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EXISTENCE OF POSITIVE SOLUTIONS FOR SYSTEMS OF BENDING ELASTIC BEAM EQUATIONS

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Abstract. This article discusses the existence of positive solutions for systems of bending elastic beam equations. In mechanics, the problem describes the deformations of two elastic beams in equilibrium state, whose two ends are simply supported.

1. Introduction

The deformations of two elastic beams in equilibrium state, whose two ends are simply supported, can be described by the systems of bending elastic beam equations:

$$\begin{gathered}
u^{(4)}(t) = f_1(t, u(t), v(t), u''(t), v''(t)), \quad 0 < t < 1, \\
v^{(4)}(t) = f_2(t, u(t), v(t), u''(t), v''(t)), \quad 0 < t < 1, \\
u(0) = u(1) = u''(0) = u''(1) = 0, \\
v(0) = v(1) = v''(0) = v''(1) = 0,
\end{gathered} \tag{1.1}$$

where $f_1, f_2 : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^+$ are continuous functions, and the u'', v'' in f_1 and f_2 are the bending moment terms which represent bending effect, $I = [0, 1], \mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}^- = (-\infty, 0]$.

In recent years, due to its importance in physics, some authors (see [3, 4, 6, 7]) have studied the existence of solutions to the equation

$$\begin{gathered}
u^{(4)}(t) = f(t, u(t), u''(t)), \quad 0 < t < 1, \\
u(0) = u(1) = u''(0) = u''(1) = 0.
\end{gathered} \tag{1.2}$$

Naturally, further study in this specific field is on the system of fourth-order ordinary differential equations. However, to our knowledge, results for systems of fourth-order ordinary differential equations are rarely seen (see [5, 8]). For example, In [5], by applying the fixed-point theorem of cone expansion and compression type due to Krasnosel'skii, the authors show the existence of single and multiple

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positive solutions of the singular boundary value problems for systems of nonlinear fourth-order differential equations of the form

$$\begin{gathered}
u^{(4)}(t)=a_1(t)f_1(t,u(t),v(t),u''(t),v''(t))+b_1(t)g_1(t,u(t),v(t),u''(t),v''(t)),\\
v^{(4)}(t)=a_2(t)f_2(t,u(t),v(t),u''(t),v''(t))+b_2(t)g_2(t,u(t),v(t),u''(t),v''(t)),\\
0<t<1,\\
u(0)=u(1)=v(0)=v(1)=0,\\
\alpha_1u''(0)-\beta_1u'''(0)=0,\quad \gamma_1u''(1)+\delta_1u'''(1)=0,\\
\alpha_2v''(0)-\beta_2v'''(0)=0,\quad \gamma_2v''(1)+\delta_2v'''(1)=0,
\end{gathered}\tag{1.3}$$

where f_i, g_i satisfied some weaker conditions and are continuous; $a_i(t)$ and $b_i(t)$ are allowed to be singular at $t=0$ or $t=1, i=1,2$.

In the above articles, it is always supposed that the nonlinear terms satisfy the superlinear and sublinear conditions, or some weaker conditions which are similar to them (see [5, 8]). Therefore, the purpose of this paper is to improve these results. We shall employ the theory of the fixed point index in cones to present some precise conditions on f_1 and f_2 guaranteeing the existence of positive solutions of the system (1.1).

Moreover, in this paper, we study the existence of positive solutions for system (1.1) in the case that the nonlinear terms have the different features. However, it is difficult to directly construct proper open sets in a single cone in product space. Therefore, we will construct a cone $K_1\times K_2$ which is the Cartesian product of two cones in space $C^2[0,1]$ and choose the proper open sets $O=O_1\times O_2\subset K_1\times K_2$. Applying the product formula for the fixed point index on product cone and the fixed point index theory, we obtain the existence of positive solutions for system (1.1).

This paper is organized as follows. In Section 2, we present some preliminaries and main result. In Section 3, we present some basic lemmas that will be used to prove our main result. In Section 4, we will prove the main result in Section 2.

2. Preliminaries and main result

In this Section, we will give some useful preliminary results and change the system (1.1) into the fixed point problem in a cone which is the Cartesian product of two cones.

We shall consider the Banach space $C^2[0,1]$ equipped with the norm

$$\|u\|_2=\|u\|+\|u''\|=\max_{0\le t\le 1}|u(t)|+\max_{0\le t\le 1}|u''(t)|,$$

and the Banach space $C^2[0,1]\times C^2[0,1]$ equipped with the norm

$$\|(u,v)\|_2=\|u\|_2+\|v\|_2.$$

Let $G(t,s)$ be the Green function to the linear boundary value problem

$$-u''=0,\quad u(0)=u(1)=0.$$

which is explicitly expressed by

$$G(t,s)=\begin{cases}t(1-s), & 0\le t\le s\le 1,\\ s(1-t), & 0\le s\le t\le 1.\end{cases}\tag{2.1}$$

It is clear that

$$\begin{gathered} G(t,s)>0, \quad 0<t,s<1, \\ G(t,t)G(s,s)\le G(t,s)\le G(s,s)=s(1-s), \quad t,s\in I. \end{gathered} \tag{2.2}$$

For convenience, we now introduce the notation, for $r>0$,

$$K_r=\{u\in K:\|u\|_2<r\}, \quad \partial K_r=\{u\in K:\|u\|_2=r\}, \overline{K_r}=\{u\in K:\|u\|_2\le r\},$$

and

$$\begin{aligned} K=\big\{&u\in C^2[0,1]:u(t)\ge 0, u''(t)\le 0,\ u(t)\ge q(t)\|u\|, \\ &u''(t)\le -q(t)\|u''\|,\ t\in I\big\}, \end{aligned}$$

where $q(t)=t(1-t)$. It is easy to prove that K is a cone in $C^2[0,1]\times C^2[0,1]$.

Let us list the following assumptions:

(H1) $f_1,f_2:I\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^-\times\mathbb{R}^-\to\mathbb{R}^+$ are continuous functions;

(H2) there exist $h_1\in C(I\times\mathbb{R}^+\times\mathbb{R}^-,\mathbb{R}^+)$, such that

$$f_1(t,x,y,r,s)\ge h_1(t,x,r), \quad \forall t\in I,\ x,y\in\mathbb{R}^+,\ r,s\in\mathbb{R}^-,$$

where

$$\liminf_{|x|+|r|\to+\infty}\min_{t\in I}\frac{h_1(t,x,r)}{|x|+|r|}>\frac{\pi^4}{1+\pi^2};$$

(H3) there exist $h_2\in C(I\times\mathbb{R}^+\times\mathbb{R}^-,\mathbb{R}^+)$, such that

$$f_2(t,x,y,r,s)\le h_2(t,y,s), \quad \forall t\in I, x,y\in\mathbb{R}^+,\ r,s\in\mathbb{R}^-,$$

where

$$\limsup_{|y|+|s|\to+\infty}\max_{t\in I}\frac{h_2(t,y,s)}{|y|+|s|}<\frac{\pi^4}{1+\pi^2};$$

(H4) there exist α_1, $\beta_1\ge 0$, with $\frac{\alpha_1}{\pi^4}+\frac{\beta_1}{\pi^2}<1$, and $r_0>0$, such that

$$f_1(t,x,y,r,s)\le\alpha_1 x-\beta_1 r, \quad \forall t\in I,\ x\in[0,r_0],\ r\in[-r_0,0],\ y\in\mathbb{R}^+,\ s\in\mathbb{R}^-;$$

(H5) there exist $\alpha_2>0$, $\beta_2\ge 0$, $\frac{\alpha_2}{\pi^4}+\frac{\beta_2}{\pi^2}>1$, and $r_0^*>0$, such that

$$f_2(t,x,y,r,s)\ge\alpha_2 y-\beta_2 s, \quad \forall t\in I,\ y\in[0,r_0^*],\ s\in[-r_0^*,0],\ x\in\mathbb{R}^+,\ r\in\mathbb{R}^-.$$

We obtain the following results concerned with positive solutions for system (1.1).

Theorem 2.1. *Assume that* (H1)–(H5) *hold. Then* (1.1) *has at least one positive solution.*

It is easy to see that conditions (H4) and (H5) are weaker than the superlinear and sublinear conditions.

3. Basic lemmas

For $\lambda \in I$, $u, v \in C^2[0,1]$, we define two operators $A_\lambda, B_\lambda : C^2[0,1] \times C^2[0,1] \to C^2[0,1]$ by

$$\begin{aligned} A_\lambda(u,v)(t) &= \int_0^1 \int_0^1 G(t,s)G(s,\tau)[\lambda f_1(\tau, u(\tau), v(\tau), u''(\tau), v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau, u(\tau), u''(\tau))]\mathrm{d}\tau \mathrm{d}s, \\ B_\lambda(u,v)(t) &= \int_0^1 \int_0^1 G(t,s)G(s,\tau)[\lambda f_2(\tau, u(\tau), v(\tau), u''(\tau), v''(\tau)) \\ &\quad + (1-\lambda)h_2(\tau, v(\tau), v''(\tau))]\mathrm{d}\tau \mathrm{d}s. \end{aligned} \tag{3.1}$$

Then we define an operator $T_\lambda : C^2[0,1] \times C^2[0,1] \to C^2[0,1] \times C^2[0,1]$ by

$$T_\lambda(u,v) = (A_\lambda(u,v), B_\lambda(u,v)), \ (u,v) \in C^2[0,1] \times C^2[0,1]. \tag{3.2}$$

Lemma 3.1. *Assume that* (H1) *holds. Then*

1. $T_\lambda : C^2[0,1] \times C^2[0,1] \to C^2[0,1] \times C^2[0,1]$ *is completely continuous.*
2. $T_\lambda : K \times K \to K \times K$ *is completely continuous.*
3. *If* $(u,v) \in K \times K$ *is a nontrivial fixed point of* T_1*, then* (u,v) *is a positive solution of system* (1.1).

Proof. (1) The proof is similar to that of [5, Lemma 2.1], and we omit it. (2) By (1), we only need to prove that operator $T_\lambda : K \times K \to K \times K$. In fact, for any $(u,v) \in K \times K$, it follows from (3.1) that

$$\begin{aligned} &A_\lambda(u,v)(t) \ge 0, \quad A_\lambda(u,v)''(t) \le 0, \quad t \in [0,1], \\ \|A_\lambda(u,v)\| &\le \int_0^1 \int_0^1 s(1-s)G(s,\tau)[\lambda f_1(\tau, u(\tau), v(\tau), u''(\tau), v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau, u(\tau), u''(\tau))]\mathrm{d}\tau \mathrm{d}s, \\ \|A_\lambda''(u,v)\| &\le \int_0^1 G(s,s)[\lambda f_1(s, u(s), v(s), u''(s), v''(s)) \\ &\quad + (1-\lambda)h_1(s, u(s), u''(s))]\mathrm{d}s. \end{aligned} \tag{3.3}$$

On the other hand, for any $(u,v) \in K \times K$ and any $0 \le t \le 1$, It follows from (2.2), (3.1), and (3.3) that

$$\begin{aligned} A_\lambda(u,v)(t) &= \int_0^1 \int_0^1 G(t,s)G(s,\tau)[\lambda f_1(\tau, u(\tau), v(\tau), u''(\tau), v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau, u(\tau), u''(\tau))]\mathrm{d}\tau \mathrm{d}s \\ &\ge q(t) \int_0^1 \int_0^1 s(1-s)G(s,\tau)[\lambda f_1(\tau, u(\tau), v(\tau), u''(\tau), v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau, u(\tau), u''(\tau))]\mathrm{d}\tau \mathrm{d}s \\ &\ge q(t)\|A_\lambda(u,v)\|, \end{aligned}$$

and

$$\begin{aligned} A_\lambda''(u,v)(t) &= -\int_0^1 G(t,s)[\lambda f_1(s, u(s), v(s), u''(s), v''(s)) \\ &\quad + (1-\lambda)h_1(s, u(s), u''(s))]\mathrm{d}s \end{aligned}$$

$$\begin{aligned}
&\le -q(t)\int_0^1 G(s,s)[\lambda f_1(s,u(s),v(s),u''(s),v''(s))\\
&\quad + (1-\lambda)h_1(s,u(s),u''(s))]\mathrm{d}s\\
&\le -q(t)\|A''_\lambda(u,v)\|.
\end{aligned}$$

In a similar way, it follows that

$$B_\lambda(u,v)(t)\ge 0,\quad B_\lambda(u,v)''(t)\le 0,\quad t\in I,$$

and

$$B_\lambda(u,v)(t)\ge q(t)\|B_\lambda(u,v)\|,\quad B''_\lambda(u,v)(t)\le -q(t)\|B''_\lambda(u,v)\|,\quad \forall\ t\in I.$$

From the above, we assert that $T_\lambda(u,v)=(A_\lambda(u,v),B_\lambda(u,v))\in K\times K$; that is, $T_\lambda: K\times K\to K\times K$.

(3) Let $(u,v)\in K\times K$ is a fixed point of T_1, Then

$$\begin{aligned}
u(t)&=A_1(u,v)(t)\\
&=\int_0^1\Big[\int_0^1 G(t,s)G(s,\tau)f_1(\tau,u(\tau),v(\tau),u''(\tau),v''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s,\quad t\in I,\\
v(t)&=B_1(u,v)(t)\\
&=\int_0^1\Big[\int_0^1 G(t,s)G(s,\tau)f_2(\tau,u(\tau),v(\tau),u''(\tau),v''(\tau))\mathrm{d}\tau\Big]\mathrm{d}s,\quad t\in I.
\end{aligned}$$

After direct computations, we obtain

$$\begin{gathered}
u''(t)=-\int_0^1 G(t,s)f_1(s,u(s),v(s),u''(s),v''(s))\mathrm{d}s,\\
u'''(t)=\int_0^t sf_1(s,u(s),v(s),u''(s),v''(s))\mathrm{d}s\\
-\int_t^1(1-s)f_1(s,u(s),v(s),u''(s),v''(s))\mathrm{d}s,\\
u^{(4)}(t)=f_1(t,u(t),v(t),u''(t),v''(t)),\\
v''(t)=-\int_0^1 G(t,s)f_2(s,u(s),v(s),u''(s),v''(s))\mathrm{d}s,\\
v'''(t)=\int_0^t sf_2(s,u(s),v(s),u''(s),v''(s))\mathrm{d}s\\
-\int_t^1(1-s)f_2(s,u(s),v(s),u''(s),v''(s))\mathrm{d}s,\\
v^{(4)}(t)=f_2(t,u(t),v(t),u''(t),v''(t)).
\end{gathered}$$

Moreover, since $G(0,s)=G(1,s)=0$, we see that $u(0)=u(1)=u''(0)=u''(1)=v(0)=v(1)=v''(0)=v''(1)=0$.

Therefore, (u,v) is a solution of (1.1). Moreover, since the graphs of $u\in K$ and $v\in K$ are concave down on I, we assert that (u,v) is a positive solution of system (1.1). This completes the proof. $\square$

Remark 3.2. Denoting $T(\lambda,u,v)(t)=T_\lambda(u,v)(t)$, we see that $\overline{T(\lambda\times K\times K)}$ is a compact set by the Arzela-Ascoli theorem.

Lemma 3.3 ([2, 9]). *Let E be a real Banach space and let P be a closed convex cone in E. Ω be a bounded open set of E, $\theta \in \Omega$, $A : P \cap \overline{\Omega} \to P$ be completely continuous. Then the following conclusions are valid:*

(i) *if $\mu Au \neq u$ for every $u \in P \cap \partial\Omega$ and $\mu \in (0, 1]$, then $i(A, P \cap \Omega, P) = 1$;*
(ii) *if mapping A satisfies the following two conditions:*
  (a) $\inf_{u \in P \cap \partial\Omega} \|Au\| > 0$,
  (b) $\mu Au \neq u$ for every $u \in P \cap \partial\Omega$ and $\mu \geq 1$, then $i(A, P \cap \Omega, P) = 0$.

Lemma 3.4 ([1]). *Let E be a Banach space and let $K_i \subset E (i = 1, 2)$ be a closed convex cone in E. For $r_i > 0$ $(i = 1, 2)$, denote $K_{r_i} = \{u \in K_i : \|u\| < r_i\}$, $\partial K_{r_i} = \{u \in K_i : \|u\| = r_i\}$. Let $A_i : K_i \to K_i$ be completely continuous. If $A_i u_i \neq u_i$, for all $u \in \partial K_{r_i}$, then*

$$i(A, K_{r_1} \times K_{r_2}, K_1 \times K_2) = i(A_1, K_{r_1}, K_1) \times i(A_2, K_{r_2}, K_2),$$

where $A(u, v) = (A_1 u, A_2 v)$, for all $(u, v) \in K_1 \times K_2$.

4. Proof of main result

We separate the proof of Theorem 2.1 into five steps.

Step 1. For each $r_1 \in (0, r_0)$, we will prove that

$$\mu A_\lambda(u, v) \neq u, \quad \forall \mu \in (0, 1],\ (u, v) \in \partial K_{r_1} \times K. \tag{4.1}$$

In fact, if there exist $\mu_0 \in (0, 1]$ and $(u_0, v_0) \in \partial K_{r_1} \times K$, such that $\mu_0 A_\lambda(u_0, v_0) = u_0$, then $u_0(t)$ satisfies the differential equation

$$\begin{gathered} u_0^{(4)}(t) = \mu_0[\lambda f_1(t, u_0(t), v_0(t), u_0''(t), v_0''(t)) + (1 - \lambda)h_1(t, u_0(t), u_0''(t))], \\ u_0(0) = u_0(1) = u_0''(0) = u_0''(1) = 0. \end{gathered}$$

Since $0 \leq u_0(t), -u_0''(t) \leq \|u_0\|_2 = r_1 < r_0$, from (H2) and (H4), we obtain

$$\begin{aligned} u_0^{(4)}(t) &\leq \lambda f_1(t, u_0(t), v_0(t), u_0''(t), v_0''(t)) + (1 - \lambda)h_1(t, u_0(t), u_0''(t)) \\ &\leq \alpha_1 u_0(t) - \beta_1 u_0''(t), \end{aligned}$$

Multiplying both sides of this inequality by $\sin(\pi t)$ and integrating on I, then using integrating by parts, we obtain

$$\pi^4 \int_0^1 u_0(t) \sin(\pi t) \mathrm{d}t \leq (\alpha_1 + \beta_1 \pi^2) \int_0^1 u_0(t) \sin(\pi t) \mathrm{d}t. \tag{4.2}$$

By [4, Lemma 1],

$$\frac{\pi^3 + \pi^5}{4} \int_0^1 u_0(t) \sin(\pi t) \mathrm{d}t \geq \|u_0\| + \|u_0''\| = \|u_0\|_2 = r_1 > 0. \tag{4.3}$$

Hence, $\int_0^1 u_0(t) \sin(\pi t) \mathrm{d}t > 0$. From (4.2) and (4.3), we obtain that $\pi^4 \leq (\alpha_1 + \beta_1 \pi^2)$, which is a contradiction.

Step 2. From (H2), there exist $\epsilon > 0$, $m > 0$, $C > 0$, such that

$$h_1(t, u, u'') \geq (\frac{\pi^4}{1 + \pi^2} + \epsilon)(|u| + |u''|), \quad \forall t \in I,\ |u| + |u''| \geq m, \tag{4.4}$$

and

$$h_1(t, u, u'') \geq (\frac{\pi^4}{1 + \pi^2} + \epsilon)(|u| + |u''|) - C, \quad \forall t \in I,\ u \in \mathbb{R}^+. \tag{4.5}$$

We will prove that there exist $R_1 > r_1$, such that

$$\mu A_\lambda(u,v) \neq u, \quad \inf_{u\in\partial K_{R_1}} \|A_\lambda(u,v)\|_2 > 0, \quad \forall \mu \geq 1,\ (u,v)\in \partial K_{R_1}\times K. \tag{4.6}$$

In fact, if there exist $\mu_1 \geq 1$ and $(u_1,v_1)\in\partial K_{R_1}\times K$, such that $\mu_1 A_\lambda(u_1,v_1) = u_1$, then $u_1(t)$ satisfies the differential equation

$$\begin{gathered} u_1^{(4)}(t) = \mu_1[\lambda f_1(t,u_1(t),v_1(t),u_1''(t),v_1''(t)) + (1-\lambda)h_1(t,u_1(t),u_1''(t))] \\ u_1(0) = u_1(1) = u_1''(0) = u_1''(1) = 0. \end{gathered}$$

In combination with (4.5) and the condition (H2), we obtain that

$$\begin{aligned} u_1^{(4)}(t) &\geq \lambda f_1(t,u_1(t),v_1(t),u_1''(t),v_1''(t)) + (1-\lambda)h_1(t,u_1(t),u_1''(t)) \\ &= \lambda(f_1(t,u_1(t),v_1(t),u_1''(t),v_1''(t)) - h_1(t,u_1(t),u_1''(t))) + h_1(t,u_1(t),u_1''(t)) \\ &\geq h_1(t,u_1(t),u_1''(t)) \\ &\geq (\frac{\pi^4}{1+\pi^2}+\epsilon)(u_1 - u_1'') - C, \quad \forall t\in I. \end{aligned}$$

Multiplying the both sides of this inequality by $\sin(\pi t)$ and integrating on I, then using integrating by parts, we obtain

$$\pi^4\int_0^1 u_1(t)\sin(\pi t)\mathrm{d}t \geq (\frac{\pi^4}{1+\pi^2}+\epsilon)(1+\pi^2)\int_0^1 u_1(t)\sin(\pi t)\mathrm{d}t - \frac{2C}{\pi}.$$

Hence

$$\int_0^1 u_1(t)\sin(\pi t)\mathrm{d}t \leq \frac{1}{(1+\pi^2)\epsilon}\frac{2C}{\pi}.$$

In combination with (4.3), we obtain

$$\|u_1\|_2 \leq \frac{\pi^3+\pi^5}{4(1+\pi^2)\epsilon}\frac{2C}{\pi} = \frac{\pi^2 C}{2\epsilon} := R^*.$$

So, as $R_1 > R^*$, we have $\mu A_\lambda(u,v)\neq u$, for all $(u,v)\in\partial K_{R_1}\times K$ and $\mu\geq 1$. In addition, if $R_1 > \frac{5C}{\epsilon}$, by (4.5), we know that for all $(u,v)\in\partial K_{R_1}\times K$,

$$\begin{aligned} &A_\lambda(u,v)(\frac{1}{2}) \\ &= \int_0^1\int_0^1 G(\frac{1}{2},s)G(s,\tau)[\lambda f_1(\tau,u(\tau),v(\tau),u''(\tau),v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau,u(\tau),u''(\tau))]\mathrm{d}\tau\mathrm{d}s \\ &\geq \frac{1}{4}\int_0^1\int_0^1 G(s,s)G(s,\tau)[\lambda f_1(\tau,u(\tau),v(\tau),u''(\tau),v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau,u(\tau),u''(\tau))]\mathrm{d}\tau\mathrm{d}s \\ &\geq \frac{1}{4}\int_0^1 G(s,s)\int_0^1 q(s)G(\tau,\tau)[\lambda f_1(\tau,u(\tau),v(\tau),u''(\tau),v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau,u(\tau),u''(\tau))]\mathrm{d}\tau\mathrm{d}s \\ &= \frac{1}{4}\int_0^1 G(s,s)q(s)\mathrm{d}s\int_0^1 G(\tau,\tau)[\lambda f_1(\tau,u(\tau),v(\tau),u''(\tau),v''(\tau)) \\ &\quad + (1-\lambda)h_1(\tau,u(\tau),u''(\tau))]\mathrm{d}\tau \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{120}\int_0^1 G(\tau,\tau)[(\frac{\pi^4}{1+\pi^2}+\epsilon)(u(\tau)-u''(\tau))]\mathrm{d}\tau - \frac{1}{120}\int_0^1 G(\tau,\tau)\mathrm{d}\tau\cdot C\\
&\geq \frac{1}{120}\Big[\epsilon\int_0^1 G(\tau,\tau)q(\tau)(\|u\|+\|u''\|)\mathrm{d}\tau\Big] - \frac{1}{720}C\\
&= \frac{1}{120}\frac{\epsilon}{30}\|u\|_2 - \frac{1}{720}C > 0,
\end{aligned}
$$

which follows that $\inf_{u\in\partial K_{R_1}}\|A_\lambda(u,v)\|_2 > 0$. Hence, we choose

$$
R_1 > \max\{R^*, \frac{5C}{\epsilon}, r_1\}. \tag{4.7}
$$

Step 3. For each $r_2 \in (0, r_0^*)$, we will prove that

$$
\mu B_\lambda(u,v)\neq v, \quad \inf_{v\in\partial K_{r_2}}\|B_\lambda(u,v)\|_2 > 0, \quad \forall \mu\geq 1,\ (u,v)\in K\times\partial K_{r_2}. \tag{4.8}
$$

From (H3) and (H5),

$$
\begin{gathered}
\lambda f_2(t,u,v,u'',v'') + (1-\lambda)h_2(t,v,v'') \geq \alpha_2 v - \beta_2 v'',\\
\forall t\in I,\ v\in[0,r_2],\ v''\in[-r_2,0],\ u\in\mathbb{R}^+,\ u''\in\mathbb{R}^-.
\end{gathered} \tag{4.9}
$$

By (4.9) and a proof similar to Step 1 and 2, we deduce that (4.8) holds.

Step 4. We will prove that

$$
\mu B_\lambda(u,v)\neq v, \quad \forall\mu\in(0,1],\ (u,v)\in K\times\partial K_{R_2}. \tag{4.10}
$$

From (H3),we know that there exist $\epsilon>0$, $m>0$, $C>0$, such that

$$
\begin{gathered}
\lambda f_2(t,u,v,u'',v'') + (1-\lambda)h_2(t,v,v'') \leq (\frac{\pi^4}{1+\pi^2}-\epsilon)(|v|+|v''|),\\
\forall t\in I,\ |v|+|v''|\geq m,\ u\in\mathbb{R}^+,\ u''\in\mathbb{R}^-;\\
\lambda f_2(t,u,v,u'',v'') + (1-\lambda)h_2(t,v,v'') \leq (\frac{\pi^4}{1+\pi^2}-\epsilon)(|v|+|v''|)+C,\\
\forall t\in I,\ u\in\mathbb{R}^+,\ v\in\mathbb{R}^+,\ u''\in\mathbb{R}^-,\ v''\in\mathbb{R}^-.
\end{gathered}
$$

Then the proof similar to Step 2. If we choose $R_2>\max\{R^*, r_2\}$, we deduce that (4.10) holds.

Step 5. We choose an open set $D=(K_{R_1}\backslash\overline{K_{r_1}})\times(K_{R_2}\backslash\overline{K_{r_2}})$. By (4.1), (4.6), (4.8), and (4.10), it is easy to verify that $\{T_\lambda\}_{\lambda\in I}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D; on the other hand, in combination with the classical fixed point index results (see Lemma 3.3), we have

$$
\begin{gathered}
i(A_0,K_{r_1},K) = i(B_0,K_{R_2},K) = 1,\\
i(A_0,K_{R_1},K) = i(B_0,K_{r_2},K) = 0.
\end{gathered}
$$

Applying the homotopy invariance of fixed point index and the product formula for the fixed point index (see Lemma 3.4), we obtain

$$
\begin{aligned}
&i(T_1,D,K\times K) = i(T_0,D,K\times K)\\
&= i(A_0,K_{R_1}\backslash\overline{K_{r_1}},K)\times i(B_0,K_{R_2}\backslash\overline{K_{r_2}},K)\\
&= [i(A_0,K_{R_1},K)-i(A_0,K_{r_1},K)]\times[i(B_0,K_{R_2},K)-i(B_0,K_{r_2},K)] = -1.
\end{aligned}
$$

Thus, T_1 has at least a fixed point$(u^*,v^*)\in(K_{R_1}\backslash\overline{K_{r_1}})\times(K_{R_2}\backslash\overline{K_{r_2}})$. Hence, by Lemma 3.1, system (1.1) has at least one positive solution (u^*,v^*). The proof of Theorem 2.1 is complete.

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