(3.13)

$$A(x) = \frac{M\sqrt{\lambda_1}\coth(\sqrt{\lambda_1}l)}{\rho\cosh^2(\sqrt{\lambda_1}(x-l))}$$

maximizes the functional $A \mapsto \lambda_1(A)$ *on the set*

$$ad = \Big\{A : 0 < A(r) < \infty, x \in [0,l], \int_0^l A(r)dr = \frac{M}{\rho}\Big\}$$

Proof. By (3.4) we know that $\lambda_1(A)$ is bounded. A is the unique solution of the optimality system given by (3.10)-(3.13). Suppose that A is not a maximizer. By Theorem 4.2, the associated design can be improved and the first eigenvalue increased by replacing A with its increasing rearrangement A_*. Since A is already an increasing function of x, the design cannot be improved. Hence A maximizes the functional. □

5. Numerical comparison of cooling properties

The product $z = \sqrt{\lambda} l$ is a function of the ratio M/M_0 in both the constant area case (3.21) and the optimal case (3.19). The equation (3.21) was solved numerically. Similar to the presentation of [2], a comparison of constant and variable cross-section is shown in Figure 1 as a function of M/M_0. Extended surfaces with more mass, M, than the base mass, M_0, are not used in engineering practice, and hence values of $M/M_0 > 1$ in the graph are shown for illustrative purposes only.

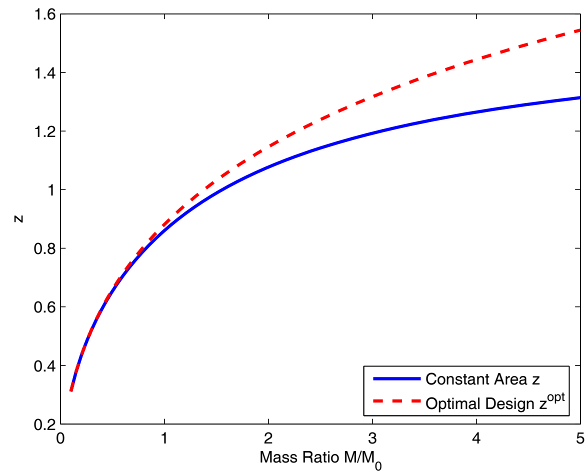

Figure 1. Comparision of optimal design and constant area design

Numerical results show that the advantage of the optimum cross-section over the constant cross-section is small and becomes less so as the base mass, M_0, increases. This is physically reasonable. Indeed, recall that convective heat transfer from the side of the area has been neglected. Hence addition of the extended surface does little but move the boundary condition at $x = l$ that distance from M_0. Furthermore, as M becomes small compared to M_0, its very presence becomes negligible and hence its shape does not matter.

In each case $z^{\text{opt}} \geq z$, as shown in the Appendix. This numerical observation is certainly in agreement with the general results of Sections 3 and 4. However, the effect is not large because of the physical reasons explained above. Moreover, for $M/M_0 \to 0$ the optimum and constant area results merge. This numerical observation is in agreement with the asymptotic formula (7.5).

6. Conclusion

We have found the optimal distribution of the cross-section area of a bar in the form of a surface of revolution of a given total mass with a point mass attached at the end such that the heat transfer rate is a maximum. That rate is defined by the least eigenvalue of the corresponding Sturm-Liouville problem. This is of independent interest because the spectral parameter appears not only in the differential equation but also in the boundary condition. The bar will produce the maximum cooling per unit mass and may be considered the optimum. The optimal distribution coincides with one found by Taylor [24] and M.J. Turner [26] for the design of a bar having a maximum lowest eigenfrequency with the given mass. Numerical results show that the advantage of the optimal design over the constant cross-section is small and decreases as the base mass increases. We believe this to be a result of the fact that our model neglects heat transfer from the side of the bar.

We should emphasize that we have considered a special case of the heat transfer assuming that convective heat transfer from the side of the bar is neglected and only conduction through the length of the bar is considered (see Section 1).

We expect that the solution of the optimal design problem for the more general problem will show a more noticeable difference between the optimal design and the constant case. If we were to include the heat transfer phenomenon from the sides of the bar, we would have to consider the partial differential equation

$$a^2(x)\frac{\partial T}{\partial t} = \frac{k}{\rho c}\frac{\partial}{\partial x}\Big(a^2(x)\frac{\partial T}{\partial x}\Big) - \frac{ha(x)\sqrt{1+(a'(x)^2}}{\rho c}(T - T_\infty), \tag{6.1}$$

where $(x,t) \in (0,l)\times(0,\infty)$, and $a(x)$ would be the radius of the body of revolution that represents the bar. The corresponding Sturm-Liouville problem has a discrete spectrum and a complete set of eigenfunctions. We could derive a Rayleigh-Ritz ratio for the least eigenvalue similar to expression (2.14). But the technique of the Calculus of Variations used in Section 3 will not lead to an explicit form of the cross-section area and the least eigenvalue. For that problem, we had hoped to use a numerical approach based on the discretization of our bar that would reduce the problem of the optimal design to the problem of optimization for a function of several variables (this idea was developed for the optimal design of mechanical systems in [6]). We have recently learned that an equation with similar appearance of the function $a(x)$ is optimized in [11]. The techniques used there may be applicable when we consider the more general heat transfer model. This will be will be considered in a future paper.

7. Appendix

Having derived an explicit formula for the optimal rate of cooling and an equation for the rate for the bar with the constant cross-section area, we may compare them directly. We prove below that the optimal rate is greater than the rate for the bar of the same mass but with the constant cross-section area. This inequality clearly is demonstrated in our numerical results above, but is proven here for completeness.

Lemma 7.1. *The inequality*

$$z^{\rm opt} > z \tag{7.1}$$

where

$$z^{\rm opt} = \ln\Big(\mu + \sqrt{\mu^2+1}\Big) \tag{7.2}$$

and z is the minimal positive root of the equation

$$z\tan z = \mu^2 \tag{7.3}$$

holds for any positive quantity

$$\mu \equiv \sqrt{\frac{M}{M_0}}. \tag{7.4}$$

Proof. We consider both $z^{\rm opt}$ and z as functions of $\mu > 0$. Note first that $z^{\rm opt}(\mu) \asymp \mu, z(\mu) \asymp \mu$, and hence

$$\lim_{\mu\to 0} z^{\rm opt}(\mu) = \lim_{\mu\to 0} z(\mu) = 0. \tag{7.5}$$

Hence the inequality (7.1) holds if the derivative of $z^{\rm opt}(\mu)$ is greater or equal to the derivative of $z(\mu)$. It is easy to prove that the first positive solution of (7.3) is a uniquely defined function on $z \in (0, \pi/2)$ with the derivative

$$z'(\mu) = \frac{2\mu}{\tan z + \frac{z}{\cos^2 z}} = \frac{2\mu}{\frac{\mu^2}{z} + z + \frac{\mu^4}{z}} \tag{7.6}$$

where we used (7.3) to get the final form. The derivative of $z^{\rm opt}$ can be easily found from (7.2). We finally come up with the necessity to prove the following inequality

$$F(\mu) \equiv \frac{1}{\sqrt{1+\mu^2}} - \frac{2\mu z}{\mu^2 + \mu^4 + z^2} \geq 0. \tag{7.7}$$

We find first

$$\mu^2 + \mu^4 + z^2 \geq 2\sqrt{\mu^2 + \mu^4}z = 2\mu\sqrt{1+\mu^2}z. \tag{7.8}$$

Hence

$$F(\mu) \geq \frac{1}{\sqrt{1+\mu^2}} - \frac{2\mu z}{2\mu\sqrt{1+\mu^2}z} = 0 \tag{7.9}$$

which proves (7.7) and, along with (7.5), proves (7.1). $\square$

Acknowledgments. The first author was supported in part by a University of Tennessee at Chattanooga Faculty Research Grant. The third author was supported in part by NSF DMS # 0209562 and # 0531865. The authors are grateful to the referee for pointing out the work of Henrot and Privat.

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