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LOWER SEMICONTINUITY OF PULLBACK ATTRACTORS FOR A SINGULARLY NONAUTONOMOUS PLATE EQUATION

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Abstract. We show the lower semicontinuity of the family of pullback attractors for the singularly nonautonomous plate equation with structural damping

$$u_{tt} + a(t,x)u_t + (-\Delta)u_t + (-\Delta)^2 u + \lambda u = f(u),$$

in the energy space $H_0^2(\Omega) \times L^2(\Omega)$ under small perturbations of the damping term a.

1. Introduction

In this paper, we shall continue the study started in [5] about the asymptotic behavior under perturbations of the nonautonomous plate equation

$$\begin{gathered} u_{tt} + a_\epsilon(t,x)u_t + (-\Delta)u_t + (-\Delta)^2 u + \lambda u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\lambda > 0$ and $f \in C^2(\mathbb{R})$ is a nonlinearity satisfying

$$(i) \quad |f'(s)| \leqslant c(1 + |s|^{\rho-1}),\ \forall s \in \mathbb{R},\ \text{with } \begin{cases} 1 < \rho < \frac{n+4}{n-4} & \text{if } n \geq 5, \\ \rho \in (1,\infty), & \text{if } n = 1,2,3,4; \end{cases} \tag{1.2}$$

$$(ii) \quad f(s)s < 0,\ \forall s \in \mathbb{R}.$$

The map $\mathbb{R} \ni t \mapsto a_\epsilon(t,\cdot) \in L^\infty(\Omega)$ is supposed to be Hölder continuous with exponent $0 < \beta < 1$ and constant C uniformly in $\epsilon \in [0,1]$, $0 < \alpha_0 \leqslant a_\epsilon(t,x) \leqslant \alpha_1$, for $(t,x,\epsilon) \in \mathbb{R} \times \Omega \times [0,1]$, and $a_\epsilon(t,x) \xrightarrow{\epsilon \to 0} a_0(t,x)$, uniformly in $\mathbb{R} \times \Omega$. Such problems arise on models of vibration of elastic systems, see for example [6, 7, 8, 10, 11].

Writing $A := (-\Delta)^2$ with domain $D(A) = \{u \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta u_{|\partial\Omega} = 0\}$, it is well known that A is a positive self-adjoint operator in $L^2(\Omega)$ with compact resolvent. For $\alpha \geqslant 0$, we consider the scale of Hilbert spaces $E^\alpha := \big(D(A^\alpha), \|A^\alpha \cdot \|_{L^2(\Omega)} + \|\cdot\|_{L^2(\Omega)}\big)$, where $A^0 = I$. It is of special interest the case $\alpha = \frac{1}{2}$, where

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$-A^{1/2}$ is the Laplace operator with homogeneous Dirichlet boundary conditions; i.e., $A^{1/2} = -\Delta$ with domain $E^{1/2} = H^2(\Omega) \cap H_0^1(\Omega)$.

Setting the Hilbert space $X^0 := E^{1/2} \times E^0$, let $\mathcal{A}_\epsilon(t) : D(\mathcal{A}_\epsilon(t)) \subset X^0 \to X^0$ be defined by

$$\mathcal{A}_\epsilon(t) := \begin{bmatrix} 0 & -I \\ A + \lambda I & A^{1/2} + a_\epsilon(t) I \end{bmatrix},$$

with domain $D(\mathcal{A}_\epsilon(t)) := E^1 \times E^{1/2}$ (independent on t and ϵ). We also define $X^\alpha := E^{\frac{\alpha+1}{2}} \times E^{\frac{\alpha}{2}}$.

In this framework was shown in [5] that the problem (1.1) can be written as an ordinary differential system

$$\frac{d}{dt}(u,v) + \mathcal{A}_\epsilon(t)(u,v) = F((u,v)), \quad (u(\tau), v(\tau)) = (u_0, v_0) \in X^0, t \geqslant \tau \in \mathbb{R}, \tag{1.3}$$

where $F((u,v)) = (0, f^e(u))$ and f^e is the Nemitskiĭ operator associated to f. This equation yields an evolution process $\{S_\epsilon(t,\tau) : t \geqslant \tau\}$ in X^0 which is given by

$$S_\epsilon(t,\tau)x = L_\epsilon(t,\tau)x + \int_\tau^t L_\epsilon(t,s) F(S_\epsilon(s,\tau)x)\, ds, \quad \forall t \geqslant \tau \in \mathbb{R},\ x \in X^0, \tag{1.4}$$

being $\{L_\epsilon(t,\tau) : t \geqslant \tau \in \mathbb{R}\}$ the linear evolution process associated to the homogeneous system

$$\frac{d}{dt}(u,v) + \mathcal{A}_\epsilon(t)(u,v) = (0,0), \quad (u(\tau), v(\tau)) = (u_0, v_0) \in X^0,\ t \geqslant \tau. \tag{1.5}$$

Furthermore the evolution process $\{S_\epsilon(t,\tau) : t \geqslant \tau\}$ has a pullback attractor $\{\mathbb{A}_\epsilon(t) : t \in \mathbb{R}\}$ with the property that

$$\cup_{\epsilon \in [0,\epsilon_0]} \cup_{t \in \mathbb{R}} \mathbb{A}_\epsilon(t) \subset X^0 \text{ is bounded.} \tag{1.6}$$

Recalling the Hausdorff semi-distance of two subsets $A, B \subset X$

$$\operatorname{dist}_H(A,B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_{X^0},$$

also was shown the upper semicontinuity of the family $\{\mathbb{A}_\epsilon(t) : t \in \mathbb{R}\}$ at $\epsilon = 0$; i.e.,

$$\operatorname{dist}_H(\mathbb{A}_\epsilon(t), \mathbb{A}_0(t)) \xrightarrow{\epsilon \to 0} 0.$$

Our aim in this paper is to prove its lower semicontinuity at $\epsilon = 0$; i.e.,

$$\operatorname{dist}_H(\mathbb{A}_0(t), \mathbb{A}_\epsilon(t)) \xrightarrow{\epsilon \to 0} 0.$$

To achieve this propose we proceed in the following way: We assume there exists only a many finite number of equilibrium e^* of (1.3), all of them hyperbolic in the sense that the linearized operator of (1.3) around e^* admits an exponential dichotomy. Then we write the limit attractor as an unstable manifold of the equilibria set, allowing us to obtain the lower semicontinuity as in [3].

This article follows closely [1, 2], and it is organized as follows: In Section 2 we derive some additional stability properties of the solutions starting in the pullback attractors. In Section 3 we get the characterization of the pullback attractor as a unstable manifold of the equilibria set, and in Section 4, we show the hyperbolicity property of the equilibria of (1.1) and we derive the lower semicontinuity of the pullback attractors.

2. Stability of the process on the attractor

In this section we prove an asymptotically stability result of the evolution processes starting on the attractors. First we recall from (1.6) that

$$\{\mathbb{A}_\epsilon(t) : t \in \mathbb{R}\} = \{\xi \in C(\mathbb{R}, X^0) : \xi \text{ is bounded and } S_\epsilon(t,\tau)\xi(\tau) = \xi(t)\}.$$

Therefore if $\xi(t) \in \mathbb{A}_\epsilon(t)$ for all $t \in \mathbb{R}$, then

$$\xi(t) := (u(t), u_t(t)) = L_\epsilon(t,\tau)\xi(\tau) + \int_\tau^t L_\epsilon(t,s)F(\xi(s))\,ds,$$

and by the exponential decay of $L_\epsilon(t,\tau)$ [5, Theorem 3.1], we can write

$$\xi(t) = \int_{-\infty}^t L_\epsilon(t,s)F(\xi(s))\,ds. \tag{2.1}$$

For $w_0 = \xi(\tau)$ fixed, consider

$$U(t,\tau) := (w(t), w_t(t)) = \int_\tau^t L_\epsilon(t,s)F(S_\epsilon(s,\tau)w_0)\,ds,$$

and note that

$$\begin{gathered} w_{tt} + a_\epsilon(t,x)w_t + (-\Delta)w_t + (-\Delta)^2 w + \lambda w = f(u(t,\tau,w_0)), \\ w(\tau) = w_t(\tau) = 0. \end{gathered} \tag{2.2}$$

Also notice that by [5, Theorem 3.2], $\{U(t,\tau) : t \geqslant \tau\}$ is a bounded subset of X^0. Therefore using the fact that f^e maps bounded subsets of $E^{1/2}$ to bounded subsets of $E^{-\frac{1}{2}+\tilde\gamma}$, for some $\tilde\gamma > 0$ [5, Lemma 2.5], we can state the problem (1.3) in $X^{2\gamma} = E^{\frac{1}{2}+\gamma} \times E^\gamma$ with $0 < \gamma < \tilde\gamma$ (note that $U(0,0) = (0,0) \in E^{\frac{1}{2}+\gamma} \times E^\gamma$), and we have [4] the estimate

$$\begin{aligned} \|U(t,\tau)\|_{X^{1+2\gamma}} &\leqslant \int_\tau^t \|L_\epsilon(t,s)\|_{\mathcal{L}(X^{1+2\gamma}, X^{-1+2\tilde\gamma})} \|F(S_\epsilon(s,\tau))w_0\|_{E^{\frac{1}{2}+\gamma}\times E^{-\frac{1}{2}+\gamma}}\,ds \\ &\leqslant K\int_\tau^t (t-s)^{-1+2\tilde\gamma-2\gamma} e^{-\alpha(t-s)}\,ds. \end{aligned}$$

Noticing that $-1 + 2\tilde\gamma > -1$, from (2.1) it follows that

$$\sup_{\epsilon\in[0,1]}\sup_{t\in\mathbb{R}}\sup_{\xi\in\mathcal{A}_\epsilon(t)} \|\xi(t)\|_{E^{\frac{1}{2}+\gamma}\times E^\gamma} < \infty.$$

From the compact embedding $E^{\frac{1}{2}+\gamma} \times E^\gamma \overset{cc}{\hookrightarrow} E^{1/2} \times E^0$, the set $\overline{\cup_{\epsilon\in[0,1]}\cup_{t\in\mathbb{R}}\mathbb{A}_\epsilon}$ is a compact subset of X^0.

The rest of the section is dedicated to show asymptotically stability of those solutions starting on the attractors. Since the map $t \mapsto a_0(t,x)$ is a bounded and Lipschitz function uniform in $x \in \Omega$, given a sequence $\{t_n\} \subset \mathbb{R}$, we have for each $t \in \mathbb{R}$ fixed, that the sequence $\{a_n(t,x) := a_0(t+t_n,x)\}$ has a subsequence convergent $a_n(t,x) \to \bar a(t,x)$, uniformly in compact subsets of $\mathbb{R}$ and $x \in \Omega$. Therefore $\bar a$ inherits the same boundedness and Lipschitz properties of a_0. This allows us to consider the following two problems:

$$\begin{gathered} u_{tt} + a_n(t,x)u_t + (-\Delta)u_t + (-\Delta)^2 u + \lambda u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \\ u(\tau) = u_0 \in H^2(\Omega)\cap H^1_0(\Omega), \quad u_t(\tau) = v_0 \in L^2(\Omega), \end{gathered} \tag{2.3}$$

and

$$\begin{gathered} u_{tt} + \bar{a}(t,x)u_t + (-\Delta)u_t + (-\Delta)^2 u + \lambda u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \\ u(\tau) = u_0 \in H^2(\Omega) \cap H_0^1(\Omega), \quad u_t(\tau) = v_0 \in L^2(\Omega). \end{gathered} \tag{2.4}$$

We want to compare solutions of the above problems with initial data $(u_0, v_0) \in \mathbb{A}_n(\tau)$, where $\{\mathbb{A}_n(t) : t \in \mathbb{R}\}$ and $\{\mathbb{A}_\infty(t) : t \in \mathbb{R}\}$ are the pullback attractors of (2.3) and (2.4) respectively. Proceeding as above we obtain that

$$\overline{\cup_{n\in\mathbb{N}} \cup_{t\in\mathbb{R}} \mathbb{A}_n(t) \cup \mathbb{A}_\infty(t)} \text{ is a compact subset of } X^0.$$

For $(u_0, v_0) \in \mathbb{A}_n(\tau)$, let $\xi_n(t)$ and $\bar{\xi}(t)$ be the solutions of (2.3) and (2.4) respectively. Defining $w(t) := \xi_n(t) - \bar{\xi}(t)$, we have

$$\begin{gathered} w_{tt} = \bar{a}(t,x)\bar{\xi}_t - a_n(t,x)\xi_t + \Delta w_t - \Delta^2 w - \lambda w + f(\xi) - f(\bar{\xi}) \\ w(\tau) = w_t(\tau) = 0. \end{gathered} \tag{2.5}$$

Define $Z(u,v)) = \frac{1}{2}(\|u\|_{1/2}^2 + \|v\|_{L^2(\Omega)}^2)$. Since that f^e is Lipschitz in bounded sets from $E^{1/2}$ to E^0, and ξ, $\bar{\xi}$, ξ_t, $\bar{\xi}_t$ are bounded, Young's Inequality leads to

$$\begin{aligned} &\frac{d}{dt} Z((w, w_t)) \\ &= \langle w, w_t\rangle_{E^{1/2}} + \langle w_t, w_{tt}\rangle_{L^2(\Omega)} \\ &= \langle \Delta w, \Delta w_t\rangle_{L^2(\Omega)} + \lambda\langle w, w_t\rangle_{L^2(\Omega)} + \langle w_t, w_{tt}\rangle_{L^2(\Omega)} \\ &= \langle \Delta^2 w + \lambda w + w_{tt}, w_t\rangle_{L^2(\Omega)} \\ &= \langle \bar{a}(t,x)\bar{\xi}_t - a_n(t,x)\xi_t + \Delta w_t + f(\xi) - f(\bar{\xi}), w_t\rangle_{L^2(\Omega)} \\ &= \langle -\bar{a}(t,x)w_t + (\bar{a}(t,x) - a_n(t,x))\xi_t, w_t\rangle_{L^2(\Omega)} - \|\nabla w_t\|_{L^2(\Omega)}^2 \\ &\quad + \langle f(\xi) - f(\bar{\xi}), w_t\rangle_{L^2(\Omega)} \\ &\leqslant -\alpha_0\|w_t\|_{L^2(\Omega)}^2 + \|\bar{a} - a_n\|_{L^\infty([\tau,t]\times\Omega)}\|\xi_t\|_{L^2(\Omega)}\|w_t\|_{L^2(\Omega)} \\ &\quad + K(\|w\|_{L^2(\Omega)}^2 + \|w_t\|_{L^2(\Omega)}^2) \\ &\leqslant \tilde{K} Z((w, w_t)) + \tilde{K}\|\bar{a} - a_n\|_{L^\infty([\tau,t]\times\Omega)}. \end{aligned}$$

Therefore,

$$\begin{aligned} Z((w, w_t)) &\leqslant \tilde{K}\int_\tau^t Z((w(s), w_t(s)))ds + \tilde{K}(t-\tau)\|\bar{a} - a_n\|_{L^\infty([\tau,t]\times\Omega)} \\ &\quad + Z\big((w(\tau), w_t(\tau))\big) \\ &\leqslant \tilde{\tilde{K}}\int_\tau^t Z((w, w_t))ds + \tilde{\tilde{K}}(t-\tau)\|\bar{a} - a_n\|_{L^\infty([\tau,t]\times\Omega)}, \end{aligned}$$

where $\tilde{\tilde{K}} = \max\big\{\tilde{K}, \frac{Z((w(\tau), w_t(\tau)))}{(\alpha_1 - \alpha_0)}\big\}$. Gronwall's Inequality yields

$$\|\xi_n(t) - \bar{\xi(t)}\|_{X^0}^2 \leqslant \tilde{\tilde{\tilde{K}}}\|\bar{a} - a_n\|_{L^\infty([\tau,t]\times\Omega)}\int_\tau^t e^{K(t-s)}\, ds \to 0, \tag{2.6}$$

as $n \to \infty$ in compact subsets of $\mathbb{R}$.

3. Structure of the pullback attractor

We will assume that there exist only finitely many $\{u_1^*, \dots, u_r^*\}$ solutions of the problem

$$\begin{gathered}(-\Delta)^2 u + \lambda u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{3.1}$$

Defining $\mathcal{E} = \{e_1^*, \dots, e_r^*\}$, where $e_i^* := (u_i^*, 0)$, we will show that

$$\mathbb{A}_0(t) = \cup_{i=1}^r W^u(e_i^*)(t), \quad \text{for all } t \in \mathbb{R}, \tag{3.2}$$

where

$$\begin{aligned}W^u(e_i^*) = \big\{(\tau,\zeta) \in \mathbb{R} \times X^0 :\ &\text{there exists a backwards solution } \xi(t,\tau,\zeta) \text{ of (1.3)} \\ &(\epsilon = 0) \text{ satisfying } \xi(\tau,\tau,\zeta) = \zeta \text{ and } \|\xi(t,\tau,\zeta) - e_i^*\|_{X^0} \overset{t\to-\infty}{\longrightarrow} 0\big\},\end{aligned}$$

and $W^u(e_i^*)(t) = \{\zeta \in X^0 : (t,\zeta) \in W^u(e_i^*)\}$.

Consider the norms in $E^{1/2}$ and X^0 given respectively by:

$$\|u\|_{1/2} := [\|\Delta u\|^2_{L^2(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)}]^{1/2} \text{ and } \|(u,v)\|_{X^0} = [\|u\|^2_{1/2} + \|v\|^2_{L^2(\Omega)}]^{1/2}.$$

For any $0 < b \leqslant 1/4$ fixed we have

$$\frac{1}{4}\|(u,v)\|^2_{X^0} \leqslant \frac{1}{2}\|(u,v)\|^2_{X^0} + 2b\lambda^{1/2}\langle u,v\rangle_{L^2(\Omega)} \leqslant \frac{3}{4}\|(u,v)]\|^2_{X^0}.$$

Let us consider the Lyapunov functional $V : X^0 \to \mathbb{R}$ defined by

$$V((u,v)) = \frac{1}{2}\|(u,v)\|^2_{X^0} + 2b\lambda^{1/2}\langle u,v\rangle_{L^2(\Omega)} - \int_\Omega \mathcal{F}^e(u)\,dx, \tag{3.3}$$

where $\mathcal{F}^e$ is the Nemitskiĭ map associated to a primitive of f, $\mathcal{F}(s) = \int_0^s f(t)\,dt$.

If $u = u(t)$ is a solution of the equation (1.1) ($\epsilon = 0$) then

$$\begin{aligned}&\frac{d}{dt}V((u,u_t)) \\ &= \langle \Delta u, \Delta u_t\rangle_{L^2(\Omega)} + \lambda\langle u,u_t\rangle_{L^2(\Omega)} + \langle u_t,u_{tt}\rangle_{L^2(\Omega)} + 2b\lambda^{1/2}\langle u_t,u_t\rangle_{L^2(\Omega)} \\ &\quad + 2b\lambda^{1/2}\langle u,u_{tt}\rangle_{L^2(\Omega)} - \int_\Omega f(u)u_t dx \\ &= \langle \Delta u, \Delta u_t\rangle_{L^2(\Omega)} + \lambda\langle u,u_t\rangle_{L^2(\Omega)} + \langle u_t, -a_\epsilon(t,x)u_t - (-\Delta)^2 u \\ &\quad - (-\Delta)u_t - \lambda u + f(u)\rangle_{L^2(\Omega)} + 2b\lambda^{1/2}\langle u_t,u_t\rangle_{L^2(\Omega)} + 2b\lambda^{1/2}\langle u, -a_\epsilon(t,x)u_t \\ &\quad - (-\Delta)^2 u - (-\Delta)u_t - \lambda u + f(u)\rangle_{L^2(\Omega)} - \int_\Omega f(u)u_t dx \\ &\leqslant -(\alpha_0 - 2b\lambda^{1/2} - b\lambda^{1/2} - \frac{b\alpha_1\lambda^{1/2}}{\eta})\|u_t\|^2_{L^2(\Omega)} + \lambda^{1/2}(b\alpha_1\eta - b\lambda)\|u\|^2_{L^2(\Omega)} \\ &\quad - b\lambda^{1/2}(\|\Delta u\|^2_{L^2(\Omega)} + \lambda\|u\|^2_{L^2(\Omega)}) + 2b\lambda^{1/2}\int_\Omega f(u)u dx,\end{aligned}$$

for all $\eta > 0$. The choice $\eta = \frac{\lambda}{\alpha_1}$ leads to

$$\begin{aligned}\frac{d}{dt}V((u,u_t)) \leqslant -(\alpha_0 - 2b\lambda^{1/2} - b\lambda^{1/2} - \frac{b\alpha_1^2}{\lambda^{1/2}})\|u_t\|^2_{L^2(\Omega)} - b\lambda^{1/2}\|u\|_{1/2} \\ + 2b\lambda^{1/2}\int_\Omega f(u)u dx \le 0,\end{aligned}$$

which means that V is non-increasing on solutions of (1.1) and the global solutions where V is constant must be an equilibrium. This implies in particular, that in $\mathcal{E}$ there is no homoclinic structure.

Finally, we show that all solutions in the pullback attractor $\{\mathbb{A}_0 : t \in \mathbb{R}\}$ are forwards and backwards asymptotic to equilibria.

Let $\{\xi(t) : t \in \mathbb{R}\} \subset \{\mathbb{A}_0(t) : t \in \mathbb{R}\}$ a global solution in the attractor. Since it lies in a compact set of X^0, $V(\xi(t+r)) \overset{t\to-\infty}{\longrightarrow} \omega_1$ and $V(\xi(t+r)) \overset{t\to+\infty}{\longrightarrow} \omega_2$, for some ω_1, $\omega_2 \in \mathbb{R}$ and $r \in \mathbb{R}$.

We can choose a sequence $t_n \overset{n\to\infty}{\longrightarrow} \infty$ such that $a_0(t_n + r, x) \overset{n\to\infty}{\longrightarrow} \bar{a}(r, x)$, uniformly for r in compact subsets of $\mathbb{R}$ and $x \in \Omega$. Therefore, the solution (ζ, ζ_t) of the problem

$$\begin{gathered} u_{tt} + \bar{a}(t,x)u_t + (-\Delta)u_t + (-\Delta)^2 u + \lambda u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.4}$$

satisfies $V((\zeta, \zeta_t)) = \omega_2$, for all $t \in \mathbb{R}$. Hence $(\zeta, \zeta_t) \in \mathcal{E}$ and $\xi(t + r) \overset{t\to\infty}{\longrightarrow} (\zeta, \zeta_t)$. Taking $\tilde{t}_n \overset{n\to\infty}{\longrightarrow} -\infty$ we obtain a similar result.

Now we show that this convergence does not depend on the particular choice of subsequences. In fact, suppose that there are sequences $\{t_n\}, \{s_n\} \overset{n\to\infty}{\longrightarrow} \infty$, such that $\xi(t_n) \overset{n\to\infty}{\longrightarrow} e_i^* \neq e_j^* \overset{n\to\infty}{\longleftarrow} \xi(s_n)$. Reindexing if necessary we can suppose that $t_{n+1} > s_n > t_n$, for all $n \in \mathbb{N}$.

If $\tau_n \in (t_n, s_n)$, then $\tau_n \overset{n\to\infty}{\longrightarrow} \infty$ and (taking subsequence if necessary), $a_0(\tau_n + r) \overset{n\to\infty}{\longrightarrow} \bar{a}(r)$. Therefore we also have that $\xi(\tau_n + r) \overset{n\to\infty}{\longrightarrow} \bar{\zeta}(r)$, which is a solution of

$$\begin{gathered} u_{tt} + \bar{a}(t,x)u_t + (-\Delta)u_t + (-\Delta)^2 u + \lambda u = f(u) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.5}$$

with $V(\bar{\zeta}, \bar{\zeta}_t) = \omega_2$ for all $t \in \mathbb{R}$. Consequently, $\bar{\zeta}(t) \equiv e_m^* \in \mathcal{E} \setminus \{e_i^*, e_j^*\}$.

Choosing $\tilde{\tau}_n \in (\tau_n, s_n)$ we can repeat the argument that leads to a contradiction with the fact that there are only finitely many equilibria. Therefore we can write the pullback attractor as in (3.2).

4. Lower semicontinuity of attractors

Definition 4.1. We say that a linear evolution process $\{L(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ in a Banach space X has an exponential dichotomy with exponent ω and constant M if there is a family of bounded linear projections $\{P(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$ such that

(i) $P(t)L(t,\tau) = L(t,\tau)P(\tau)$, for all $t \geqslant \tau$;
(ii) The restriction $L(t,\tau)_{|P(\tau)X}$, is an isomorphism from $P(\tau)X$ into $P(t)X$, for all $t \geqslant \tau$;
(iii) There are constants $\omega > 0$ and $M > 1$ such that

$$\begin{gathered} \|L(t,\tau)(I - P(\tau))\|_{\mathcal{L}(X)} \leqslant Me^{-\omega(t-\tau)},\ t \geqslant \tau, \\ \|L(t,\tau)P(\tau)\|_{\mathcal{L}(X)} \leqslant Me^{\omega(t-\tau)},\ t \leqslant \tau. \end{gathered}$$

To see that the linear process $\{L_\epsilon(t,\tau) : t \geqslant \tau\}$ has an exponential dichotomy, given u_ϵ the global solution of (1.3), define $z_\epsilon(t) := u_\epsilon(t) - e_j^*$, for any $e_j^* \in \mathcal{E}$. Then we have

$$\begin{gathered} z_{\epsilon tt} + a_\epsilon(t,x) z_{\epsilon t} + (-\Delta) z_{\epsilon t} + (-\Delta)^2 z_\epsilon + \lambda z_\epsilon - f'(e_j^*) z_\epsilon = h(z_\epsilon) \\ z_\epsilon(\tau) = z_{\epsilon 0}, \quad z_{\epsilon t}(\tau) = z_{\epsilon 1} \end{gathered} \tag{4.1}$$

where $h(u) = f(u + e_j^*) - f(e_j^*) - f'(e_j^*)u$. Note that $h(0) = 0$ as well $Dh(0) = 0 \in \mathcal{L}(X^0)$.

Let us consider the system

$$\frac{d}{dt}(u,v) + \bar{A}_\epsilon(t)(u,v) = (0, h(u)), \tag{4.2}$$

where

$$\bar{A}_\epsilon(t) := \begin{bmatrix} 0 & -I \\ (-\Delta)^2 - \lambda I - f'(e_j^*) & -\Delta + a_\epsilon(t) I \end{bmatrix}.$$

Under the hypothesis on the map $t \mapsto a_\epsilon(t)$, it follows from [9, Theorem 7.6.11] that the process $\{L_\epsilon(t,\tau) : t \geqslant \tau\}$ has an exponential dichotomy, for all $\epsilon \in [0, \epsilon_0]$, for some $\epsilon_0 > 0$ sufficiently small.

Therefore, the proof of the lower semicontinuity of the family $\{\mathbb{A}_\epsilon : t \in \mathbb{R}\}$, based on the proof of the continuity of the sets $W^u(e_i^*)$ and $W^u(e_{i,\epsilon}^*)$, is achieved thanks to the following Theorem from [3].

Theorem 4.2 ([3, Theorem 3.1]). *Let X be a Banach space and consider a family $\{S_\epsilon(t,\tau) : t \geq \tau\}_{\epsilon \in [0,1]}$, of evolution process in X. Assume that for any x in a compact subset of X, $\|S_\epsilon(t,\tau)x - S_0(t,\tau)x\|_X \overset{\epsilon \to 0}{\longrightarrow} 0$, for $[\tau, t] \subset \mathbb{R}$ and suppose that for each $\epsilon \in [0,1]$ there exist a pullback attractor $\{\mathbb{A}_\epsilon(t) : t \in \mathbb{R}\}$, such that $\cup_{t \in \mathbb{R}} \cup_{\epsilon \in [0,\epsilon_0]} \mathbb{A}_\epsilon(t) \subset X$ is relatively compact and $\{\mathbb{A}_0(t) : t \in \mathbb{R}\}$ is given as (3.2). Further, assume that for each $e_i^* \in \mathcal{E}_0$:*

- (i) *Given $\delta > 0$, there exist $\epsilon_{i,\delta}$ such that for all $0 < \epsilon < \epsilon_{i,\delta}$ there is a global hyperbolic solution $\xi_{i,\epsilon}$ of (1.3) that satisfies $\sup_{t \in \mathbb{R}} \|\xi_{i,\epsilon}(t) - e_i^*\| < \delta$;*
- (ii) *The local unstable manifold of $\xi_{i,\epsilon}$ behaves continuously at $\epsilon = 0$; i.e.,*

$$\max[\operatorname{dist}_H(W^u_{0,\mathrm{loc}}(e_i^*), W^u_{\epsilon,\mathrm{loc}}(e_{i,\epsilon}^*)), \operatorname{dist}_H(W^u_{\epsilon,\mathrm{loc}}(e_{i,\epsilon}^*), W^u_{0,\mathrm{loc}}(e_i^*))] \overset{\epsilon \to 0}{\longrightarrow} 0,$$

where $W^u_{\mathrm{loc}}(\cdot) = W^u(\cdot) \cap B_X(\cdot, \rho)$, for some $\rho > 0$.

Then the family $\{\mathbb{A}_\epsilon(t) : t \in \mathbb{R}\}_{\epsilon \in [0,\epsilon_0]}$ is lower semicontinuous at $\epsilon = 0$.

References

[1] T. Caraballo, A. N. Carvalho, J. A. Langa and F. Rivero; *A gradient-like non-autonomous evolution processes*, International Journal of Bifurcation and Chaos, 20 (9) 2751–2760 (2010).

[2] T. Caraballo, A. N. Carvalho, J. A. Langa, and F. Rivero; *A non-autonomous strongly damped wave equation: existence and continuity of the pullback attractor*, Nonlinear Analysis, 74, 2272–2283, (2011).

[3] A.N. Carvalho, J.A. Langa, J.C. Robinson; *Lower semicontinuity of attractors for non-autonomous dynamical systems*, Ergodic Theory Dynam. Systems 29, (6), 1765–1780, (2009).

[4] A. N. Carvalho, M. J. D. Nascimento; *Singularly non-autonomous semilinear parabolic problems with critical exponents*, Discrete Contin. Dyn. Syst. Ser. S, Vol. 2, No. 3, 449-471 (2009).

[5] V. L. Carbone, M. J. D. Nascimento, K. Schiabel-Silva, R .P. Silva; *Pullback attractors for a singularly nonautonomous plate equation*, Electronic Journal of Differential Equations v. 2011, 77, 1–13, (2011).

[6] S. Chen, D. L. Russell; *A mathematical model for linear elastic systems with structural damping*, Quarterly of Applied Mathematics, v. 39, **4**, 433-454 (1981).

[7] S. Chen, R. Triggiani; *Proof of extensions of two conjectures on structural damping for elastic systems*, Pacific Journal of Mathematics, v. 136, **1**, 15-55 (1989).

[8] G. Di Blasio, K. Kunisch, E. Sinestrari; *Mathematical models for the elastic beam with structural damping*, Appl. Anal. **48**, 133-156 (1993).
[9] D. Henry; *Geometric Theory of Semilinear Parabolic Equations*. Springer-Verlag, Berlin, (1981).
[10] F. Huang; *On the mathematical model for linear elastic systems with analytic damping*, SIAM J. Control and Optimization, **126**, (3) (1988).
[11] T. Xiao, J. Liang; *Semigroups Arising from Elastic Systems with Dissipation*, Computers Math. Applic., **33**, (10), 1-9 (1997).

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