Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 192, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLINEAR DISCRETE INCLUSIONS

NICU MARCU, GIOVANNI MOLICA BISCI

Abstract. A non-smooth abstract result is used for proving the existence of at least one nontrivial solution of an algebraic discrete inclusion. Successively, a multiplicity theorem for the same class of discrete problems is also established by using a locally Lipschitz continuous version of the famous Brézis-Nirenberg theoretical result in presence of splitting. Some applications to tridiagonal, fourth-order and partial difference inclusions are pointed out.

1. Introduction

A considerable number of problems, which are strictly connected both with boundary value differential problems and numerical simulations of some mathematical models arising from many research areas (biological, physical and computer science) can be formulated as special cases of nonlinear algebraic systems (see, for instance [28]).

In this article, motivated by this large interest, we investigate the existence of solutions for discrete algebraic inclusions. More precisely, let $T > 1$ be a positive integer and let $g_k : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function, for every $k \in \mathbb{Z}[1, T] := \{1, 2, \dots, T\}$. We are interested either on the existence or in multiple solutions for the discrete inclusion

$$\sum_{l=1}^{T} a_{kl} u_l \in [g_k^-(u_k), g_k^+(u_k)], \quad (\forall\, k \in \mathbb{Z}[1, T]), \tag{1.1}$$

where $A := (a_{ij})_{T\times T}$ is a real symmetric positive definite matrix and

$$g_k^-(t) := \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|\xi-t|<\delta} g_k(\xi), \quad g_k^+(t) := \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|\xi-t|<\delta} g_k(\xi),$$

for every $k \in \mathbb{Z}[1, T]$.

It is clear that if the functions g_k are continuous (instead of locally essentially bounded) problem (1.1) becomes a more familiar nonlinear algebraic system

$$Au = g(u),$$

in which $u = (u_1, \dots, u_T)^t \in \mathbb{R}^T$ and $g(u) := (g_1(u_1), \dots, g_T(u_T))^t$.

2000 *Mathematics Subject Classification.* 38J20, 39A10, 34B15, 49J40.
Key words and phrases. Nonlinear algebraic systems; difference equations; non-smooth problems; variational methods.

Submitted September 4, 2012. Published November 6, 2012.

However, to the best of our knowledge, for discrete difference inclusions there are only few papers involving the second-order difference operator. For instance, in [1], the existence of at least one solution was obtained via the set-valued mapping theory, while in [31], existence results for suitable second-order discrete discontinuous equations have been investigated by variational methods. The aim of this paper is to establish existence and multiplicity results for algebraic discrete inclusions like problem (1.1).

The main existence result contained here (see Theorem 3.1) is obtained using a non-smooth critical points theorem contained in [3, Theorem 2.1; part (a)]. This theoretical argument represents a non-smooth refinement of the quoted variational principle of Ricceri (see [21]).

Through this variational approach, we are able to prove the existence of one solution for problem (1.1) just requiring that there is a real constant $\bar{\gamma} > 0$ such that

$$\frac{\bar{\gamma}^2}{\sum_{k=1}^{T} \max_{|\xi|\leq\bar{\gamma}} \int_0^{\xi} g_k(t)dt} > \frac{2}{\lambda_1},$$

where λ_1 is the 1-th eigenvalue of the symmetric and positive definite matrix A (see Theorem 3.1 as well as Remarks 3.2 and 3.3).

Successively, a two solutions result for the algebraic inclusion (1.1) is proved (see Theorem 4.1 and Corollary 4.2). Our proof in this case is based on an extension of the famous Brézis-Nirenberg result [4, Theorem 4] obtained by Wu in [26, Theorem 2.3] for locally Lipschitz continuous functionals.

Due to the generality of (1.1), remarkable applications are easily achieved. Indeed, Theorem 4.1 can be used proving either existence or multiplicity of solutions for discrete inclusions involving certain tridiagonal matrices, fourth-order discrete problems and partial difference inclusions (see Example 3.4 and Section 5).

A special case of Theorem 3.1 reads as follows.

Theorem 1.1. *Let $g_k : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded and nonnegative function, for every $k \in \mathbb{Z}[1,T]$. Assume that*

$$\limsup_{\gamma\to+\infty} \frac{\gamma^2}{\sum_{k=1}^{T} \int_0^{\gamma} g_k(t)dt} = +\infty,$$

and $g_k^-(0) > 0$, for some $k \in \mathbb{Z}[1,T]$. Then problem (1.1) admits at least one nontrivial solution.

Moreover, denoting by $\lambda_\ell^{(4)}$ and $\lambda_{\ell+1}^{(4)}$ respectively the ℓ-th and $(\ell+1)$-th eigenvalue of the discrete problem

$$\begin{gathered}\Delta^4 u_{k-2} = \lambda u_k, \quad (\forall\, k \in \mathbb{Z}[1,T]) \\ u_{-2} = u_{-1} = u_0 = 0 \\ u_{T+1} = u_{T+2} = u_{T+3} = 0,\end{gathered}$$

one has the following multiplicity property.

Theorem 1.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be locally essentially bounded positive function and consider the usual forward difference operator $\Delta u_{k-1} := u_k - u_{k-1}$. Assume that*

(H1) $\limsup_{|t|\to\infty} \frac{h(t)}{t} = 0$;

(H2) *There exists an integer* $\ell \in \mathbb{Z}[1, T-1]$ *such that*

$$\frac{\lambda_\ell^{(4)}}{T} \le \lim_{t\to 0} \frac{h(t)}{t} \le \frac{\lambda_{\ell+1}^{(4)}}{T}, \quad (\forall\, k \in \mathbb{Z}[1,T])$$

Then the forth-order discrete inclusion

$$\begin{gathered} \Delta^4 u_{k-2} \in [h^-(u_k), h^+(u_k)], \quad (\forall\, k \in \mathbb{Z}[1,T]) \\ u_{-2} = u_{-1} = u_0 = 0 \\ u_{T+1} = u_{T+2} = u_{T+3} = 0. \end{gathered} \tag{1.2}$$

admits at least two nontrivial solutions.

We emphasize that our results are also new in the continuous setting. In this case, the existence and multiplicity of solutions was investigated in a large number of other papers under various assumptions (see for instance [27, 28, 29, 30, 31] and references therein). See also the recent papers [5, 13, 14, 16] for related topics.

The plan of the paper is as follows. In the next section we introduce our abstract framework. Successively, in sections 3 and 4, we show our existence and multiplicity results. A concrete example of an application of our abstract results to discrete partial inclusions is presented in the last section.

2. Basic definitions and preliminary results

Let $(X, \|\cdot\|)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X.

A function $J : X \to \mathbb{R}$ is called locally Lipschitz continuous if to every $x \in X$ there corresponds a neighborhood V_x of x and a constant $L_x \ge 0$ such that

$$|J(z) - J(w)| \le L_x \|z - w\|, \quad (\forall\, z, w \in V_x)\,.$$

If $x, z \in X$, we write $J^0(x; z)$ for the generalized directional derivative of J at the point x along the direction z; i.e.,

$$J^0(x; z) := \limsup_{w\to x,\, t\to 0^+} \frac{J(w + tz) - J(w)}{t}\,.$$

The generalized gradient of the function J in x, denoted by $\partial J(x)$, is the set

$$\partial J(x) := \{x^* \in X^* : \langle x^*, z\rangle \le J^0(x; z),\ \forall\, z \in X\}.$$

The basic properties of generalized directional derivative and generalized gradient were studied in [7, 9].

We recall that if J is continuously Gâteaux differentiable at u, then J is locally Lipschitz at u and $\partial J(u) = \{J'(u)\}$, where $J'(u)$ stands for the first derivative of J at u. Further, a point u is called a (generalized) critical point of the locally Lipschitz continuous function J if $0_{X^*} \in \partial J(u)$; i.e.,

$$J^0(u; z) \ge 0,$$

for every $z \in X$. Clearly, if J is a continuously Gâteaux differentiable at u, then u becomes a (classical) critical point of J, that is $J'(u) = 0_{X^*}$.

A locally Lipschitz functional $J : X \to \mathbb{R}$ is said to fulfill the Palais-Smale condition if

(PS) Every sequence $\{x_n\} \subset X$ such that $\{J(x_n)\}$ is bounded and

$$J^0(x_n; x - x_n) \geq -\varepsilon_n \|x - x_n\|, \quad (\forall\, x \in X)$$

where $\varepsilon_n \to 0^+$, possesses a convergent subsequence.

For an complete overview on the non-smooth calculus we mention the monograph [19]. Further, we cite a very recent book [15] as a general reference on this subject.

Our main tool will be the following two abstract critical point theorems, for locally Lipschitz continuous functions.

Theorem 2.1 ([3, Theorem 2.1; part (a)])**.** *Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $\rho > \inf_X \Phi$, put*

$$\varphi(\rho) := \inf_{u \in \Phi^{-1}(]-\infty, \rho[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty,\rho[)} \Psi(v) - \Psi(u)}{\rho - \Phi(u)}.$$

Then, for every $\rho > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(\rho)[$, the restriction of the functional $J_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, \rho[)$ admits a global minimum, which is a critical point (*local minimum*) *of J_λ in X.*

Theorem 2.2. [26, Theorem 2.3] *Suppose that $X := X_1 \oplus X_2$ with* $\dim(X_1) > 0$ *and* $0 < \dim(X_2) < \infty$. *Let J be a locally Lipschitz continuous functional satisfying the* (PS) *condition and such that*

$$J(u) \leq 0, \quad (\forall\, u \in \bar{B}(0, \rho) \cap X_2),$$
$$J(u) \geq 0, \quad (\forall\, u \in \bar{B}(0, \rho) \cap X_1)$$

for some $\rho > 0$. Assume also that J is bounded from below and $\inf_{u \in X} J(u) < 0$. *Then J has at least two nonzero critical points.*

Remark 2.3. As pointed out in Introduction Theorem 2.1 can be view as a non-smooth version of the quoted variational principle of Ricceri; see the paper [21]. Further, Theorem 2.2 represents an extension, to the case of locally Lipschitz continuous functionals, of the celebrated critical point theorem in presence of splitting established by Brézis and Nirenberg [4, Theorem 4]. See for completeness the work [26, Theorem 2.3].

Here, as the ambient space X, we consider the T-dimensional Banach space $\mathbb{R}^T$ endowed with the norm

$$\|u\| := \Big(\sum_{k=1}^{T} u_k^2\Big)^{1/2},$$

induced by the standard Euclidean inner product $\langle u, v\rangle_X := \sum_{k=1}^{T} u_k v_k$.

Set $\mathfrak{X}_T$ to be the class of all symmetric and positive definite matrices of order T. Further, we denote by $\lambda_1, \dots, \lambda_T$ the eigenvalues of A (ordered as $0 < \lambda_1 \leq \dots \leq \lambda_T$) and by $\xi_1, \dots, \xi_T$ the corresponding orthonormal eigenvectors. It is well-known that if $A \in \mathfrak{X}_T$, for every $u \in X$, then one has

$$\lambda_1 \|u\|^2 \leq u^t A u \leq \lambda_T \|u\|^2, \tag{2.1}$$

$$\|u\|_\infty \leq \frac{1}{\sqrt{\lambda_1}} (u^t A u)^{1/2}, \tag{2.2}$$

where $\|u\|_\infty := \max_{k \in \mathbb{Z}[1,T]} |u_k|$.

For the rest of this article, we assume that $A \in \mathfrak{X}_T$. For every $u \in X$, we put

$$\Phi(u) := \frac{u^t A u}{2}, \quad \Psi(u) := \sum_{k=1}^{T} G_k(u_k), \quad J(u) := \Phi(u) - \Psi(u),$$

where $G_k(t) := \int_0^t g_k(\xi)d\xi$, for every $(k,t) \in \mathbb{Z}[1,T] \times \mathbb{R}$. It is easy to verify that Φ is continuously Gâteaux differentiable, while Ψ is locally Lipschitz continuous.

Proposition 2.1. *Assume that $u \in X$ is a critical point of the functional J. Then u is a solution of problem* (1.1).

Proof. If u is a critical point of J, bearing in mind of [9, Propositions 2.3.1 and 2.3.3], it follows that

$$\Phi'(u)(z) \leq \Psi^0(u;z) \leq \Big(\sum_{k=1}^{T} G_k^0(u_k;z_k)\Big), \tag{2.3}$$

for every $z \in X$. Moreover,

$$\Phi'(u)(z) = \frac{\langle \nabla(u^t A u), z\rangle_X}{2}, \tag{2.4}$$

for every $z \in X$.

For every $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}[1,T]$, by putting in (2.3) the vector $z = \xi e_k$, where e_k are the canonical unit vectors of X, and taking in mind (2.4), we obtain

$$\langle \sum_{l=1}^{T} a_{kl} u_l, \xi\rangle_{\mathbb{R}} = \Phi'(u)(z) \leq G_k^0(u_k;\xi),$$

namely

$$\sum_{l=1}^{T} a_{kl} u_l \in \partial G_k(u_k).$$

Finally, since it is well-known that

$$\partial G_k(u_k) = [g_k^-(u_k), g_k^+(u_k)],$$

for every $k \in \mathbb{Z}[1,T]$ (see for instance [9, Example 2.2.5]) it follows that

$$\sum_{l=1}^{T} a_{kl} u_l \in [g_k^-(u_k), g_k^+(u_k)], \quad (\forall\, k \in \mathbb{Z}[1,T]).$$

Therefore our assertion is proved. □

3. A nontrivial solution

The main result of this section reads as follows.

Theorem 3.1. *Let $g_k : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded function, for every $k \in \mathbb{Z}[1,T]$. Assume that*

$$\sup_{\gamma>0} \frac{\gamma^2}{\sum_{k=1}^{T} \max_{|\xi|\leq\gamma} G_k(\xi)} > \frac{2}{\lambda_1}. \tag{3.1}$$

Then problem (1.1) *admits at least one solution. Moreover if, in addition to the above condition, one has $g_k^-(0) > 0$, for some $k \in \mathbb{Z}[1,T]$, the obtained solution is nontrivial.*

Proof. Since condition (3.1) holds, there exists $\bar{\gamma} > 0$ such that

$$\frac{\bar{\gamma}^2}{\sum_{k=1}^{T} \max_{|\xi| \le \bar{\gamma}} G_k(\xi)} > \frac{2}{\lambda_1}. \tag{3.2}$$

Hence, take $\bar{\rho} := \frac{\lambda_1 \bar{\gamma}^2}{2}$ and apply Theorem 2.1. Clearly, $\inf_{u \in X} \Phi(u) < \bar{\rho}$ and

$$\varphi(\bar{\rho}) := \inf_{u \in \Phi^{-1}(]-\infty, \bar{\rho}[)} \frac{\sup_{v \in \Phi^{-1}(]-\infty, \bar{\rho}[)} \Psi(v) - \Psi(u)}{\bar{\rho} - \Phi(u)} \le \frac{\sup_{v \in \Phi^{-1}(]-\infty, \bar{\rho}[)} \Psi(v)}{\bar{\rho}},$$

taking into account that $0_X \in \Phi^{-1}(]-\infty, \bar{\rho}[)$ and $\Phi(0_X) = \Psi(0_X) = 0$.

Now, using condition (2.2), it follows that

$$\Phi^{-1}(]-\infty, \bar{\rho}[) \subseteq \{u \in X : \|u\|_\infty \le \gamma\}.$$

Thus, the above remarks imply that

$$\varphi(\bar{\rho}) \le \frac{2}{\lambda_1} \frac{\sum_{k=1}^{T} \max_{|\xi| \le \bar{\gamma}} G_k(\xi)}{\bar{\gamma}^2}. \tag{3.3}$$

Consequently, by (3.2) and (3.3) one has $\varphi(\bar{\rho}) < 1$. Hence, since $1 \in]0, 1/\varphi(\bar{\rho})[$, Theorem 2.1 ensures that the functional J admits at least one critical point (local minima) $\widetilde{u} \in \Phi^{-1}(]-\infty, \bar{\rho}[)$.

Due to Proposition 2.1, $\widetilde{u} \in X$ is a solution of (1.1). Under the additional assumption $g_k^-(0) > 0$ (for some $k \in \mathbb{Z}[1, T]$) the obtained solution is clearly non-trivial. □

Remark 3.2. If in Theorem 3.1 the functions g_k are nonnegative, condition (3.1) assumes the more simple and significative form

$$\sup_{\gamma > 0} \frac{\gamma^2}{\sum_{k=1}^{T} G_k(\gamma)} > \frac{2}{\lambda_1}. \tag{3.4}$$

Moreover, if

$$\limsup_{\gamma \to +\infty} \frac{\gamma^2}{\sum_{k=1}^{T} G_k(\gamma)} > \frac{2}{\lambda_1},$$

condition (3.4) automatically holds. Hence, Theorem 1.1 in Introduction is an immediate consequence of Theorem 3.1 taking into account the considerations above.

Remark 3.3. Let $\bar{\gamma} > 0$ be a real constant such that

$$\frac{\bar{\gamma}^2}{\sum_{k=1}^{T} \max_{|\xi| \le \bar{\gamma}} G_k(\xi)} > \frac{2}{\lambda_1},$$

and said $\widetilde{u} \in X$ be the solution of problem (1.1) obtained by using Theorem 3.1. Hence, since $\widetilde{u} \in \Phi^{-1}(]-\infty, \bar{\rho}[)$, it follows that $\|\widetilde{u}\|_\infty \le \bar{\gamma}$.

Example 3.4. Let $T \ge 3$ and $(a, b) \in \mathbb{R}^- \times \mathbb{R}^+$ be such that

$$\cos\big(\frac{\pi}{T+1}\big) < -\frac{b}{2a}.$$

Set

$$\operatorname{Trid}_T(a, b, a) = \begin{pmatrix} b & a & 0 & \dots & 0 \\ a & b & a & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & a & b & a \\ 0 & \dots & 0 & a & b \end{pmatrix}_{T \times T},$$

and consider the discrete problem

$$L_{\mathrm{Trid}}(u) \in [j_k^-(u_k), j_k^+(u_k)], \quad (\forall\, k \in \mathbb{Z}[1,T]) \tag{3.5}$$

where

$$L_{\mathrm{Trid}}(u) := \begin{cases} bu_1 + au_2 \\ au_{k-1} + bu_k + au_{k+1}, & (\forall\, k \in \{2,\dots,T-1\}) \\ au_{T-1} + bu_T, \end{cases}$$

and the functions $j_k : \mathbb{R} \to \mathbb{R}$ are assumed to be locally essentially bounded. Hence, Theorem 3.1 ensures that if

$$\sup_{\gamma>0} \frac{\gamma^2}{\sum_{k=1}^T \max_{|\xi|\le\gamma} \int_0^\xi j_k(t)dt} > \frac{2}{b + 2a\cos\left(\frac{\pi}{T+1}\right)},$$

problem (3.5) admits one solution (see [22, Example 9; p.179] for details). Moreover if, in addition to our algebraic inequality, one also have $j_k^-(0) > 0$, for some $k \in \mathbb{Z}[1,T]$, the obtained solution is nontrivial. The above result can be applied to second-order difference inclusions. Indeed, it is well-know that the $T \times T$ matrix

$$\mathrm{Trid}_T(-1,2,-1) := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

in $\mathfrak{X}_T$, is associated to the second-order discrete boundary value problem

$$\begin{gathered} -\Delta^2 u_{k-1} \in [j_k^-(u_k), j_k^+(u_k)], \quad \forall k \in \mathbb{Z}[1,T] \\ u_0 = u_{T+1} = 0, \end{gathered} \tag{3.6}$$

where $\Delta^2 u_{k-1} := \Delta(\Delta u_{k-1})$, and, as usual, $\Delta u_{k-1} := u_k - u_{k-1}$ denotes the forward difference operator.

4. Two nontrivial solutions

With the above notation and assumptions, the main result reads as follows.

Theorem 4.1. *Assume that*

(G1) $\limsup_{|\xi|\to\infty} \frac{G_k(\xi)}{\xi^2} < \frac{\lambda_1}{2}$, *for all* $k \in \mathbb{Z}[1,T]$,

and that there exists an integer $\ell \in \mathbb{Z}[1,T-1]$ *such that*

(G2) $\liminf_{\xi\to 0} \frac{G_k(\xi)}{\xi^2} \ge \frac{\lambda_\ell}{2}$, *for all* $k \in \mathbb{Z}[1,T]$.

Further, suppose that

(G3) $\limsup_{\xi\to 0} \frac{G_k(\xi)}{\xi^2} \le \frac{\lambda_{\ell+1}}{2}$, *for all* $k \in \mathbb{Z}[1,T]$.

Then problem (1.1) *possesses at least two nontrivial solutions.*

Proof. Our aim is to apply Theorem 2.2. From (G1), since X is a finite dimensional space, it is easy to see that J satisfies condition (PS).

Indeed, using condition (G1), there are constants $\epsilon \in]0, \lambda_1/2[$ and $\sigma > 0$ such that

$$\frac{G_k(\xi)}{\xi^2} < \frac{\lambda_1}{2} - \epsilon,$$

for every $|t| \geq \sigma$ and $k \in \mathbb{Z}[1,T]$. Let us put

$$M_1 := \max_{(k,\xi)\in\mathbb{Z}[1,T]\times[-\sigma,\sigma]} G_k(\xi).$$

Therefore, for every $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}[1,T]$, one has

$$G_k(\xi) \leq M_1 + M_2\xi^2,$$

where

$$M_2 := \frac{\lambda_1}{2} - \epsilon.$$

Moreover, the following inequality holds

$$J(u) \geq \frac{u^tAu}{2} - \sum_{k=1}^{T}\Big[M_1 + M_2u_k^2\Big], \quad (\forall\, u \in X).$$

Hence

$$J(u) \geq \frac{u^tAu}{2} - M_2\|u\|^2 - TM_1, \quad (\forall\, u \in X).$$

Thus, by using (2.1), one has

$$J(u) \geq \epsilon\|u\|^2 - TM_1, \quad (\forall\, u \in X) \tag{4.1}$$

which clearly shows that

$$\lim_{\|u\|\to\infty} J(u) = +\infty.$$

From this, and taking into account that X is a finite T-dimensional Hilbert space, it follows that the functional J satisfies the (PS) condition.

We will prove now that, for some $\rho_1 > 0$, $J(u) \leq 0$ for every $u \in X_2 \cap \bar{B}(0,\rho_1)$, where $X_2 = \overline{\mathrm{Span}}\{\xi_1,\dots,\xi_\ell\}$. Thus, by condition (G2), there exists $\delta > 0$ such that

$$G_k(\xi) \geq \frac{\lambda_\ell}{2}\xi^2, \quad (\forall\, k \in \mathbb{Z}[1,T]),$$

provided $0 < |\xi| \leq \delta$. Now, taking into account the discrete Cauchy-Schwarz inequality, one has

$$|u_k| \leq \sum_{k=1}^{T}|u_k| \leq T^{1/2}\Big(\sum_{k=1}^{T}|u_k|^2\Big)^{1/2},$$

for every $k \in \mathbb{Z}[1,T]$. Then

$$\|u\|_\infty \leq T^{1/2}\|u\|, \quad (\forall\, u \in X).$$

Hence, for every $u \in \bar{B}(0,\rho_1) \cap X_2$, it follows that

$$\|u\|_\infty \leq T^{1/2}\rho_1.$$

Consequently, if we take $\rho_1 \leq \delta/T^{1/2}$, we obtain

$$G_k(u_k) \geq \frac{\lambda_\ell}{2}u_k^2, \tag{4.2}$$

for every $k \in \mathbb{Z}[1,T]$. Moreover, if $u \in X_2$, there exists $a_k \in \mathbb{R}$ for every $k \in \mathbb{Z}[1,T]$, such that

$$u = \sum_{k=1}^{\ell} a_k\xi_k \quad \text{and} \quad u^tAu = \sum_{k=1}^{\ell}\lambda_k a_k^2 \leq \lambda_\ell\sum_{k=1}^{\ell}a_k^2.$$

Hence,

$$u^tAu \leq \lambda_\ell\|u\|^2, \quad (\forall\, u \in X_2). \tag{4.3}$$

Putting together (4.2) and (4.3), we have

$$J(u) \leq \frac{\lambda_\ell}{2}(\|u\|^2 - \|u\|^2) = 0, \quad (\forall\, u \in \bar{B}(0,\rho_1) \cap X_2);$$

that is,

$$J(u) \leq 0, \quad (\forall\, u \in \bar{B}(0,\rho_1) \cap X_2).$$

At this point, it remains to show that there exists $\rho_2 > 0$ such that

$$J(u) \geq 0, \quad (\forall\, u \in \bar{B}(0,\rho_2) \cap X_1)$$

where $X_1 := \overline{\operatorname{Span}}\{\xi_{\ell+1}, \dots, \xi_T\}$.

Fix $u \in X_1$, for suitable $b_k \in \mathbb{R}$ and for every $k \in \mathbb{Z}[\ell+1, T]$, one has that $u = \sum_{k=\ell+1}^{T} b_k \xi_k$ and

$$J(u) \geq \sum_{k=\ell+1}^{T} \frac{\lambda_k}{2} b_k^2 - \sum_{k=1}^{T} G_k(u_k).$$

Thus

$$J(u) \geq \frac{\lambda_{\ell+1}}{2}\|u\|^2 - \sum_{k=1}^{T} G_k(u_k).$$

Further, from (G3), there exists $\sigma > 0$ such that

$$G_k(\xi) \leq \frac{\lambda_{\ell+1}}{2}\xi^2, \quad (\forall\, k \in \mathbb{Z}[1,T])$$

provided $0 < |\xi| \leq \sigma$. Then, taking $\rho_2 \leq \sigma/N^{1/2}$, for every $u \in \bar{B}(0,\rho_2) \cap X_1$, we have

$$J(u) \geq \frac{\lambda_{\ell+1}}{2}(\|u\|^2 - \|u\|^2) = 0.$$

Therefore, choosing $\rho \leq \min\{\rho_1, \rho_2\}$, if $\inf_{u \in X} J(u) < 0 = J(0)$ our claim is proved.

On the other hand, if $\inf_{u \in X} J(u) = 0$, we argue as above; that is, every $u \in X_2$ with $\|u\|_2 \leq \rho$ is solution of problem (1.1). So, our goal is achieved. □

The following result is a direct consequence of Theorem 4.1.

Corollary 4.2. *Let $\alpha : \mathbb{Z}[1,T] \to \mathbb{R}$ be a nonnegative (not identically zero) function and let $h : \mathbb{R} \to \mathbb{R}$ be a locally essentially bounded map. Assume that there exists an integer $\ell \in \mathbb{Z}[1, T-1]$ such that*

- (G0) $\frac{\lambda_\ell}{\sum_{k=1}^{T} \alpha_k} \leq \lim_{t \to 0} \frac{h(t)}{t} \leq \frac{\lambda_{\ell+1}}{\sum_{k=1}^{T} \alpha_k}$, *for all $k \in \mathbb{Z}[1,T]$;*
- (G4) $\limsup_{|t| \to \infty} \frac{h(t)}{t} < \frac{\lambda_1}{\sum_{k=1}^{T} \alpha_k}$.

Then the discrete problem

$$\sum_{l=1}^{T} a_{kl} u_l \in \alpha_k [h^-(u_k), h^+(u_k)], \quad (\forall\, k \in \mathbb{Z}[1,T]) \tag{4.4}$$

admits at least two nontrivial solutions.

Proof. It is elementary to observe that from condition (G0) immediately (G2) and (G3) hold. We proceed by proving that condition (G4) implies (G1). Indeed, by (G4), there are constants $\epsilon' \in \left]0, \lambda_1/\left(\sum_{k=1}^{T} \alpha_k\right)\right[$ and $\sigma > 0$ such that

$$\frac{h(t)}{t} < \frac{\lambda_1}{\sum_{k=1}^{T} \alpha_k} - \epsilon',$$

for every $|t| \geq \sigma$. Since h is a locally essentially bounded function, we also have

$$M := \operatorname{ess\,sup}_{t\in[-\sigma,\sigma]} |h(t)| < +\infty.$$

Therefore, if $\xi \geq \sigma$, it follows that

$$\int_0^\xi h(t)dt = \int_0^\sigma h(t)dt + \int_\sigma^\xi h(t)dt \leq M\sigma + \frac{1}{2}\Big(\lambda_1/\Big(\sum_{k=1}^T \alpha_k\Big) - \epsilon'\Big)\xi^2,$$

while, for $\xi \leq -\sigma$, one has

$$\int_0^\xi h(t)dt = -\Big[\int_\xi^{-\sigma} h(t)dt + \int_{-\sigma}^0 h(t)dt\Big] \leq M\sigma + \frac{1}{2}\Big(\lambda_1/\Big(\sum_{k=1}^T \alpha_k\Big) - \epsilon'\Big)\xi^2.$$

Consequently,

$$\int_0^\xi h(t)dt \leq M\sigma + \frac{1}{2}\Big(\lambda_1/\Big(\sum_{k=1}^T \alpha_k\Big) - \epsilon'\Big)\xi^2, \quad (\forall\,\xi \in \mathbb{R}). \tag{4.5}$$

Hence, by using the above inequality, we can write

$$\limsup_{|\xi|\to\infty} \frac{\int_0^\xi \alpha_k h(t)dt}{\xi^2} = \alpha_k \limsup_{|\xi|\to\infty} \frac{H(\xi)}{\xi^2} \leq \frac{1}{2}\Big(\sum_{k=1}^T \alpha_k\Big)\Big(\lambda_1/\Big(\sum_{k=1}^T \alpha_k\Big) - \epsilon'\Big) < \frac{\lambda_1}{2},$$

for every $k \in \mathbb{Z}[1,T]$. So, it is clear that condition (G2) holds. In conclusion, our claim is verified and the proof is complete. $\square$

Remark 4.3. Boundary value problems involving fourth-order difference inclusions such as

$$\begin{gathered} \Delta^4 u_{k-2} \in [g_k^-(u_k), g_k^+(u_k)], \quad (\forall\, k \in \mathbb{Z}[1,T]) \\ u_{-2} = u_{-1} = u_0 = 0 \\ u_{T+1} = u_{T+2} = u_{T+3} = 0, \end{gathered} \tag{4.6}$$

can also be expressed as problem (1.1), where A is the real symmetric and positive definite matrix of the form

$$A := \begin{pmatrix} 6 & -4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & 6 \end{pmatrix}$$

in $\mathfrak{X}_T$. Then, it is easily seen that Theorem 1.2 in the introduction is a direct consequence of Corollary 4.2.

5. Partial algebraic inclusions

Nonlinear inclusions of the form (1.1) arise in many applications such as boundary value problems involving partial difference equations. For instance, we just point out that our results can be applied to the following problem

$$\begin{aligned} &4u(i,j) - u(i+1,j) - u(i-1,j) - u(i,j+1) - u(i,j-1) \\ &\in [f^-_{(i,j)}(u(i,j)), f^+_{(i,j)}(u(i,j))] \end{aligned} \tag{5.1}$$

for every $(i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]$, with boundary conditions

$$\begin{aligned} u(i,0) = u(i,n+1) = 0, \quad (\forall\, i \in \mathbb{Z}[1,m]), \\ u(0,j) = u(m+1,j) = 0, \quad (\forall\, j \in \mathbb{Z}[1,n]) \end{aligned}$$

where every $f_{(i,j)} : \mathbb{R} \to \mathbb{R}$ denotes a locally essentially bounded function.

Let $z : \mathbb{Z}[1,m] \times \mathbb{Z}[1,n] \to \mathbb{Z}[1,mn]$ be the bijection defined by $z(i,j) := i + m(j-1)$, for every $(i,j) \in \mathbb{Z}[1,m] \times \mathbb{Z}[1,n]$. Let us denote $w_k := u(z^{-1}(k))$ and $g_k(w_k) := f_{z^{-1}(k)}(w_k)$, for every $k \in \mathbb{Z}[1,mn]$.

With the above notation, problem (5.1) can be written as a nonlinear algebraic inclusion of the form

$$\sum_{l=1}^{T} b_{kl} w_l \in [g^-_k(w_k), g^+_k(w_k)], \quad (\forall\, k \in \mathbb{Z}[1,mn]), \tag{5.2}$$

where

$$B := (b_{ij}) = \begin{pmatrix} L & -I_m & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -I_m & L & -I_m & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -I_m & L & -I_m & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & L & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & L & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -I_m & L & -I_m & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -I_m & L & -I_m \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_m & L \end{pmatrix}$$

in $\mathfrak{X}_{mn}$, in which L is the $m \times m$ matrix

$$L := \begin{pmatrix} 4 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 4 \end{pmatrix}$$

and I_m is the $m \times m$ identity matrix.

Finally for completeness, we observe that the existence of multiple solutions for the nonlinear discrete problems can be used in the study of numerical methods applied to some mathematical models; see for instance the recent article [20].

References

[1] R. P. Agarwal, D. O'Regan and V. Lakshmikantham; *Discrete Second Order Inclusions*, J. Difference Equ. Appl. **9** (2003), 879-885.

[2] G. Bonanno and P. Candito; *Nonlinear difference equations investigated via critical methods*, Nonlinear Anal. **70** (2009), 3180-3186.

[3] G. Bonanno and G. Molica Bisci; *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl. (2009), Art. ID 670675, 1-20.

[4] H. Brézis and L. Nirenberg; *Remarks on finding critical points,* Comm. Pure Appl. Math. **44** (1991), 939–963.

[5] P. Candito and G. Molica Bisci; *Existence of two solutions for a nonlinear second-order discrete boundary value problem*, Adv. Nonlinear Stud. **11** (2011), 443-453.

[6] P. Candito and G. Molica Bisci; *Existence of positive solutions for nonlinear algebraic systems with a parameter*, Appl. Math. Comput. **218** (2012), 11700-11707.

[7] K.-C. F. Chang; *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. **80** (1981), 102-129.

[8] S. S. Cheng; *Partial difference equations*, Taylor & Francis, London (2003).

[9] F. H. Clarke; *Optimization and Nonsmooth Analysis*, Classics Appl. Math. SIAM, **5**, (1990).

[10] M. Galewski and A. Orpel; *On the existence of solutions for discrete elliptic boundary value problems*, Appl. Anal. **89** (2010), 1879-1891.

[11] M. Imbesi and G. Molica Bisci; *Some existence results for partial discrete problems with Dirichlet boundary conditions*, preprint.

[12] J. Ji and B. Yang; *Eigenvalue comparisons for boundary value problems of the discrete elliptic equation*, Commun. Appl. Anal. **12** (2008), 189-197.

[13] A. Kristály, M. Mihăilescu and V. Rădulescu; *Discrete boundary value problems involving oscillatory nonlinearities: small and large solutions*, J. Difference Equ. Appl. **17** (2011), 1431-1440.

[14] A. Kristály, M. Mihăilescu, V. Rădulescu and S. Tersian; *Spectral estimates for a nonhomogeneous difference problem*, Commun. Contemp. Math. **12** (2010), 1015-1029.

[15] A. Kristály, V. Rădulescu and Cs. Varga; Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, Encyclopedia of Mathematics and its Applications, No. 136, Cambridge University Press, Cambridge, 2010.

[16] M. Mihăilescu, V. Rădulescu and S. Tersian; *Eigenvalue Problems for Anisotropic Discrete Boundary Value Problems*, J. Difference Equ. Appl. **15** (2009), 557-567.

[17] G. Molica Bisci and D. Repovš; *Nonlinear algebraic systems with discontinuous terms*, J. Math. Anal. (2012), doi:10.1016/j.jmaa.2012.09.046.

[18] G. Molica Bisci and D. Repovš; *On some variational algebraic problems*, preprint.

[19] D. Motreanu and V. Rădulescu; *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Nonconvex Optimization and its Applications, 67, Kluwer Academic Publishers, Dordrecht, 2003.

[20] A. A. Pisano, P. Fuschi, and D. De Domenico; *A layered limit analysis of pinned-joints composite laminates: Numerical versus experimental findings*, Composites part B: Engineering, Vol. **43**, Issue 3 (2012), 940-952.

[21] B. Ricceri; *A general variational principle and some of its applications*, J. Comput. Appl. Math. **133** (2000), 401-410.

[22] J. T. Scheick; *Linear Algebra with Applications,* McGraw-Hill international editions, Mathematics & Statistics Series (1997).

[23] Y. Yang and J. Zhang; *Existence results for a nonlinear system with a parameter*, J. Math. Anal. Appl. **340** (2008), 658-668.

[24] Y. Yang and J. Zhang; *Existence and multiple solutions for a nonlinear system with a parameter*, Nonlinear Anal. **70** (2009), 2542-2548.

[25] G. Wang and S.S. Cheng; *Elementary variational approach to zero-free solutions of a non linear eigenvalue problem*, Nonlinear Anal. **69** (2008), 3030-3041.

[26] X. Wu; *A new critical point theorem for locally Lipschitz functionals with applications to differential equations*, Nonlinear Anal. **66** (2007), 624-638.

[27] G. Zhang; *Existence of non-zero solutions for a nonlinear system with a parameter*, Nonlinear Anal. **66** (6) (2007), 1410-1416.

[28] G. Zhang and L. Bai; *Existence of solutions for a nonlinear algebraic system*, Discrete Dyn. Nat. Soc. (2009), 1-28.

[29] G. Zhang and S.S. Cheng; *Existence of solutions for a nonlinear algebraic system with a parameter*, J. Math. Anal. Appl. **314** (2006), 311-319.

[30] G. Zhang and W. Feng; *On the number of positive solutions of a nonlinear algebraic system*, Linear Algebra Appl. **422** (2007), 404-421.

[31] G. Zhang, W. Zhang and S. Liu; *Multiplicity result for a discrete eigenvalue problem with discontinuous nonlinearitites*, J. Math. Anal. Appl. **328** (2007), 1068-1074.

Nicu Marcu
Department of Finance, University of Craiova, Street A.I. Cuza No. 13, 200585, Craiova, Romania
E-mail address: marcu.nicu@yahoo.com

Giovanni Molica Bisci
Dipartimento MECMAT, University of Reggio Calabria, Via Graziella, Feo di Vito, 89124 Reggio Calabria, Italy
E-mail address: gmolica@unirc.it