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STABILIZATION OF A LINEAR TIMOSHENKO SYSTEM WITH INFINITE HISTORY AND APPLICATIONS TO THE TIMOSHENKO-HEAT SYSTEMS

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Abstract. In this article, we, first, consider a vibrating system of Timoshenko type in a one-dimensional bounded domain with an infinite history acting in the equation of the rotation angle. We establish a general decay of the solution for the case of equal-speed wave propagation as well as for the nonequal-speed case. We, also, discuss the well-posedness and smoothness of solutions using the semigroup theory. Then, we give applications to the coupled Timoshenko-heat systems (under Fourier's, Cattaneo's and Green and Naghdi's theories). To establish our results, we adopt the method introduced, in [13] with some necessary modifications imposed by the nature of our problems since they do not fall directly in the abstract frame of the problem treated in [13]. Our results allow a larger class of kernels than those considered in [28, 29, 30], and in some particular cases, our decay estimates improve the results of [28, 29]. Our approach can be applied to many other systems with an infinite history.

1. Introduction

In the present work, we are concerned with the well-posedness, smoothness and asymptotic behavior of the solution of the Timoshenko system

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) + \int_0^\infty g(s)\psi_{xx}(t-s)ds = 0,\\
\varphi(0,t)=\psi(0,t)=\varphi(L,t)=\psi(L,t)=0,\\
\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x),\\
\psi(x,-t)=\psi_0(x,t), \quad \psi_t(x,0)=\psi_1(x),
\end{gathered}
\tag{1.1}
$$

where $(x,t)\in]0,L[\times\mathbb{R}_+$, $\mathbb{R}_+=[0,+\infty[$, $g:\mathbb{R}_+\to\mathbb{R}_+$ is a given function (which will be specified later on), L, ρ_i, k_i $(i=1,2)$ are positive constants, φ_0, φ_1, ψ_0 and ψ_1 are given initial data, and (φ,ψ) is the state of (1.1). The infinite integral in (1.1) represents the infinite history. The derivative of a generic function f with respect to a variable y is denoted by f_y or $\partial_y f$. When f has only one variable y,

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the derivative of f is noted by f'. To simplify the notation, we omit, in general, the space and time variables (or we note only the time variable when it is necessary).

In 1921, Timoshenko [41] introduced the system (1.1) with $g = 0$ to describe the transverse vibration of a thick beam, where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, φ is the transverse displacement of the beam, and $-\psi$ is the rotation angle of the filament of the beam. The positive constants ρ_1, ρ_2, k_1 and k_2 denote, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, the shear modulus and Young's modulus of elasticity times the moment of inertia of a cross section.

During the last few years, an important amount of research has been devoted to the issue of the stabilization of the Timoshenko system and search for the minimum dissipation by which the solutions decay uniformly to the stable state as time goes to infinity. To achieve this goal, diverse types of dissipative mechanisms have been introduced and several stability results have been obtained. Let us mention some of these results (for further results, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

In the presence of controls on both the rotation angle and the transverse displacement, studies show that the Timoshenko system is stable for any weak solution and without any restriction on the constants ρ_1, ρ_2, k_1 and k_2. Many decay estimates were obtained in this case; see for example [17, 24, 38, 42, 43, 44].

In the case of only one control on the rotation angle, the rate of decay depends heavily on the constants ρ_1, ρ_2, k_1 and k_2. Precisely, if

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \tag{1.2}$$

holds (that is, the speeds of wave propagation are equal), the results show that we obtain similar decay rates as in the presence of two controls. We quote in this regard [1, 3, 14, 15, 25, 26, 29, 33, 34, 35, 40]. However, if (1.2) does not hold, a situation which is more interesting from the physics point of view, then it has been shown that the Timoshenko system is not exponentially stable even for exponentially decaying relaxation functions. Whereas, some polynomial decay estimates can be obtained for the strong solution in the presence of dissipation. This has been demonstrated in [1] for the case of an internal feedback, and in [29, 30] for the case of an infinite history.

For Timoshenko system coupled with the heat equation, we mention the pioneer work of Muñoz and Racke [32], where they considered the system

$$\begin{gathered}
\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, \quad \text{in }]0, L[\times\mathbb{R}_+, \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, \quad \text{in }]0, L[\times\mathbb{R}_+, \\
\rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, \quad \text{in }]0, L[\times\mathbb{R}_+.
\end{gathered}$$

Under appropriate conditions on σ, ρ_i, b, k and γ they established well posedness and exponential decay results for the linearized system with several boundary conditions. They also proved a non exponential stability result for the case of different wave speeds. In addition, the nonlinear case was discussed and an exponential decay was established. These results were later pushed by Messaoudi et al. [23] to the situation, where the heat propagation is given by Cattaneo's law, and by Messaoudi and Said-Houari [28] to the situation, where the heat propagation is given by Green and Naghdi's theory [9, 10, 11].

The main problem concerning the stability in the presence of infinite history is determining the largest class of kernels g which guarantee the stability and the best relation between the decay rates of g and the solutions of the considered system.

When g satisfies

$$\exists\delta_1,\,\delta_2>0:-\delta_1 g(s)\le g'(s)\le-\delta_2 g(s),\quad \forall s\in\mathbb{R}_+, \tag{1.3}$$

Muñoz and Fernández Sare [30] proved that (1.1) is exponentially stable if and only if (1.2) holds, and it is polynomially stable in general. In addition, the decay rate depends on the smoothness of the initial data. When g satisfies

$$\exists\delta>0,\,\exists p\in[1,\frac{3}{2}[:g'(s)\le-\delta\, g^p(s),\quad \forall s\in\mathbb{R}_+, \tag{1.4}$$

it was proved in [20] that (1.1) is exponentially stable when $p=1$ and (1.2) holds, and it is polynomially stable otherwise, where the decay rate is better in the case (1.2) than in that of opposite case. No relationship between the decay rate and the smoothness of the initial data was given in [20]. Similar results were proved for (6.1) (see Section 6) and (7.1) (see Section 7), respectively, in [8] under (1.3) and [28] under (1.4). Recently Ma et al. [20] proved the exponential stability of (7.1) under (1.2) and (1.3) using the semigroup method. On the other hand, Fernández Sare and Racke [8] proved that (6.4) (see Section 6) is not exponentially stable even if (1.2) holds and g satisfies (1.3).

The infinite history was also used to stabilize the semigroup associated to a general abstract linear equation in [5, 13, 31, 36] (see also the references therein for more details on the existing results in this direction). In [31], some decay estimates were proved depending on the considered operators provided that g satisfies (1.3), while in [36], it was proved that the exponential stability still holds even if g has horizontal inflection points or even flat zones provided that g is equal to a negative exponential except on a sufficiently small set where g is flat. In [13], the weak stability was proved for the (much) larger class of g satisfying (H2) below. The author of [5] proved that the exponential stability does not hold if the following condition is not satisfied:

$$\exists\delta_1\ge1,\,\exists\delta_2>0:g(t+s)\le\delta_1 e^{-\delta_2 t}g(s),\quad \forall t\in\mathbb{R}_+,\text{ for a.e. } s\in\mathbb{R}_+. \tag{1.5}$$

The stability of Timoshenko systems with a finite history (that is the infinite integral $\int_0^{+\infty}$ in (1.1) is replaced with the finite one $\int_0^t$) have attracted a considerable attention in the recent years and many authors have proved different decay estimates depending on the relation (1.2) and the growth of the kernel g at infinity (see for example [11] and the references therein for more details). Using an approach introduced in [21] for a viscoelastic equation, a general estimate of stability of (1.1) with finite history and under (1.2) was obtained in [15] for kernels satisfying

$$g'(s)\le-\xi(s)g(s),\quad \forall s\in\mathbb{R}_+, \tag{1.6}$$

where ξ is a positive and non-increasing function. The decay result in [15] improves earlier ones in the literature in which only the exponential and polynomial decay rates are obtained (see [15]). The case where (1.2) does not hold was studied in [16] for kernels satisfying

$$g'(s)\le-\xi(s)g^p(s),\quad \forall s\in\mathbb{R}_+, \tag{1.7}$$

where ξ is a positive and non-increasing function and $p \geq 1$.

Concerning the stability of abstract equations with a finite history, we mention the results in [2] (see the references therein for more results), where a general and sufficient condition under which the energy converges to zero at least as fast as the kernel at infinity was given by assuming the following condition:

$$g'(s) \leq -H(g(s)), \quad \forall s \in \mathbb{R}_+, \tag{1.8}$$

where H is a non-negative function satisfying some hypotheses. Recently, the asymptotic stability of Timoshenko system with a finite history was considered in [27] under (1.8) with weaker conditions on H than those imposed in [2]. The general relation between the decay rate for the energy and that of g obtained in [27] holds without imposing restrictive assumptions on the behavior of g at infinity.

Condition (1.4) implies that g converges to zero at infinity faster than t^{-2}. For Timoshenko system with an infinite history, (1.4) is, to our best knowledge, the weakest condition considered in the literature [28, 29] on the growth of g at infinity.

Our aim in this work is to establish a general decay estimate for the solutions of systems (1.1) in the case (1.2) as well as in the opposite one, and give applications to coupled Timoshenko-heat systems (6.1)-(6.4) (see Section 6) and Timoshenko-thermoelasticity systems of type III (7.1)-(7.2) (see Section 7). We prove that the stability of these systems holds for kernels g having more general decay (which can be arbitrary close to t^{-1}), and we obtain general decay results from which the exponential and polynomial decay results of [8, 28, 29, 30] are only special cases. In addition, we improve the results of [28, 29] by getting, in some particular cases, a better decay rate of solutions. The proof is based on the multipliers method and a new approach introduced by the first author in [13] for a class of abstract hyperbolic systems with an infinite history.

The paper is organized as follows. In Section 2, we state some hypotheses and present our stability results for (1.1). The proofs of these stability results for (1.1) will be given in Sections 3 when (1.2) holds, and in Section 4 when (1.2) does not hold. In Section 5, we discuss the well-posedness and smoothness of the solution of (1.1). Our stability results of (6.1) – (6.4) and (7.1) – (7.2) will be given and proved in Sections 6 and 7, respectively. Finally, we conclude our paper by giving some general comments in Section 8.

2. Preliminaries

In this section, we state our stability results for problem (1.1). For this purpose, we start with the following hypotheses:

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function such that $g(0) > 0$ and

$$l = k_2 - \int_0^{+\infty} g(s)ds > 0. \tag{2.1}$$

(H2) There exists an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty$$

such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{2.2}$$

Remark 2.1. Hypothesis (H2), which was introduced by the first author in [13], is weaker than the classical one (1.4) considered in [28, 29]. Indeed, (1.4) implies (H2) with $G(t) = t^{2p}$ (because (1.4) implies that $\int_0^{+\infty} \sqrt{g(t)}dt < +\infty$; see [29]). On the other hand, for example, for $g(t) = q_0(1+t)^{-q}$ with $q_0 > 0$ and $q \in]1,2]$, (H2) is satisfied with $G(t) = t^r$ for all $r > \frac{q+1}{q-1}$, but (1.4) is not satisfied.

In general, any positive function g of class $C^1(\mathbb{R}_+)$ with $g' < 0$ satisfies (H2) if it is integrable on $\mathbb{R}_+$. However, it does not satisfy (1.4) if it does not converge to zero at infinity faster than t^{-2}. Because the integrability of g on $\mathbb{R}_+$ is necessary for the well-posedness of (1.1), then (H2) seems to be very realistic. In addition, (H2) allows us to improve the results of [28, 29] by getting, in some particular cases, stronger decay rates (see Examples 2.6–8.5 below).

We consider, as in [28, 29], the classical energy functional associated with (1.1) as follows:

$$E(t) = \frac{1}{2}\int_0^L \Big(\rho_1\varphi_t^2+\rho_2\psi_t^2+k_1(\varphi_x+\psi)^2+\Big(k_2-\int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big)dx+\frac{1}{2}g\circ\psi_x, \tag{2.3}$$

where, for $v : \mathbb{R} \to L^2(]0,L[)$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\phi \circ v = \int_0^L \int_0^{+\infty} \phi(s)(v(t) - v(t-s))^2 \, ds \, dx. \tag{2.4}$$

Thanks to (2.1), the expression $\int_0^L \Big(k_1(\varphi_x+\psi)^2 + \Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big)dx$ defines a norm on $\big(H_0^1(]0,L[)\big)^2$, for (φ,ψ), equivalent to the one induced by $\big(H^1(]0,L[)\big)^2$, where $H_0^1(]0,L[) = \{v \in H^1(]0,L[),\ v(0) = v(L) = 0\}$.

Here, we define the energy space $\mathcal{H}$ (for more details see Section 5) by:

$$\mathcal{H} := \big(H_0^1(]0,L[)\big)^2 \times \big(L^2(]0,L[)\big)^2 \times L_g$$

with

$$L_g = \{v : \mathbb{R}_+ \to H_0^1(]0,L[),\ \int_0^L \int_0^{+\infty} g(s)v_x^2(s)\, ds\, dx < +\infty\}.$$

Now, we give our first main stability result, which concerns the case (1.2).

Theorem 2.2. *Assume that* (1.2), (H1) *and* (H2) *are satisfied, and let* $U_0 \in \mathcal{H}$ *(see Section 5) such that*

$$\exists M_0 \ge 0 : \|\eta_{0x}(s)\|_{L^2(]0,L[)} \le M_0, \quad \forall s > 0. \tag{2.5}$$

Then there exist positive constants c', c'' *and* ϵ_0 *(depending continuously on* $E(0)$*) for which* E *satisfies*

$$E(t) \le c'' G_1^{-1}(c't), \quad \forall t \in \mathbb{R}_+, \tag{2.6}$$

where

$$G_1(s) = \int_s^1 \frac{1}{\tau G'(\epsilon_0\tau)} d\tau \ (s \in]0,1]). \tag{2.7}$$

Remark 2.3. 1. Because $\lim_{t\to 0^+} G_1(t) = +\infty$, we have the strong stability of (1.1); that is,

$$\lim_{t\to+\infty} E(t) = 0. \tag{2.8}$$

2. The decay rate given by (2.6) is weaker than the exponential decay

$$E(t) \le c'' e^{-c't}, \quad \forall t \in \mathbb{R}_+. \tag{2.9}$$

The estimate (2.6) coincides with (2.9) when $G = Id$.

Now, we treat the case when (1.2) does not hold.

Theorem 2.4. *Assume that* (H1) *and* (H2) *are satisfied, and let* $U_0 \in D(A)$ *(see Section 5) such that*

$$\exists M_0 \geq 0 : \max\{\|\eta_{0x}(s)\|_{L^2(]0,L[)}, \|\partial_s \eta_{0x}(s)\|_{L^2(]0,L[)}\} \leq M_0, \quad \forall s > 0. \tag{2.10}$$

Then there exist positive constants C *and* ϵ_0 *(depending continuously on* $\|U_0\|_{D(A)}$*) such that*

$$E(t) \leq G_0^{-1}(\frac{C}{t}), \quad \forall t > 0, \tag{2.11}$$

where

$$G_0(s) = sG'(\epsilon_0 s) \ (s \in \mathbb{R}_+). \tag{2.12}$$

Remark 2.5. Estimate (2.11) implies (2.8) but it is weaker than

$$E(t) \leq \frac{C}{t}, \quad \forall t > 0, \tag{2.13}$$

which, in turn, coincides with (2.11) when $G = Id$. When g satisfies the classical condition (1.4) with $p = 1$ (that is g converges exponentially to zero at infinity), it is well known (see [30]) that (2.9) and (2.13) are satisfied (without the restrictions (2.5) and (2.10)).

Example 2.6. Let us give two examples to illustrate our general decay estimates and show how they generalize and improve the ones known in the literature. For other examples, see [13].

Let $g(t) = \frac{d}{(2+t)(\ln(2+t))^q}$ for $q > 1$, and $d > 0$ small enough so that (2.1) is satisfied. The classical condition (1.4) is not satisfied, while (H2) holds with

$$G(t) = \int_0^t s^{1/p} e^{-s^{-1/p}} ds \quad \text{for any } p \in]0, q-1[.$$

Indeed, here (2.2) depends only on the growth of G near zero. Using the fact that $G(t) \leq t^{\frac{1}{p}+1} e^{-t^{-1/p}}$, we can see that $G(t(\ln t)^r g(t)) \leq -g'(t)$, for t near infinity and for any $r \in]1, q-p[$, which implies (2.2). Then (2.6) takes the form

$$E(t) \leq \frac{C}{(\ln(t+2))^p}, \quad \forall t \in \mathbb{R}_+, \ \forall p \in]0, q-1[. \tag{2.14}$$

Because $G_0(s) \geq e^{-cs^{-1/p}}$, for some positive constant c and for s near zero, then also (2.11) implies (2.14).

2. Let $g(t) = de^{-(\ln(2+t))^q}$, for $q > 1$, and $d > 0$ small enough so that (2.1) is satisfied. Hypothesis (H2) holds with

$$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{p}} e^{-(-\ln s)^{1/p}} ds \quad \text{for } t \text{ near zero and for any } p \in]1, q[,$$

since condition (2.2) depends only on the growth of G at zero, and when t goes to infinity and $p \in]1, q[$, $G(t^r g(t)) \leq -g'(t)$, for any $r > 1$. Then (2.6) becomes

$$E(t) \leq ce^{-C(\ln(1+t))^p}, \quad \forall t \in \mathbb{R}_+, \ \forall p \in]1, q[. \tag{2.15}$$

Condition (2.3) holds also with $G(s) = s^p$, for any $p > 1$. Then (2.11) gives

$$E(t) \leq \frac{C}{(t+1)^{\frac{1}{p}}}, \quad \forall t \in \mathbb{R}_+, \ \forall p > 1. \tag{2.16}$$

Here, the decay rates of E in (2.15) and (2.16) are arbitrary close to the one of g and t^{-1}, respectively. This improves the results of [28, 29] in case (1.2), where only the polynomial decay was obtained.

3. Proof of Theorem 2.2

We will use c (sometimes c_τ, which depends on some parameter τ), throughout this paper, to denote a generic positive constant, which depends continuously on the initial data and can be different from step to step.

Following the classical energy method (see [1, 3, 8, 15, 28, 29, 30, 31] for example), we will construct a Lyapunov function F, equivalent to E and satisfying (3.23) below. For this purpose we establish several lemmas, for all $U_0 \in D(A)$ satisfying (2.5), so all the calculations are justified. By a simple density argument ($D(A)$ is dense in $\mathcal{H}$; see Section 5), (2.6) remains valid, for any $U_0 \in \mathcal{H}$ satisfying (2.5). On the other hand, if $E(t_0) = 0$, for some $t_0 \in \mathbb{R}_+$, then $E(t) = 0$, for all $t \geq t_0$ (E is non-increasing thanks to (3.1) below) and thus the stability estimates (2.6) and (2.11) are satisfied. Therefore, without loss of generality, we assume that $E(t) > 0$, for all $t \in \mathbb{R}_+$.

To obtain estimate (3.18) below, we prove Lemmas 3.1-3.10, where the proofs are inspired from the classical multipliers method used in [1, 3, 7, 8, 15, 18, 19, 20, 21, 28, 29, 30, 31]. Our main contribution in this section is the use of the new approach of [13] to prove (3.19) below under assumption (H2).

Lemma 3.1. *The energy functional E defined by* (2.3) *satisfies*

$$E'(t) = \frac{1}{2} g' \circ \psi_x \leq 0. \tag{3.1}$$

Proof. By multiplying the first two equations in (1.1), respectively, by φ_t and ψ_t, integrating over $]0, L[$, and using the boundary conditions, we obtain (3.1) (note that g is non-increasing). The estimate (3.1) shows that (1.1) is dissipative, where the entire dissipation is given by the infinite history. $\square$

Lemma 3.2 ([14])**.** *The following inequalities hold, where $g_0 = \int_0^{+\infty} g(s)ds$:*

$$\Big(\int_0^{+\infty} g(s)(\psi_x(t) - \psi_x(t-s))ds\Big)^2 \leq g_0 \int_0^{+\infty} g(s)(\psi_x(t) - \psi_x(t-s))^2 ds, \tag{3.2}$$

$$\Big(\int_0^{+\infty} g'(s)(\psi_x(t) - \psi_x(t-s))ds\Big)^2 \leq -g(0) \int_0^{+\infty} g'(s)(\psi_x(t) - \psi_x(t-s))^2 ds. \tag{3.3}$$

As in [15, 29], we consider the following case.

Lemma 3.3 ([26, 29])**.** *The functional*

$$I_1(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s))\,ds\,dx \tag{3.4}$$

satisfies, for any $\delta > 0$,

$$\begin{aligned} I_1'(t) \leq & -\rho_2\Big(\int_0^{+\infty} g(s)ds - \delta\Big) \int_0^L \psi_t^2 dx \\ & + \delta \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2)dx + c_\delta g \circ \psi_x - c_\delta g' \circ \psi_x. \end{aligned} \tag{3.5}$$

As in [29, 30], we consider the following result.

Lemma 3.4 ([3, 26, 29]). *The functional*

$$I_2(t) = -\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)dx$$

satisfies

$$I_2'(t) \le -\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2)dx + \int_0^L (k_1(\varphi_x+\psi)^2 + c\psi_x^2)dx + cg\circ\psi_x. \tag{3.6}$$

Similarly to [3], we consider the following result.

Lemma 3.5. *The functional*

$$I_3(t) = \rho_2\int_0^L \psi_t(\varphi_x+\psi)dx + \frac{k_2\rho_1}{k_1}\int_0^L \psi_x\varphi_t dx - \frac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^{+\infty} g(s)\psi_x(t-s)\,ds\,dx$$

satisfies, for any $\epsilon > 0$,

$$\begin{aligned} I_3'(t) \le{}& \frac{1}{2\epsilon}\Big(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\Big)^2 \\ &+ \frac{1}{2\epsilon}\Big(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\Big)^2 \\ &+ \frac{\epsilon}{2}(\varphi_x^2(L,t)+\varphi_x^2(0,t)) - k_1\int_0^L(\varphi_x+\psi)^2dx + \rho_2\int_0^L \psi_t^2dx \\ &+ \epsilon\int_0^L \varphi_t^2dx - c_\epsilon g'\circ\psi_x + (\frac{k_2\rho_1}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx. \end{aligned} \tag{3.7}$$

Proof. Using the equations in (1.1) and arguing as before, we have

$$\begin{aligned} I_3'(t) ={}& \rho_2\int_0^L (\varphi_{xt}+\psi_t)\psi_t dx + \frac{k_2\rho_1}{k_1}\int_0^L \psi_{xt}\varphi_t dx \\ &+ \int_0^L (\varphi_x+\psi)\Big(k_2\psi_{xx} - \int_0^{+\infty} g(s)\psi_{xx}(t-s)ds - k_1(\varphi_x+\psi)\Big)dx \\ &+ k_2\int_0^L \psi_x(\varphi_x+\psi)_x dx - \int_0^L (\varphi_x+\psi)_x\Big(\int_0^{+\infty} g(s)\psi_x(t-s)ds\Big)dx \\ &- \frac{\rho_1}{k_1}\int_0^L \varphi_t\Big(g(0)\psi_x + \int_0^{+\infty} g'(s)\psi_x(t-s)ds\Big)dx \\ ={}& -k_1\int_0^L (\varphi_x+\psi)^2dx + \rho_2\int_0^L \psi_t^2dx + (\frac{k_2\rho_1}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx \\ &+ \Big[\Big(k_2\psi_x - \int_0^{+\infty} g(s)\psi_x(t-s)ds\Big)(\varphi_x+\psi)\Big]_{x=0}^{x=L} \\ &+ \frac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^{+\infty} g'(s)(\psi_x(t)-\psi_x(t-s))\,ds\,dx. \end{aligned}$$

By using (3.3) and Young's inequality (for the last three terms of this equality), (3.7) is established. ∎

To estimate the boundary terms in (3.7), we proceed as in [3].

Lemma 3.6 ([3]). *Let $m(x) = 2 - \frac{4}{L}x$. Then, for any $\epsilon > 0$, the functionals*

$$I_4 = \rho_2 \int_0^L m(x)\psi_t\Big(k_2\psi_x - \int_0^{+\infty} g(s)\psi_x(t-s)ds\Big)dx,$$

$$I_5 = \rho_1 \int_0^L m(x)\varphi_t\varphi_x dx$$

satisfy

$$\begin{aligned} I_4'(t) \le & -\Big(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\Big)^2 \\ & - \Big(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\Big)^2 + \epsilon k_1 \int_0^L (\varphi_x+\psi)^2 dx \\ & + c(1+\frac{1}{\varepsilon})\int_0^L \psi_x^2 dx + c_\epsilon g\circ\psi_x + c\int_0^L \psi_t^2 dx - cg'\circ\psi_x \end{aligned} \tag{3.8}$$

and

$$I_5'(t) \le -k_1(\varphi_x^2(L,t) + \varphi_x^2(0,t)) + c\int_0^L (\varphi_t^2 + \varphi_x^2 + \psi_x^2)dx. \tag{3.9}$$

Lemma 3.7. *For any $\epsilon \in]0,1[$, the functional*

$$I_6(t) = I_3(t) + \frac{1}{2\epsilon}I_4(t) + \frac{\epsilon}{2k_1}I_5(t)$$

satisfies

$$\begin{aligned} I_6'(t) \le & -(\frac{k_1}{2} - c\epsilon)\int_0^L (\varphi_x+\psi)^2 dx + c\epsilon\int_0^L \varphi_t^2 dx + \frac{c}{\epsilon}\int_0^L \psi_t^2 dx \\ & + \frac{c}{\epsilon^2}\int_0^L \psi_x^2 dx + c_\epsilon(g\circ\psi_x - g'\circ\psi_x) + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx. \end{aligned} \tag{3.10}$$

Proof. By using Poincaré's inequality for ψ, we have

$$\int_0^L \varphi_x^2 dx \le 2\int_0^L (\varphi_x+\psi)^2 dx + 2\int_0^L \psi^2 dx.$$

Then (3.7)-(3.9) imply (3.10). □

Lemma 3.8. *The functional $I_7(t) = I_6(t) + \frac{1}{8}I_2(t)$ satisfies*

$$\begin{aligned} I_7'(t) \le & -\frac{k_1}{4}\int_0^L (\varphi_x+\psi)^2 dx - \frac{\rho_1}{16}\int_0^L \varphi_t^2 dx + c\int_0^L (\psi_t^2 + \psi_x^2)dx \\ & + c(g\circ\psi_x - g'\circ\psi_x) + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx. \end{aligned} \tag{3.11}$$

Proof. Inequalities (3.10) (with $\epsilon \in]0,1[$ small enough) and (3.6) imply (3.11). □

Now, as in [3], we use a function w to get a crucial estimate.

Lemma 3.9. *The function*

$$w(x,t) = -\int_0^x \psi(y,t)dy + \frac{1}{L}\Big(\int_0^L \psi(y,t)dy\Big)x \tag{3.12}$$

satisfies the estimates

$$\int_0^L w_x^2 dx \le c \int_0^L \psi^2 dx, \quad \forall t \ge 0, \tag{3.13}$$

$$\int_0^L w_t^2 dx \le c \int_0^L \psi_t^2 dx, \quad \forall t \ge 0. \tag{3.14}$$

Proof. We just have to calculate w_x and use Hölder's inequality to get (3.13). Applying (3.13) to w_t, we obtain

$$\int_0^L w_{xt}^2 dx \le c \int_0^L \psi_t^2 dx, \quad \forall t \ge 0.$$

Then, using Poincaré's inequality for w_t (note that $w_t(0,t) = w_t(L,t) = 0$), we arrive at (3.14). $\square$

Lemma 3.10 ([3, 29]). *For any $\epsilon \in]0,1[$, the functional*

$$I_8(t) = \int_0^L (\rho_2 \psi \psi_t + \rho_1 w \varphi_t) dx$$

satisfies

$$I_8'(t) \le -\frac{l}{2} \int_0^L \psi_x^2 dx + \frac{c}{\epsilon} \int_0^L \psi_t^2 dx + \epsilon \int_0^L \varphi_t^2 dx + c g \circ \psi_x, \tag{3.15}$$

where l is defined by (2.1).

Now, for $N_1, N_2, N_3 > 0$, let

$$I_9(t) = N_1 E(t) + N_2 I_1(t) + N_3 I_8(t) + I_7(t). \tag{3.16}$$

By combining (3.1), (3.5), (3.11) and (3.15), taking $\delta = \frac{k_1}{8N_2}$ in (3.5) and noting that $g_0 = \int_0^{+\infty} g(s) ds < +\infty$ (thanks to (H1)), we obtain

$$\begin{aligned} I_9'(t) \le& -(\frac{lN_3}{2} - c) \int_0^L \psi_x^2 dx - (\frac{\rho_1}{16} - \epsilon N_3) \int_0^L \varphi_t^2 dx \\ &- \int_0^L \Big(N_2 \rho_2 g_0 - \frac{cN_3}{\epsilon} - c \Big) \psi_t^2 dx - \frac{k_1}{8} \int_0^L (\varphi_x + \psi)^2 dx \\ &+ c_{N_2,N_3} g \circ \psi_x + (\frac{N_1}{2} - c_{N_2}) g' \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx. \end{aligned} \tag{3.17}$$

At this point, we choose N_3 large enough so that $\frac{lN_3}{2} - c > 0$, then $\epsilon \in]0,1[$ small enough so that $\frac{\rho_1}{16} - \epsilon N_3 > 0$. Next, we pick N_2 large enough so that $N_2 \rho_2 g_0 - \frac{cN_3}{\epsilon} - c > 0$.

On the other hand, by definition of the functionals $I_1 - I_8$ and E, there exists a positive constant β satisfying $|N_2 I_1 + N_3 I_8 + I_7| \le \beta E$, which implies that

$$(N_1 - \beta) E \le I_9 \le (N_1 + \beta) E,$$

then we choose N_1 large enough so that $\frac{N_1}{2} - c_{N_2} \ge 0$ and $N_1 > \beta$ (that is $I_9 \sim E$). Consequently, using the definition (2.3) of E, from (3.17) we obtain

$$I_9'(t) \le -cE(t) + c g \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx. \tag{3.18}$$

Now, we estimate the term $g \circ \psi_x$ in (3.18) in function of E' by exploiting (H2). This is the main difficulty in treating the infinite history term.

Lemma 3.11. *For any $\epsilon_0 > 0$, the following inequality holds:*
$$G'(\epsilon_0 E(t)) g \circ \psi_x \leq -cE'(t) + c\epsilon_0 E(t) G'(\epsilon_0 E(t)). \tag{3.19}$$

Proof. This lemma was proved by the first author (see [13, Lemma 3.4]) for an abstract system with infinite history. The proof is based on some classical properties of convex functions (see [4, 7] for example), in particular, the general Young inequality. Let us give a short proof of (3.19) in the particular case (1.1) (see [13] for details).

Because E is non-increasing, then (η_0 is defined in Section 5)
$$\begin{aligned}\int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx &\leq 4 \sup_{\tau \in \mathbb{R}} \int_0^L \psi_x^2(\tau) dx \\ &\leq 4 \sup_{\tau > 0} \int_0^L \psi_{0x}^2(\tau) dx + cE(0) \\ &\leq c \sup_{\tau > 0} \int_0^L \eta_{0x}^2(\tau) dx + cE(0).\end{aligned}$$
Thus, thanks to (2.5), there exists a positive constant $m_1 = c(M_0^2 + E(0))$ (where M_0 is defined in (2.5)) such that
$$\int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx \leq m_1, \quad \forall t,\, s \in \mathbb{R}_+.$$
Let $\epsilon_0, \tau_1, \tau_2 > 0$ and $K(s) = s/G^{-1}(s)$ which is non-decreasing. Then,
$$K\Big(-\tau_2 g'(s) \int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx\Big) \leq K(-m_1 \tau_2 g'(s)).$$
Using this inequality, we arrive at
$$\begin{aligned} g \circ \psi_x &= \frac{1}{\tau_1 G'(\epsilon_0 E(t))} \int_0^{+\infty} G^{-1}\Big(-\tau_2 g'(s) \int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx\Big) \\ &\quad \times \frac{\tau_1 G'(\epsilon_0 E(t)) g(s)}{-\tau_2 g'(s)} K\Big(-\tau_2 g'(s) \int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx\Big) ds \\ &\leq \frac{1}{\tau_1 G'(\epsilon_0 E(t))} \int_0^{+\infty} G^{-1}\Big(-\tau_2 g'(s) \int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx\Big) \\ &\quad \times \frac{\tau_1 G'(\epsilon_0 E(t)) g(s)}{-\tau_2 g'(s)} K(-m_1 \tau_2 g'(s)) ds \\ &\leq \frac{1}{\tau_1 G'(\epsilon_0 E(t))} \int_0^{+\infty} G^{-1}\Big(-\tau_2 g'(s) \int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx\Big) \\ &\quad \times \frac{m_1 \tau_1 G'(\epsilon_0 E(t)) g(s)}{G^{-1}(-m_1 \tau_2 g'(s))} ds. \end{aligned}$$
We denote by G^* the dual function of G defined by
$$G^*(t) = \sup_{s \in \mathbb{R}_+} \{ts - G(s)\} = t{G'}^{-1}(t) - G({G'}^{-1}(t)), \quad \forall t \in \mathbb{R}_+.$$
Using Young's inequality: $t_1 t_2 \leq G(t_1) + G^*(t_2)$, for
$$t_1 = G^{-1}\Big(-\tau_2 g'(s) \int_0^L (\psi_x(t) - \psi_x(t-s))^2 dx\Big), \quad t_2 = \frac{m_1 \tau_1 G'(\epsilon_0 E(t)) g(s)}{G^{-1}(-m_1 \tau_2 g'(s))},$$

we obtain

$$g\circ\psi_x \le \frac{-\tau_2}{\tau_1 G'(\epsilon_0 E(t))} g'\circ\psi_x + \frac{1}{\tau_1 G'(\epsilon_0 E(t))}\int_0^{+\infty} G^*\Big(\frac{m_1\tau_1 G'(\epsilon_0 E(t))g(s)}{G^{-1}(-m_1\tau_2 g'(s))}\Big)ds.$$

Using (3.1) and the fact that $G^*(t)\le tG'^{-1}(t)$, we obtain

$$\begin{aligned} g\circ\psi_x &\le \frac{-2\tau_2}{\tau_1 G'(\epsilon_0 E(t))}E'(t)\\ &\quad + m_1\int_0^{+\infty}\frac{g(s)}{G^{-1}(-m_1\tau_2 g'(s))}G'^{-1}\Big(\frac{m_1\tau_1 G'(\epsilon_0 E(t))g(s)}{G^{-1}(-m_1\tau_2 g'(s))}\Big)ds. \end{aligned}$$

Thanks to (2.2), $\sup_{s\in\mathbb{R}_+}\frac{g(s)}{G^{-1}(-g'(s))}=m_2<+\infty$. Then, using the fact that G'^{-1} is non-decreasing (thanks to (H2)) and choosing $\tau_2=1/m_1$, we obtain

$$g\circ\psi_x \le \frac{-2}{m_1\tau_1 G'(\epsilon_0 E(t))}E'(t) + m_1 G'^{-1}\Big(m_1 m_2\tau_1 G'(\epsilon_0 E(t))\Big)\int_0^{\infty}\frac{g(s)}{G^{-1}(-g'(s))}ds.$$

Now, choosing $\tau_1=\frac{1}{m_1m_2}$ and using the fact that $\int_0^{+\infty}\frac{g(s)}{G^{-1}(-g'(s))}ds=m_3<+\infty$ (thanks to (2.2)), we obtain

$$g\circ\psi_x \le \frac{-c}{G'(\epsilon_0 E(t))}E'(t)+c\epsilon_0 E(t),$$

which implies (3.19) with $c=\max\{2m_2, m_1m_3\}$. $\square$

Now, going back to the proof of Theorem 2.2, multiplying (3.18) by $G'(\epsilon_0 E(t))$ and using (3.19), we obtain

$$\begin{aligned} G'(\epsilon_0 E(t))I_9'(t) &\le -(c-c\epsilon_0)E(t)G'(\epsilon_0 E(t)) - cE'(t)\\ &\quad + (\frac{\rho_1 k_2}{k_1}-\rho_2)G'(\epsilon_0 E(t))\int_0^L \varphi_t\psi_{xt}dx. \end{aligned}$$

Choosing ϵ_0 small enough, we obtain

$$\begin{aligned} G'(\epsilon_0 E(t))I_9'(t)+cE'(t) &\le -cE(t)G'(\epsilon_0 E(t))\\ &\quad + (\frac{\rho_1 k_2}{k_1}-\rho_2)G'(\epsilon_0 E(t))\int_0^L \varphi_t\psi_{xt}dx. \end{aligned} \tag{3.20}$$

Let

$$F=\tau\Big(G'(\epsilon_0 E)I_9+cE\Big),$$

where $\tau>0$. We have $F\sim E$ (because $I_9\sim E$ and $G'(\epsilon_0 E)$ is non-increasing) and, using (3.20),

$$F'(t)\le -c\tau E(t)G'(\epsilon_0 E(t)) + \tau(\frac{\rho_1 k_2}{k_1}-\rho_2)G'(\epsilon_0 E(t))\int_0^L \varphi_t\psi_{xt}dx. \tag{3.21}$$

Now, thanks to (1.2), the last term of (3.21) vanishes. Then, for $\tau>0$ small enough such that

$$F\le E \quad\text{and}\quad F(0)\le 1, \tag{3.22}$$

we obtain, for $c'=c\tau>0$,

$$F'\le -c'FG'(\epsilon_0 F). \tag{3.23}$$

This implies that $(G_1(F))'\ge c'$, where G_1 is defined by (2.7). Then, by integrating over $[0,t]$, we obtain

$$G_1(F(t))\ge c't+G_1(F(0)).$$

Because $F(0) \leq 1$, $G_1(1) = 0$ and G_1 is decreasing, we obtain $G_1(F(t)) \geq c't$, which implies that $F(t) \leq G_1^{-1}(c't)$. The fact that $F \sim E$ gives (2.6). This completes the proof of Theorem 2.2.

4. Proof of Theorem 2.4

In this section, we treat the case when (1.2) does not hold, which is more realistic from the physics point of view. We will estimate the last term of (3.21) using the following system resulting from differentiating (1.1) with respect to time,

$$
\begin{gathered}
\rho_1 \varphi_{ttt} - k_1(\varphi_{xt} + \psi_t)_x = 0, \\
\rho_2 \psi_{ttt} - k_2 \psi_{xxt} + k_1(\varphi_{xt} + \psi_t) + \int_0^{+\infty} g(s) \psi_{xxt}(t-s) ds = 0, \\
\varphi_t(0,t) = \psi_t(0,t) = \varphi_t(L,t) = \psi_t(L,t) = 0.
\end{gathered} \tag{4.1}
$$

System (4.1) is well posed for initial data $U_0 \in D(A)$ (see Section 5). Let $\tilde{E}$ be the second-order energy (the energy of (4.1)) defined by $\tilde{E}(t) = E(U_t(t))$, where $E(U(t)) = E(t)$ and E is defined by (2.3). A simple calculation (as in (3.1)) implies that

$$
\tilde{E}'(t) = \frac{1}{2} g' \circ \psi_{xt} \leq 0. \tag{4.2}
$$

The energy of high order is widely used in the literature to estimate some terms (see [1, 12, 30, 31] for example). Our main contribution in this section is obtaining estimate (4.6) below under (H2). Now, we proceed as in [30] to establish the following lemma.

Lemma 4.1. *For any $\epsilon > 0$, we have*

$$
\Big(\frac{\rho_1 k_2}{k_1} - \rho_2\Big) \int_0^L \varphi_t \psi_{xt} dx \leq \epsilon E(t) + c_\epsilon (g \circ \psi_{xt} - g' \circ \psi_x). \tag{4.3}
$$

Proof. By recalling that $g_0 = \int_0^{+\infty} g(s) ds$, we have

$$
\begin{aligned}
\Big(\frac{\rho_1 k_2}{k_1} - \rho_2\Big) \int_0^L \varphi_t \psi_{xt} dx &= \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{g_0} \int_0^L \varphi_t \int_0^{+\infty} g(s)(\psi_{xt}(t) - \psi_{xt}(t-s))\, ds\, dx \\
&\quad + \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{g_0} \int_0^L \varphi_t \int_0^{+\infty} g(s) \psi_{xt}(t-s)\, ds\, dx.
\end{aligned} \tag{4.4}
$$

Using Young's inequality and (3.2) (for ψ_{xt} instead of ψ_x), we obtain, for all $\epsilon > 0$,

$$
\begin{aligned}
&\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{g_0} \int_0^L \varphi_t \int_0^{+\infty} g(s)(\psi_{xt}(t) - \psi_{xt}(t-s))\, ds\, dx \\
&\leq c \int_0^L |\varphi_t| \int_0^{+\infty} g(s) |\psi_{xt}(t) - \psi_{xt}(t-s)|\, ds\, dx \\
&\leq \frac{\epsilon}{2} E(t) + c_\epsilon g \circ \psi_{xt}.
\end{aligned}
$$

On the other hand, by integrating by parts and using (3.3), we obtain

$$
\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{g_0} \int_0^L \varphi_t \int_0^{+\infty} g(s) \psi_{xt}(t-s)\, ds\, dx
$$

$$
\begin{aligned}
&= \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{g_0} \int_0^L \varphi_t \Big(g(0)\psi_x + \int_0^{+\infty} g'(s)\psi_x(t-s)ds \Big) dx \\
&= \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{g_0} \int_0^L \varphi_t \int_0^{+\infty} (-g'(s))(\psi_x(t) - \psi_x(t-s))\, ds\, dx \\
&\le \frac{\epsilon}{2} E(t) - c_\epsilon g' \circ \psi_x .
\end{aligned}
$$

Inserting these last two inequalities into (4.4), we obtain (4.3). □

Now, going back to the proof of Theorem 2.4, choosing $\tau = 1$ in (3), using (3.21) and (4.3) and choosing ϵ small enough, we obtain

$$F'(t) \le -cE(t)G'(\epsilon_0 E(t)) + cG'(\epsilon_0 E(t))(g \circ \psi_{xt} - g' \circ \psi_x),$$

which implies, using (3.1) and the fact that $G'(\epsilon_0 E)$ is non-increasing,

$$E(t)G'(\epsilon_0 E(t)) \le -cG'(\epsilon_0 E(0))E'(t) - cF'(t) + cG'(\epsilon_0 E(t)) g \circ \psi_{xt}. \tag{4.5}$$

Now, we estimate the last term in (4.5). Similarly to the case of $g \circ \psi_x$ in Lemma 3.11 (for $g \circ \psi_{xt}$ instead of $g \circ \psi_x$), we obtain, using (2.10) and (4.2),

$$G'(\epsilon_0 E(t)) g \circ \psi_{xt} \le -c\tilde{E}'(t) + c\epsilon_0 E(t) G'(\epsilon_0 E(t)), \quad \forall \epsilon_0 > 0. \tag{4.6}$$

Then (4.5) and (4.6) with ϵ_0 chosen small enough imply that

$$E(t)G'(\epsilon_0 E(t)) \le -cG'(\epsilon_0 E(0))E'(t) - cF'(t) - c\tilde{E}'(t). \tag{4.7}$$

Using the fact that $F \sim E$ and $EG'(\epsilon_0 E)$ is non-increasing, we deduce that, for all $T \in \mathbb{R}_+$ (G_0 is defined by (2.12)),

$$G_0(E(T))T \le \int_0^T G_0(E(t))dt \le c(G'(\epsilon_0 E(0)) + 1)E(0) + c\tilde{E}(0), \tag{4.8}$$

which gives (2.11) with $C = c(G'(\epsilon_0 E(0)) + 1)E(0) + c\tilde{E}(0)$. This completes the proof of Theorem 2.4.

5. Well-Posedness and Smoothness

In this section, we discuss the existence, uniqueness and smoothness of solution of (1.1) under hypothesis (H1). We use the semigroup theory and some arguments of [6] (see also [29, 30]). Following the idea of [6], let

$$\eta(x,t,s) = \psi(x,t) - \psi(x,t-s) \quad \text{for } (x,t,s) \in]0,L[\times \mathbb{R}_+ \times \mathbb{R}_+ \tag{5.1}$$

(η is the relative history of ψ, and it was introduced first in [6]). This function satisfies the initial conditions

$$\eta(0,t,s) = \eta(L,t,s) = 0, \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \eta(x,t,0) = 0, \quad \text{in }]0,L[\times \mathbb{R}_+ \tag{5.2}$$

and the equation

$$\eta_t + \eta_s - \psi_t = 0, \quad \text{in }]0,L[\times \mathbb{R}_+ \times \mathbb{R}_+. \tag{5.3}$$

Then the second equation of (1.1) can be formulated as

$$\rho_2 \psi_{tt} - k_2 \psi_{xx} + \Big(\int_0^{+\infty} g(s)ds \Big) \psi_{xx} - \int_0^{+\infty} g(s)\eta_{xx} ds + k_1(\varphi_x + \psi) = 0.$$

Let $\eta_0(x,s) = \eta(x,0,s) = \psi_0(x,0) - \psi_0(x,s)$ for $(x,s) \in]0,L[\times \mathbb{R}_+$. This means that the history is considered as an initial data for η. Let

$$\mathcal{H} = \big(H_0^1(]0,L[) \big)^2 \times \big(L^2(]0,L[) \big)^2 \times L_g \tag{5.4}$$

with
$$L_g = \{v : \mathbb{R}_+ \to H_0^1(]0,L[),\ \int_0^L \int_0^{+\infty} g(s) v_x^2(s)\,ds\,dx < +\infty\}. \tag{5.5}$$
The set L_g is a Hilbert space endowed with the inner product
$$\langle v, w\rangle_{L_g} = \int_0^L \int_0^{+\infty} g(s) v_x(s) w_x(s)\,ds\,dx. \tag{5.6}$$
Then $\mathcal{H}$ is also a Hilbert space endowed with the inner product defined, for $V = (v_1, v_2, v_3, v_4, v_5)^T$, $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathcal{H}$, by
$$\begin{aligned} \langle V, W\rangle_{\mathcal{H}} &= \int_0^L \Big(\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4\Big)dx \\ &\quad + \langle v_5, w_5\rangle_{L_g} + k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)dx. \end{aligned} \tag{5.7}$$
Now, for $U = (\varphi, \psi, \varphi_t, \psi_t, \eta)^T$ and $U_0 = (\varphi_0, \psi_0(\cdot, 0), \varphi_1, \psi_1, \eta_0)^T$, (1.1) is equivalent to the abstract linear first order Cauchy problem
$$\begin{gathered} U_t(t) + AU(t) = 0 \quad \text{on } \mathbb{R}_+, \\ U(0) = U_0, \end{gathered} \tag{5.8}$$
where A is the linear operator defined by $AV = (f_1, f_2, f_3, f_4, f_5)$, for any $V = (v_1, v_2, v_3, v_4, v_5)^T \in D(A)$, where
$$\begin{gathered} f_1 = -v_3, \quad f_2 = -v_4, \quad f_3 = -\frac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2), \\ f_4 = -\frac{1}{\rho_2}\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_{xx} v_2 - \frac{1}{\rho_2}\int_0^{+\infty} g(s)\partial_{xx} v_5(s)ds + \frac{k_1}{\rho_2}(\partial_x v_1 + v_2), \\ f_5 = -v_4 + \partial_s v_5. \end{gathered}$$
The domain $D(A)$ of A given by $D(A) = \{V \in \mathcal{H},\ AV \in \mathcal{H} \text{ and } v_5(0) = 0\}$ and endowed with the graph norm
$$\|V\|_{D(A)} = \|V\|_{\mathcal{H}} + \|AV\|_{\mathcal{H}} \tag{5.9}$$
can be characterized by
$$\begin{aligned} D(A) = \Big\{& V = (v_1, v_2, v_3, v_4, v_5)^T \in \big(H^2(]0,L[) \cap H_0^1(]0,L[)\big) \times \big(H_0^1(]0,L[)\big)^3 \times \mathcal{L}_g, \\ &\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_{xx} v_2 + \int_0^{+\infty} g(s)\partial_{xx} v_5(s)ds \in L^2(]0,L[)\Big\} \end{aligned}$$
and it is dense in $\mathcal{H}$ (see also [30, 31] and the reference therein), where
$$\mathcal{L}_g = \{v \in L_g,\ \partial_s v \in L_g,\ v(x,0) = 0\}. \tag{5.10}$$
Now, we prove that $A : D(A) \to \mathcal{H}$ is a maximal monotone operator; that is $-A$ is dissipative and $Id + A$ is surjective. Indeed, a simple calculation implies that, for any $V = (v_1, v_2, v_3, v_4, v_5)^T \in D(A)$,
$$\langle AV, V\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)(\partial_x v_5(s))^2\,ds\,dx \geq 0, \tag{5.11}$$
since g is non-increasing. This implies that $-A$ is dissipative.

On the other hand, we prove that $Id + A$ is surjective; that is, for any $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, there exists $V = (v_1, v_2, v_3, v_4, v_5)^T \in D(A)$ satisfying

$$(Id + A)V = F. \tag{5.12}$$

The first two equations of system (5.12) are equivalent to

$$v_1 = v_3 + f_1 \quad \text{and} \quad v_2 = v_4 + f_2. \tag{5.13}$$

The last equation of system (5.12) is equivalent to

$$v_5 + \partial_s v_5 = v_4 + f_5,$$

then, by integrating with respect to s and noting that $v_5(0) = 0$, we obtain

$$v_5(s) = \Big(\int_0^s (v_4 + f_5(\tau))e^{\tau} d\tau\Big)e^{-s}. \tag{5.14}$$

Now, we look for $(v_3, v_4) \in \big(H_0^1(]0, L[)\big)^2$. To simplify the formulations, we put $\mathcal{H}_1 = \big(H_0^1(]0, L[)\big)^2$ and $\mathcal{H}_2 = \big(L^2(]0, L[)\big)^2$ endowed with the inner products

$$\begin{aligned} \langle (z_1, z_2)^T, (w_1, w_2)^T\rangle_{\mathcal{H}_1} &= \int_0^L \Big(k_2 - \int_0^{+\infty} e^{-s} g(s)ds\Big)\partial_x z_2 \partial_x w_2 dx \\ &\quad + k_1 \int_0^L (\partial_x z_1 + z_2)(\partial_x w_1 + w_2)dx \end{aligned} \tag{5.15}$$

and

$$\langle (z_1, z_2)^T, (w_1, w_2)^T\rangle_{\mathcal{H}_2} = \int_0^L (\rho_1 z_1 w_1 + \rho_2 z_2 w_2)dx. \tag{5.16}$$

Thanks to (2.1) and Poincaré's inequality, $\langle, \rangle_{\mathcal{H}_1}$ defines a norm on $\mathcal{H}_1$ equivalent to the norm induced by $\Big(H^1(]0, L[)\Big)^2$. On the other hand, the inclusion $\mathcal{H}_1 \subset \mathcal{H}_2$ is dense and compact.

Now, inserting (5.13) and (5.14) into the third and the fourth equations of system (5.12), multiplying them, respectively, by $\rho_1 w_3$ and $\rho_2 w_4$, where $(w_3, w_4)^T \in \mathcal{H}_1$, and then integrating their sum over $]0, L[$, we obtain, for all $(w_3, w_4)^T \in \mathcal{H}_1$,

$$\begin{aligned} &\langle (v_3, v_4)^T, (w_3, w_4)^T\rangle_{\mathcal{H}_2} + \langle (v_3, v_4)^T, (w_3, w_4)^T\rangle_{\mathcal{H}_1} \\ &= \langle (f_3, f_4)^T, (w_3, w_4)^T\rangle_{\mathcal{H}_2} - \langle (f_1, f_2)^T, (w_3, w_4)^T\rangle_{\mathcal{H}_1} \\ &\quad + \Big(\int_0^{+\infty} (1 - e^{-s})g(s)ds\Big) \int_0^L \partial_x f_2 \partial_x w_4 dx \\ &\quad - \int_0^L \partial_x \Big(\int_0^{+\infty} e^{-s} g(s)\Big(\int_0^s f_5(\tau)e^{\tau} d\tau\Big)ds\Big)\partial_x w_4 dx. \end{aligned} \tag{5.17}$$

We have just to prove that (5.17) has a solution $(v_3, v_4)^T \in \mathcal{H}_1$, and then, using (5.13), (5.14) and regularity arguments, we find (5.12). Following the method in [18, page 95], let $\mathcal{H}_1'$ be the dual space of $\mathcal{H}_1$ and $A_1 : \mathcal{H}_1 \to \mathcal{H}_1'$ be the duality mapping. We consider the map $B_1 : \mathcal{H}_1 \to \mathcal{H}_1'$ defined by

$$\langle B_1(z_1, z_2)^T, (w_1, w_2)^T\rangle_{\mathcal{H}_1', \mathcal{H}_1} = \int_0^L \partial_x z_2 \partial_x w_2 dx.$$

We identify $\mathcal{H}_2$ with its dual space $\mathcal{H}_2'$ and we set

$$\tilde{f}_5 = \Big(\int_0^{+\infty} (1 - e^{-s})g(s)ds\Big)f_2 - \int_0^{+\infty} e^{-s} g(s)\Big(\int_0^s f_5(\tau)e^{\tau} d\tau\Big)ds.$$

We have $B_1(0,\tilde f_5)^T \in \mathcal{H}_1'$ and (5.17) becomes

$$\begin{aligned} &\langle (Id + A_1)(v_3,v_4)^T, (w_3,w_4)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1} \\ &= \langle (f_3,f_4)^T - A_1(f_1,f_2)^T + B_1(0,\tilde f_5)^T, (w_3,w_4)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}, \end{aligned} \tag{5.18}$$

for all $(w_3,w_4)^T \in \mathcal{H}_1$. Let

$$(\tilde f_1,\tilde f_2)^T = (f_3,f_4)^T - A_1(f_1,f_2)^T + B_1(0,\tilde f_5)^T.$$

Because $\mathcal{H}_2 = \mathcal{H}_2' \subset \mathcal{H}_1'$, then $(\tilde f_1,\tilde f_2)^T \in \mathcal{H}_1'$. Therefore, (5.18) is well-defined and equivalent to

$$(Id + A_1)(v_3,v_4)^T = (\tilde f_1,\tilde f_2)^T. \tag{5.19}$$

It is sufficient to show that (5.19) has a solution $(v_3,v_4)^T \in \mathcal{H}_1$. Let $\Gamma : \mathcal{H}_1 \to \mathbb{R}$ defined by

$$\Gamma((z_1,z_2)^T) = \frac{1}{2}\|(z_1,z_2)^T\|_{\mathcal{H}_2}^2 + \frac{1}{2}\|(z_1,z_2)^T\|_{\mathcal{H}_1}^2 - \langle (\tilde f_1,\tilde f_2)^T, (z_1,z_2)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}. \tag{5.20}$$

The map Γ is well-defined and differentiable such that

$$\begin{aligned} &\Gamma'((z_1,z_2)^T)(w_1,w_2)^T \\ &= \langle (Id + A_1)(z_1,z_2)^T - (\tilde f_1,\tilde f_2)^T, (w_1,w_2)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}, \end{aligned} \tag{5.21}$$

for all $(z_1,z_2)^T, (w_1,w_2)^T \in \mathcal{H}_1$.

On the other hand, using Cauchy-Schwarz inequality to minimize the last term in (5.20), we have

$$\Gamma((z_1,z_2)^T) \geq \Big(\frac{1}{2}\|(z_1,z_2)^T\|_{\mathcal{H}_1} - \|(\tilde f_1,\tilde f_2)^T\|_{\mathcal{H}_1'}\Big)\|(z_1,z_2)^T\|_{\mathcal{H}_1}. \tag{5.22}$$

This implies that Γ goes to infinity when $\|(z_1,z_2)^T\|_{\mathcal{H}_1}$ goes to infinity, and therefore, Γ reaches its minimum at some point $(v_3,v_4)^T \in \mathcal{H}_1$. This point satisfies $\Gamma'((v_3,v_4)^T) = 0$, which solves (5.19) thanks to (5.21) with the choice $(z_1,z_2)^T = (v_3,v_4)^T$.

Finally, using Lummer-Phillips theorem (see [37]), we deduce that A is an infinitesimal generator of a contraction semigroup in $\mathcal{H}$, which implies the following results of existence, uniqueness and smoothness of the solution of (1.1) (see [18, 37]).

Theorem 5.1. *Assume that* (H1) *is satisfied.*

1. *For any $U_0 \in \mathcal{H}$,* (1.1) *has a unique weak solution*

$$U \in C(\mathbb{R}_+;\mathcal{H}). \tag{5.23}$$

2. *If $U_0 \in D(A^n)$ for $n \in \mathbb{N}^*$, then the solution U has the regularity*

$$U \in \cap_{j=0}^{n} C^{n-j}(\mathbb{R}_+; D(A^j)), \tag{5.24}$$

where $D(A^j)$ is endowed with the graph norm $\|V\|_{D(A^j)} = \sum_{m=0}^{j}\|A^m V\|_{\mathcal{H}}$.

6. Timoshenko-heat: Fourier's and Cattaneo's laws

In this section, we give applications of our results of Section 2 to the case of coupled Timoshenko-heat systems on $]0,L[$ under Fourier's law of heat conduction and with an infinite history acting on the second equation:

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + k_4\theta_x + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0,\\
\rho_3\theta_t - k_3\theta_{xx} + k_4\psi_{xt} = 0,\\
\varphi(0,t) = \psi(0,t) = \theta(0,t) = \varphi(L,t) = \psi(L,t) = \theta(L,t) = 0,\\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x),\\
\psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x)
\end{gathered}
\tag{6.1}
$$

and with an infinite history acting on the first equation:

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + \int_0^{+\infty} g(s)\varphi_{xx}(x,t-s)ds = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + k_4\theta_x = 0,\\
\rho_3\theta_t - k_3\theta_{xx} + k_4\psi_{xt} = 0,\\
\varphi(0,t) = \psi(0,t) = \theta_x(0,t) = \varphi(L,t) = \psi(L,t) = \theta_x(L,t) = 0,\\
\varphi(x,-t) = \varphi_0(x,t), \quad \varphi_t(x,0) = \varphi_1(x),\\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x).
\end{gathered}
\tag{6.2}
$$

We also consider systems under Cattaneo's law and with an infinite history acting on the first equation:

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + \int_0^{+\infty} g(s)\varphi_{xx}(x,t-s)ds = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + k_4\theta_x = 0,\\
\rho_3\theta_t + k_3q_x + k_4\psi_{xt} = 0,\\
\rho_4q_t + k_5q + k_3\theta_x = 0,\\
\varphi(0,t) = \psi(0,t) = q(0,t) = \varphi(L,t) = \psi(L,t) = q(L,t) = 0,\\
\varphi(x,-t) = \varphi_0(x,t), \quad \varphi_t(x,0) = \varphi_1(x),\\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x),
\end{gathered}
\tag{6.3}
$$

and with an infinite history acting on the second equation:

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + k_4\theta_x + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds = 0,\\
\rho_3\theta_t + k_3q_x + k_4\psi_{xt} = 0,\\
\rho_4q_t + k_5q + k_3\theta_x = 0,\\
\varphi(0,t) = \psi(0,t) = q(0,t) = \varphi(L,t) = \psi(L,t) = q(L,t) = 0,\\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x),\\
\psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x),
\end{gathered}
\tag{6.4}
$$

where ρ_i and k_i are also positive constants, and θ and q denote, respectively, the temperature difference and the heat flux vector. Systems (6.3) and (6.4) (Cattaneo law), with $\rho_4 = 0$, implies, respectively, (6.2) and (6.1) (Fourier's law).

6.1. **Well-posedness.** We start our analysis by showing without details, using the semigroup theory (as for (1.1) in Section 5), how to prove that (6.1)–(6.4) are well-posed under the following hypothesis:

(H3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function satisfying $g(0) > 0$ such that (2.1) holds in case (6.1) and (6.4), and
$$\int_0^{+\infty} g(s)ds < \frac{k_1 k_2}{k_2 + k_0 k_1} \tag{6.5}$$
in case (6.2) and (6.3), where k_0 is the smallest positive constant satisfying, for all $v \in H_0^1(]0, L[)$ (Poincaré's inequality),
$$\int_0^L v^2 dx \le k_0 \int_0^L v_x^2 dx. \tag{6.6}$$

Remark 6.1. Thanks to Poincaré's inequality (applied for ψ), we have
$$k_1 \int_0^L (\varphi_x + \psi)^2 dx \ge k_1(1-\epsilon) \int_0^L \varphi_x^2 dx + k_0 k_1 (1 - \frac{1}{\epsilon}) \int_0^L \psi_x^2 dx,$$
for any $0 < \epsilon < 1$. Then, thanks to (6.5), we can choose
$$\frac{k_0 k_1}{k_2 + k_0 k_1} < \epsilon < \frac{1}{k_1}\Big(k_1 - \int_0^{+\infty} g(s)ds\Big)$$
and we obtain
$$\hat{c} \int_0^L (\varphi_x^2 + \psi_x^2) dx \le \int_0^L \Big(-\Big(\int_0^{+\infty} g(s)ds\Big)\varphi_x^2 + k_2 \psi_x^2 + k_1(\varphi_x + \psi)^2 \Big) dx,$$
where $\hat{c} = \min\{k_1(1-\epsilon) - \int_0^{+\infty} g(s)ds, k_2 + k_0 k_1 (1 - \frac{1}{\epsilon})\} > 0$. Thus,
$$\int_0^L \Big(-\Big(\int_0^{+\infty} g(s)ds\Big)\varphi_x^2 + k_2 \psi_x^2 + k_1(\varphi_x + \psi)^2 \Big) dx$$
defines a norm on $\big(H_0^1(]0, L[)\big)^2$, for (φ, ψ), equivalent to the one induced by $\big(H^1(]0, L[)\big)^2$.

Now, following the idea of [6], let, as in Section 5, η be the relative history of ψ in cases of (6.1) and (6.4), and of φ in cases of (6.2) and (6.3), defined by
$$\eta(x,t,s) = \begin{cases} \psi(x,t) - \psi(x,t-s), & \text{in cases (6.1) and (6.4)} \\ \varphi(x,t) - \varphi(x,t-s), & \text{in cases (6.2) and (6.3)}, \end{cases}$$
for $(x,t,s) \in]0, L[\times \mathbb{R}_+ \times \mathbb{R}_+$. This function satisfies (5.2) and (5.3) in case (6.1) and (6.4), and satisfies (5.2) and
$$\eta_t + \eta_s - \varphi_t = 0, \quad \text{in }]0, L[\times \mathbb{R}_+ \times \mathbb{R}_+$$
in case (6.2) and (6.3). Then the second equation of (6.1) and (6.4), and the first one of (6.2) and (6.3) can be formulated, respectively, as
$$\rho_2 \psi_{tt} - \Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\psi_{xx} - \int_0^{+\infty} g(s)\eta_{xx} ds + k_1(\varphi_x + \psi) + k_4 \theta_x = 0$$

and

$$\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + \Big(\int_0^{+\infty} g(s)ds\Big)\varphi_{xx} - \int_0^{+\infty} g(s)\eta_{xx}ds = 0.$$

Let

$$\mathcal{H} = \begin{cases} \big(H_0^1(]0,L[)\big)^2 \times \big(L^2(]0,L[)\big)^3 \times L_g, & \text{for (6.1)} \\ \big(H_0^1(]0,L[)\big)^2 \times \big(L^2(]0,L[)\big)^2 \times L_*^2(]0,L[) \times L_g, & \text{for (6.2)} \\ \big(H_0^1(]0,L[)\big)^2 \times \big(L^2(]0,L[)\big)^2 \times L_*^2(]0,L[) \times L^2(]0,L[) \times L_g, & \text{for (6.3)-(6.4),} \end{cases}$$

where $L_*^2(]0,L[) = \{v \in L^2(]0,L[),\ \int_0^L v(x)dx = 0\}$ and L_g is defined by (5.5) and endowed with the inner product (5.6). Then, thanks to (2.1) and (6.5) (see Remark 6.1), $\mathcal{H}$ is also a Hilbert space endowed with the inner product defined, for $V, W \in \mathcal{H}$, by

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} &= \langle v_6, w_6\rangle_{L_g} + k_1\int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)dx \\ &\quad + \int_0^L \Big(\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_x v_2\partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4 + \rho_3 v_5 w_5\Big)dx\end{aligned}$$

in case (6.1),

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} &= \langle v_6, w_6\rangle_{L_g} + \int_0^L \Big(k_1(\partial_x v_1 + v_2)(\partial_x w_1 + w_2) + k_2\partial_x v_2\partial_x w_2\Big)dx \\ &\quad + \int_0^L \Big(-\Big(\int_0^{+\infty} g(s)ds\Big)\partial_x v_1\partial_x w_1 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4 + \rho_3 v_5 w_5\Big)dx\end{aligned}$$

in case (6.2),

$$\begin{aligned}&\langle V, W\rangle_{\mathcal{H}} \\ &= \langle v_7, w_7\rangle_{L_g} + \int_0^L \Big(k_1(\partial_x v_1 + v_2)(\partial_x w_1 + w_2) + k_2\partial_x v_2\partial_x w_2\Big)dx \\ &\quad + \int_0^L \Big(-\Big(\int_0^{+\infty} g(s)ds\Big)\partial_x v_1\partial_x w_1 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4 + \rho_3 v_5 w_5 + \rho_4 v_6 w_6\Big)dx\end{aligned}$$

in case (6.3), and

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} &= \langle v_7, w_7\rangle_{L_g} + k_1\int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)dx \\ &\quad + \int_0^L \Big(\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_x v_2\partial_x w_2 + \rho_1 v_3 w_3 \\ &\quad + \rho_2 v_4 w_4 + \rho_3 v_5 w_5 + \rho_4 v_6 w_6\Big)dx\end{aligned}$$

in case (6.4). Now, let $\eta_0(x,s) = \eta(x,0,s)$,

$$U_0 = \begin{cases} (\varphi_0, \psi_0(\cdot,0), \varphi_1, \psi_1, \theta_0, \eta_0)^T, & \text{in case (6.1)} \\ (\varphi_0(\cdot,0), \psi_0, \varphi_1, \psi_1, \theta_0, \eta_0)^T, & \text{in case (6.2)} \\ (\varphi_0(\cdot,0), \psi_0, \varphi_1, \psi_1, \theta_0, q_0, \eta_0)^T, & \text{in case (6.3)} \\ (\varphi_0, \psi_0(\cdot,0), \varphi_1, \psi_1, \theta_0, q_0, \eta_0)^T, & \text{in case (6.4)} \end{cases}$$

and

$$U=\begin{cases}(\varphi,\psi,\varphi_t,\psi_t,\theta,\eta)^T, & \text{in cases (6.1) and (6.2)}\\ (\varphi,\psi,\varphi_t,\psi_t,\theta,q,\eta)^T, & \text{in cases (6.3) and (6.4)}\end{cases}$$

Systems (6.1)–(6.4) can be rewritten as the abstract problem (5.8), where A is the linear operator defined, for any $V\in D(A)$, by $AV=F$ and

$$\begin{gathered}
f_1=-v_3,\quad f_2=-v_4,\quad f_3=-\frac{k_1}{\rho_1}\partial_x(\partial_x v_1+v_2),\\
f_4=-\frac{1}{\rho_2}\Big(k_2-\int_0^{+\infty}g(s)ds\Big)\partial_{xx}v_2+\frac{k_1}{\rho_2}(\partial_x v_1+v_2)\\
+\frac{k_4}{\rho_2}\partial_x v_5-\frac{1}{\rho_2}\int_0^{+\infty}g(s)\partial_{xx}v_6(s)ds,\\
f_5=-\frac{k_3}{\rho_3}\partial_{xx}v_5+\frac{k_4}{\rho_3}\partial_x v_4,\quad f_6=-v_4+\partial_s v_6,
\end{gathered}$$

in case (6.1),

$$\begin{gathered}
f_1=-v_3,\quad f_2=-v_4,\\
f_3=-\frac{k_1}{\rho_1}\partial_x(\partial_x v_1+v_2)+\frac{1}{\rho_1}\Big(\int_0^{+\infty}g(s)ds\Big)\partial_{xx}v_1-\frac{1}{\rho_1}\int_0^{+\infty}g(s)\partial_{xx}v_6(s)ds,\\
f_4=-\frac{k_2}{\rho_2}\partial_{xx}v_2+\frac{k_1}{\rho_2}(\partial_x v_1+v_2)+\frac{k_4}{\rho_2}\partial_x v_5,\\
f_5=-\frac{k_3}{\rho_3}\partial_{xx}v_5+\frac{k_4}{\rho_3}\partial_x v_4,\quad f_6=-v_3+\partial_s v_6,
\end{gathered}$$

in case (6.2),

$$\begin{gathered}
f_1=-v_3,\quad f_2=-v_4,\\
f_3=-\frac{k_1}{\rho_1}\partial_x(\partial_x v_1+v_2)+\frac{1}{\rho_1}\Big(\int_0^{+\infty}g(s)ds\Big)\partial_{xx}v_1-\frac{1}{\rho_1}\int_0^{+\infty}g(s)\partial_{xx}v_7(s)ds,\\
f_4=-\frac{k_2}{\rho_2}\partial_{xx}v_2+\frac{k_1}{\rho_2}(\partial_x v_1+v_2)+\frac{k_5}{\rho_2}\partial_x v_5,\quad f_5=\frac{k_3}{\rho_3}\partial_x v_6+\frac{k_5}{\rho_3}\partial_x v_4,\\
f_6=\frac{k_4}{\rho_4}v_6+\frac{k_3}{\rho_4}\partial_x v_5,\ f_7=-v_3+\partial_s v_7,
\end{gathered}$$

in case (6.3), and

$$\begin{gathered}
f_1=-v_3,\quad f_2=-v_4,\quad f_3=-\frac{k_1}{\rho_1}\partial_x(\partial_x v_1+v_2)\\
f_4=-\frac{1}{\rho_2}\Big(k_2-\int_0^{+\infty}g(s)ds\Big)\partial_{xx}v_2+\frac{k_1}{\rho_2}(\partial_x v_1+v_2)\\
+\frac{k_4}{\rho_2}\partial_x v_5-\frac{1}{\rho_2}\int_0^{+\infty}g(s)\partial_{xx}v_7(s)ds,\\
f_5=\frac{k_3}{\rho_3}\partial_x v_6+\frac{k_4}{\rho_3}\partial_x v_4,\quad f_6=\frac{k_5}{\rho_4}v_6+\frac{k_3}{\rho_4}\partial_x v_5,\quad f_7=-v_4+\partial_s v_7,
\end{gathered}$$

in case (6.4).

The domain $D(A)$ of A is endowed with the norm (5.9) and it is given by $D(A)=\{V\in\mathcal{H},\ AV\in\mathcal{H}\text{ and }v_6(0)=0\}$ in case (6.1) and (6.2), and $D(A)=\{V\in\mathcal{H},\ AV\in\mathcal{H}\text{ and }v_7(0)=0\}$ in case (6.3) and (6.4). The operator A is maximal monotone (the proof is similar to the one of Section 5), and then A is an infinitesimal generator of a contraction semigroup in $\mathcal{H}$, which implies the well-posedness results of Theorem 5.1 for (6.1)-(6.4).

6.2. **Stability.** Similarly to (1.1) and under the hypotheses (H2) and (H3), we prove that (2.6) (when (1.2) holds) and (2.11) (when (1.2) does not hold) remain valid for (6.1). For (6.2) and (6.3), we prove that (2.6) holds independently of (1.2). Finally, for (6.4), we prove only (2.11) even if (1.2) holds.

We start by the system (6.1) and we consider its energy functional defined by (we recall (2.4))

$$\begin{aligned} E_1(t) = \frac{1}{2}\int_0^L \Big(&\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\theta^2 + k_1(\varphi_x+\psi)^2 \\ &+ \Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big)dx + \frac{1}{2}g\circ\psi_x. \end{aligned} \tag{6.7}$$

Theorem 6.2 ((1.2) holds)**.** *Assume that* (1.2), (H2), (H3) *are satisfied, and let* $U_0 \in \mathcal{H}$ *satisfying* (2.5)*. Then there exist positive constants* c', c'', ϵ_0 *(depending continuously on* $E_1(0)$*) for which* E_1 *satisfies* (2.6).

Theorem 6.3 ((1.2) does not hold)**.** *Assume that* (H2) *and* (H3) *are satisfied, and let* $U_0 \in D(A)$ *satisfying* (2.10)*. Then there exist positive constants* C *and* ϵ_0 *(depending continuously on* $\|U_0\|_{D(A)}$*) such that* E_1 *satisfies* (2.11).

The energy functionals of (6.2), (6.3) and (6.4) are, respectively, defined by

$$\begin{aligned} E_2(t) = \frac{1}{2}\int_0^L \Big(&\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\tilde{\theta}^2 + k_1(\varphi_x+\psi)^2 + k_2\psi_x^2 \\ &- \Big(\int_0^{+\infty} g(s)ds\Big)\varphi_x^2\Big)dx + \frac{1}{2}g\circ\varphi_x, \end{aligned} \tag{6.8}$$

$$\begin{aligned} E_3(t) = \frac{1}{2}\int_0^L \Big(&\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\tilde{\theta}^2 + \rho_4 q^2 + k_1(\varphi_x+\psi)^2 + k_2\psi_x^2 \\ &- \Big(\int_0^{+\infty} g(s)ds\Big)\varphi_x^2\Big)dx + \frac{1}{2}g\circ\varphi_x, \end{aligned} \tag{6.9}$$

$$\begin{aligned} E_4(t) = \frac{1}{2}\int_0^L \Big(&\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\tilde{\theta}^2 + \rho_4 q^2 + k_1(\varphi_x+\psi)^2 \\ &+ \Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big)dx + \frac{1}{2}g\circ\psi_x, \end{aligned} \tag{6.10}$$

where

$$\tilde{\theta}(x,t) = \theta(x,t) - \frac{1}{L}\int_0^L \theta_0(y)dy. \tag{6.11}$$

Theorem 6.4. *Assume that* (H2) *and* (H3) *are satisfied, and let* $U_0 \in \mathcal{H}$ *satisfying* (2.5)*. Then there exist positive constants* c', c'', ϵ_0 *(depending continuously on* $E_2(0)$ *in case* (6.2)*, and on* $E_3(0)$ *in case* (6.3)*) for which* E_2 *and* E_3 *satisfy* (2.6).

Theorem 6.5. *Assume that* (H2) *and* (H3) *are satisfied, and let* $U_0 \in D(A)$ *satisfying* (2.10)*. Then there exist positive constants* C *and* ϵ_0 *(depending continuously on* $\|U_0\|_{D(A)}$*) such that* E_4 *satisfies* (2.11).

6.3. **Proof of Theorems 6.2 and 6.3.** The proofs are very similar to the ones of Theorems 2.2 and 2.4, respectively.

Lemma 6.6. *The energy functional E_1 defined by* (6.7) *satisfies*

$$E_1'(t) = \frac{1}{2} g' \circ \psi_x - k_3 \int_0^L \theta_x^2 dx \le 0. \tag{6.12}$$

Proof. By multiplying the first three equations of (6.1) by φ_t, ψ_t and θ, respectively, integrating over $]0, L[$, and using the boundary conditions, we obtain (6.12) (which implies that (6.1) is dissipative). $\square$

Now, we consider the functionals $I_1 - I_8$ defined in Section 3 and

$$I_{10}(t) = N_1 E_1(t) + N_2 I_1(t) + N_3 I_8(t) + I_7(t), \tag{6.13}$$

where $N_1, N_2, N_3 > 0$. Using (6.12) and the same computations as in Section 3 (where we keep the terms depending on θ), we obtain (instead of (3.17) and with $g_0 = \int_0^{+\infty} g(s)ds$ and $\delta = \frac{k_1}{8N_2}$ in (3.5))

$$\begin{aligned}
I_{10}'(t) \le & -(\frac{lN_3}{2} - c) \int_0^L \psi_x^2 dx - (\frac{\rho_1}{16} - \epsilon N_3) \int_0^L \varphi_t^2 dx \\
& - \Big(N_2 \rho_2 g_0 - \frac{cN_3}{\epsilon} - c\Big) \int_0^L \psi_t^2 dx - \frac{k_1}{8} \int_0^L (\varphi_x + \psi)^2 dx + c_{N_2,N_3} g \circ \psi_x \\
& + (\frac{N_1}{2} - c_{N_2}) g' \circ \psi_x - N_1 k_3 \int_0^L \theta_x^2 dx + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \\
& + k_4 \int_0^L \theta_x \Big((\frac{1}{8} - N_3)\psi - (\varphi_x + \psi) + N_2 \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s)) ds \\
& - \frac{1}{2\epsilon} m(x) \Big(k_2 \psi_x - \int_0^{+\infty} g(s) \psi_x(t-s) ds \Big) \Big) dx.
\end{aligned} \tag{6.14}$$

Using Young's inequality, (3.2), (6.6) (for ψ) and (6.7), for any $\epsilon_1 > 0$, we have

$$\begin{aligned}
& k_4 \int_0^L \theta_x \Big((\frac{1}{8} - N_3)\psi - (\varphi_x + \psi) + N_2 \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s)) ds \\
& - \frac{1}{2\epsilon} m(x) \Big(k_2 \psi_x - \int_0^{+\infty} g(s) \psi_x(t-s) ds \Big) \Big) dx \\
& \le \epsilon_1 E_1(t) + c_{N_2,N_3,\epsilon,\epsilon_1} \int_0^L \theta_x^2 dx.
\end{aligned}$$

Then, with the same choice of N_3, ϵ and N_2 as for (3.17), from (6.14), we obtain

$$\begin{aligned}
I_{10}'(t) \le & -c\Big(g \circ \psi_x + \int_0^L \Big(\varphi_t^2 + \psi_t^2 + \theta^2 + (\varphi_x + \psi)^2 + \psi_x^2 \Big) dx \Big) \\
& + \epsilon_1 E_1(t) + c g \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \\
& + c \int_0^L \theta^2 dx - (N_1 k_3 - c_{\epsilon_1}) \int_0^L \theta_x^2 dx + (\frac{N_1}{2} - c) g' \circ \psi_x.
\end{aligned}$$

Therefore, using the definition of E_1 and Poincaré's inequality (6.6) for θ,

$$\begin{aligned} I_{10}'(t) \leq &-(c-\epsilon_1)E_1(t) + cg\circ\psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx \\ &- (N_1k_3 - c_{\epsilon_1})\int_0^L \theta_x^2 dx + (\frac{N_1}{2} - c)g'\circ\psi_x. \end{aligned} \tag{6.15}$$

On the other hand, there exists a positive constant α (which does not depend on N_1) satisfying $|N_2I_1 + N_3I_8 + I_7| \leq \alpha E_1$, which implies

$$(N_1 - \alpha)E_1 \leq I_{10} \leq (N_1+\alpha)E_1.$$

Then, by choosing ϵ_1 small enough so that $c-\epsilon_1 > 0$, and N_1 large enough so that $\frac{N_1}{2} - c \geq 0$, $N_1k_3 - c_{\epsilon_1} \geq 0$ and $N_1 > \alpha$, we deduce that $I_{10} \sim E_1$ and

$$I_{10}'(t) \leq -cE_1(t) + cg\circ\psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx, \tag{6.16}$$

which is similar to (3.18). Then the proof of Theorems 6.2 and 6.3 can be completed as in Sections 3 and 4, respectively.

6.4. **Proof of Theorem 6.4.** First, we consider the case (6.2).

Lemma 6.7. *The energy functional E_2 defined by* (6.8) *satisfies*

$$E_2'(t) = \frac{1}{2}g'\circ\varphi_x - k_3\int_0^L \theta_x^2 dx \leq 0. \tag{6.17}$$

Proof. Note that, thanks to the fact that $\tilde{\theta}_x = \theta_x$ and $\tilde{\theta}_t = \theta_t$, (6.2) is also satisfied with $\tilde{\theta}$ (defined by (6.11)) and $\theta_0 - \frac{1}{L}\int_0^L \theta_0(y)dy$ instead of θ and θ_0, respectively. Then, as in case (6.12), by multiplying the first three equations of (6.2), respectively, by φ_t, ψ_t and $\tilde{\theta}$, integrating over $]0,L[$, and using the boundary conditions, we obtain (6.17). $\square$

Lemma 6.8. *The functional*

$$J_1(t) = -\rho_1\int_0^L \varphi_t\int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\,ds\,dx$$

satisfies, for any $\delta > 0$,

$$\begin{aligned} J_1'(t) \leq &-\rho_1\Big(\int_0^{+\infty} g(s)ds - \delta\Big)\int_0^L \varphi_t^2\,dx \\ &+ \delta\int_0^L (\psi_x^2 + (\varphi_x+\psi)^2)dx + c_\delta g\circ\varphi_x - c_\delta g'\circ\varphi_x. \end{aligned} \tag{6.18}$$

The proof of the above lemma is similar to the proof of Lemma 3.3, and is omitted.

Lemma 6.9. *The functional*

$$J_2(t) = \int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)dx$$

satisfies, for some positive constants c and $\tilde{c}$,

$$\begin{aligned} J_2'(t) \le \int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx - c\int_0^L ((\varphi_x+\psi)^2+\psi_x^2)dx \\ + \tilde{c}\Big(\int_0^L \tilde{\theta}^2dx + g\circ\varphi_x\Big). \end{aligned} \tag{6.19}$$

Proof. Because system (6.2) is still satisfied with $\tilde{\theta}$ and $\theta_0 - \frac{1}{L}\int_0^L \theta_0(y)dy$ instead of θ and θ_0, respectively, then, by exploiting the first two equations of (6.2) and integrating over $]0,L[$, we obtain

$$\begin{aligned} J_2'(t) &= \int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx - k_1\int_0^L (\varphi_x+\psi)^2dx \\ &\quad - k_2\int_0^L \psi_x^2dx + \int_0^L \varphi_x\int_0^{+\infty} g(s)\varphi_x(t-s)\,ds\,dx + k_4\int_0^L \psi_x\tilde{\theta}dx. \end{aligned}$$

Let $l_1 = k_1 - \int_0^{+\infty} g(s)ds$ ($l_1>0$ thanks to (6.5)). Using (3.2) for φ and Young's inequality, for any $\epsilon>0$, we obtain

$$\begin{aligned} &\int_0^L \varphi_x\int_0^{+\infty} g(s)\varphi_x(t-s)\,ds\,dx \\ &= \int_0^L \varphi_x\int_0^{+\infty} g(s)(\varphi_x(t-s)-\varphi_x(t)+\varphi_x(t))\,ds\,dx \\ &= \int_0^{+\infty} g(s)ds\int_0^L \varphi_x^2dx - \int_0^L \varphi_x(t)\int_0^{+\infty} g(s)(\varphi_x(t)-\varphi_x(t-s))dsdx \\ &\le (k_1-l_1+\epsilon)\int_0^L \varphi_x^2dx + c_\epsilon g\circ\varphi_x. \end{aligned}$$

Similarly, for any $\epsilon'>0$, we have

$$k_4\int_0^L \psi_x\tilde{\theta}dx \le \epsilon'\int_0^L \psi_x^2dx + c_{\epsilon'}\int_0^L \tilde{\theta}^2dx.$$

On the other hand, using (6.6) for ψ, for any $\epsilon''>0$, we have

$$\int_0^L \varphi_x^2dx = \int_0^L (\varphi_x+\psi-\psi)^2dx \le (1+\epsilon'')\int_0^L (\varphi_x+\psi)^2dx + (1+\frac{1}{\epsilon''})k_0\int_0^L \psi_x^2dx.$$

Inserting these three estimates into the previous equality, we obtain

$$\begin{aligned} J_2'(t) &\le \int_0^L (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx - \Big(l_1-\epsilon-\epsilon''(k_1-l_1+\epsilon)\Big)\int_0^L (\varphi_x+\psi)^2dx \\ &\quad - \Big(k_2-\epsilon'-k_0(1+\frac{1}{\epsilon''})(k_1-l_1+\epsilon)\Big)\int_0^L \psi_x^2dx + c_{\epsilon'}\int_0^L \tilde{\theta}^2dx + c_\epsilon g\circ\varphi_x. \end{aligned}$$

Thanks to (6.5), we can choose $\frac{k_0(k_1-l_1)}{k_2-k_0(k_1-l_1)} < \epsilon'' < \frac{l_1}{k_1-l_1}$ and $\epsilon,\epsilon'>0$ small enough such that

$$\min\{l_1-\epsilon-\epsilon''(k_1-l_1+\epsilon), k_2-\epsilon'-k_0(1+\frac{1}{\epsilon''})(k_1-l_1+\epsilon)\} > 0,$$

therefore, we obtain (6.19). $\square$

The Neumann boundary conditions considered on θ in (6.2) do not allow the use of Poincaré's inequality (6.6) for θ. To overcome this difficulty we use the following classical argument (see [32]): by integrating the third equation of (6.2) and using the boundary conditions, we obtain

$$\begin{aligned}\partial_t\Big(\int_0^L \theta(x,t)dx\Big) &= \int_0^L \theta_t(x,t)dx = \frac{1}{\rho_3}\int_0^L (k_3\theta_{xx}(x,t) - k_4\psi_{xt}(x,t))dx \\ &= \frac{1}{\rho_3}\Big[k_3\theta_x(x,t) - k_4\psi_t(x,t)\Big]_{x=0}^{x=L} = 0, \quad \forall t\in\mathbb{R}_+.\end{aligned}$$

Then

$$\int_0^L \theta(x,t)dx = \int_0^L \theta(x,0)dx = \int_0^L \theta_0(x)dx, \quad \forall t\in\mathbb{R}_+.$$

Therefore, the functional $\tilde\theta$ defined by (6.11) satisfies $\int_0^L \tilde\theta(x,t)dx = 0$, and then (6.6) is also applicable for $\tilde\theta$. Now, as in [32], we have the following lemma.

Lemma 6.10. *The functional*

$$J_3(t) = \rho_2\rho_3\int_0^L \psi_t\Big(\int_0^x \tilde\theta(y,t)dy\Big)dx$$

satisfies, for any $\epsilon > 0$,

$$J_3'(t) \le -(\rho_2 k_4 - \epsilon)\int_0^L \psi_t^2 + \epsilon\int_0^L (\psi_x^2 + (\varphi_x+\psi)^2)dx + c_\epsilon\int_0^L \theta_x^2 dx. \tag{6.20}$$

Following the same arguments as in [32] one can prove easily our lemma 6.10.

Now, we go back to the proof of Theorem 6.4 in case (6.2). Let $N_1, N_2, N_3, N_4 > 0$ and, as before, $g_0 = \int_0^{+\infty} g(s)ds$. We put

$$J_4(t) = N_1E_2(t) + N_2J_1(t) + N_3J_2(t) + N_4J_3(t). \tag{6.21}$$

Then, using Poincaré's inequality (6.6) for $\tilde\theta$, combining (6.17)-(6.20) and choosing $\delta = \frac{1}{N_2}$ and $\epsilon = \frac{1}{N_4}$ in (6.18) and (6.20), respectively, we find

$$\begin{aligned}J_4'(t) &\le -(cN_3 - 2)\int_0^L (\psi_x^2 + (\varphi_x+\psi)^2)dx - (N_4\rho_2k_4 - \rho_2N_3 - 1)\int_0^L \psi_t^2 dx \\ &\quad - (N_2\rho_1g_0 - \rho_1N_3 - \rho_1)\int_0^L \varphi_t^2 dx - (N_1k_3 - c_{N_3,N_4})\int_0^L \theta_x^2 dx \\ &\quad + (\frac{N_1}{2} - c_{N_2})g'\circ\varphi_x + c_{N_2,N_3}g\circ\varphi_x.\end{aligned} \tag{6.22}$$

So, we choose N_3 large enough so that $cN_3 - 2 > 0$, then N_4 large enough so that $N_4k_4\rho_2 - \rho_2N_3 - 1 > 0$. Next, we choose N_2 large enough so that

$$N_2\rho_1g_0 - \rho_1N_3 - \rho_1 > 0.$$

Consequently, using again Poincaré's inequality (6.6) for $\tilde\theta$, from (6.22), we obtain

$$\begin{aligned}J_4'(t) &\le -c\Big(g\circ\varphi_x + \int_0^L \Big(\varphi_t^2 + \psi_t^2 + \tilde\theta^2 + (\varphi_x+\psi)^2 + \psi_x^2\Big)dx\Big) \\ &\quad + cg\circ\varphi_x - (N_1k_3 - c)\int_0^L \theta_x^2 dx + (\frac{N_1}{2} - c)g'\circ\varphi_x.\end{aligned}$$

Therefore, using the definition of E_2,

$$J_4'(t) \leq -cE_2(t) + cg \circ \varphi_x - (N_1 k_3 - c) \int_0^L \theta_x^2 dx + (\frac{N_1}{2} - c) g' \circ \varphi_x.$$

Now, as in Section 3, choosing N_1 large enough so that $\frac{N_1}{2} - c \geq 0$, $N_1 k_3 - c \geq 0$ and $J_4 \sim E_2$, we deduce that

$$J_4'(t) \leq -cE_2(t) + cg \circ \varphi_x. \tag{6.23}$$

To estimate the term $g \circ \varphi_x$ in (6.23), we apply Lemma 3.11 for φ and E_2 instead of ψ and E, respectively, and we obtain (similarly to (3.19))

$$G'(\epsilon_0 E_2(t)) g \circ \varphi_x \leq -cE_2'(t) + c\epsilon_0 E_2(t) G'(\epsilon_0 E_2(t)), \quad \forall t \in \mathbb{R}_+, \ \forall \epsilon_0 > 0. \tag{6.24}$$

By multiplying (6.23) by $G'(\epsilon_0 E_2(t))$, inserting (6.24) and choosing ϵ_0 small enough, we obtain

$$G'(\epsilon_0 E_2(t)) J_4'(t) + cE_2'(t) \leq -cE_2(t) G'(\epsilon_0 E_2(t)), \tag{6.25}$$

and then the proof can be finalized exactly as for (3.20) in Section 3, which shows (2.6) for E_2.

Now, we consider the case (6.3) and we prove (2.6) for E_3.

Lemma 6.11. *The energy functional E_3 defined by* (6.9) *satisfies*

$$E_3'(t) = \frac{1}{2} g' \circ \varphi_x - k_5 \int_0^L q^2 dx \leq 0. \tag{6.26}$$

Proof. Because (6.3) holds with $\tilde{\theta}$ and $\theta_0 - \frac{1}{L}\int_0^L \theta_0(y)dy$ instead of θ and θ_0, respectively, then by multiplying the first–fourth equations of (6.3) by φ_t, ψ_t, $\tilde{\theta}$ and q, respectively, and integrating over $]0, L[$, we obtain (6.26). $\square$

As in [32] and similarly to (6.20), we prove the following estimate.

Lemma 6.12. *The functional*

$$Q_1(t) = \rho_2 \rho_3 \int_0^L \psi_t \Big(\int_0^x \tilde{\theta}(y,t) dy \Big) dx$$

satisfies, for any $\epsilon_1 > 0$,

$$\begin{aligned} Q_1'(t) \leq & -(\rho_2 k_4 - \epsilon_1) \int_0^L \psi_t^2 dx + \epsilon_1 \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx \\ & + (\frac{c}{\epsilon_1} + \rho_3 k_4) \int_0^L \tilde{\theta}^2 dx + c_{\epsilon_1} \int_0^L q^2 dx. \end{aligned} \tag{6.27}$$

Proof. By exploiting the second and the third equations of (6.3) and integrating over $]0, L[$ (note also that $\int_0^L \tilde{\theta}(x,t) dx = 0$, $\tilde{\theta}_x = \theta_x$ and $\tilde{\theta}_t = \theta_t$), we obtain

$$\begin{aligned} Q_1'(t) &= \rho_3 \int_0^L (k_2 \psi_{xx} - k_1(\varphi_x + \psi) - k_4 \tilde{\theta}_x) \Big(\int_0^x \tilde{\theta}(y,t) dy \Big) dx \\ &\quad + \rho_2 \int_0^L \psi_t \Big(\int_0^x (-k_3 q_x(y,t) - k_4 \psi_{xt}(y,t)) dy \Big) dx \\ &= -\rho_2 k_4 \int_0^L \psi_t^2 dx + \rho_3 k_4 \int_0^L \tilde{\theta}^2 dx - \rho_3 k_2 \int_0^L \psi_x \tilde{\theta} dx \end{aligned}$$

$$- \rho_3 k_1 \int_0^L (\varphi_x + \psi)\Big(\int_0^x \tilde{\theta}(y,t)dy\Big)dx - \rho_2 k_3 \int_0^L \psi_t q dx.$$

Using Young's and Hölder's inequalities to estimate the last three integrals, we obtain (6.27). □

As in [23], we consider the following functional.

Lemma 6.13. *The functional*

$$Q_2(t) = -\rho_3\rho_4 \int_0^L q\Big(\int_0^x \tilde{\theta}(y,t)dy\Big)dx$$

satisfies, for any $\epsilon_2 > 0$,

$$Q_2'(t) \le -(\rho_3 k_3 - \epsilon_2) \int_0^L \tilde{\theta}^2 dx + \epsilon_2 \int_0^L \psi_t^2 dx + c_{\epsilon_2} \int_0^L q^2 dx. \tag{6.28}$$

Proof. By using the third and the fourth equations of (6.3) and integrating over $]0, L[$ (note also that $\int_0^L \tilde{\theta}(x,t)dx = 0$, $\tilde{\theta}_x = \theta_x$ and $\tilde{\theta}_t = \theta_t$), we obtain

$$\begin{aligned} Q_2'(t) &= \rho_3 \int_0^L (k_5 q + k_3\theta_x)\Big(\int_0^x \tilde{\theta}(y,t)dy\Big)dx \\ &\quad + \rho_4 \int_0^L q\Big(\int_0^x (k_3 q_x(y,t) + k_4\psi_{xt}(y,t))dy\Big)dx \\ &= -\rho_3 k_3 \int_0^L \tilde{\theta}^2 dx + \rho_3 k_5 \int_0^L q\Big(\int_0^x \tilde{\theta}(y,t)dy\Big)dx + \rho_4 \int_0^L q(k_3 q + k_4\psi_t)dx. \end{aligned}$$

Using Young's and Hölder's inequalities to estimate the last two integrals, we obtain (6.28). □

Now, we complete the proof of Theorem 6.4 in case (6.3). Let N_1, N_2, N_3, N_4, N_5 be positive, $g_0 = \int_0^{+\infty} g(s)ds$ and

$$Q_3(t) = N_1 E_3(t) + N_2 J_1(t) + N_3 J_2(t) + N_4 Q_1(t) + N_5 Q_2, \tag{6.29}$$

where J_1 and J_2 are defined in Lemmas 6.8 and 6.9, respectively. The estimates (6.18) and (6.19) hold also for (6.3) because we used only the first two equations of (6.2) and the boundary conditions on φ and ψ, which are the same as in (6.3). Then, by combining (6.26)-(6.28), (6.18) and (6.19), choosing $\delta = 1/N_2$, $\epsilon_1 = 1/N_4$ and $\epsilon_2 = 1/N_5$ in (6.18), (6.27) and (6.28), respectively, we obtain

$$\begin{aligned} Q_3'(t) &\le -(cN_3 - 2) \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2)dx - (N_4 k_4 \rho_2 - \rho_2 N_3 - 2) \int_0^L \psi_t^2 dx \\ &\quad - (N_2\rho_1 g_0 - \rho_1 N_3 - \rho_1) \int_0^L \varphi_t^2 dx \\ &\quad - (N_5\rho_3 k_3 - \tilde{c}N_3 - cN_4^2 - \rho_3 k_4 N_4 - 1) \int_0^L \tilde{\theta}^2 dx \\ &\quad - (N_1 k_5 - c_{N_4,N_5}) \int_0^L q^2 dx + (\frac{N_1}{2} - c_{N_2}) g' \circ \varphi_x + c_{N_2,N_3} g \circ \varphi_x. \end{aligned} \tag{6.30}$$

We choose N_3 large enough so that $cN_3 - 2 > 0$, then N_4 large enough so that $N_4k_4\rho_2 - \rho_2N_3 - 2 > 0$. After, we choose N_2 large enough so that

$$N_2\rho_1 g_0 - \rho_1 N_3 - \rho_1 > 0.$$

Next, we pick N_5 large enough so that $N_5k_3\rho_3 - \tilde{c}N_3 - cN_4^2 - \rho_3k_4N_4 - 1 > 0$. Then (6.30) implies

$$\begin{aligned} Q_3'(t) \le -c\Big(g\circ\varphi_x + \int_0^L \Big(\varphi_t^2 + \psi_t^2 + \tilde{\theta}^2 + q^2 + (\varphi_x+\psi)^2 + \psi_x^2\Big)dx\Big) \\ + cg\circ\varphi_x - (N_1k_5 - c)\int_0^L q^2dx + (\frac{N_1}{2} - c)g'\circ\varphi_x. \end{aligned}$$

Now, as before, choosing N_1 large enough so that $\frac{N_1}{2} - c \ge 0$, $N_1k_5 - c \ge 0$ and $Q_3 \sim E_3$, and using the definition of E_3, we conclude that

$$Q_3'(t) \le -cE_3(t) + cg\circ\varphi_x. \tag{6.31}$$

The estimate (6.31) is similar to (6.23), and then we complete the proof exactly as for (6.2) to get (2.6) for E_3. This completes the proof of Theorem 6.4.

Proof of Theorem 6.5. As in Section 4, we will show that E_4 satisfies the inequality (6.45) below, which implies (2.11) for E_4. As for (6.26), we can see that the energy functional E_4 defined by (6.10) satisfies

$$E_4'(t) = \frac{1}{2}g'\circ\psi_x - k_5\int_0^L q^2dx \le 0. \tag{6.32}$$

As in Section 4, we consider the second-order energy $\tilde{E}_4$ of the system resulting from differentiating (6.4) with respect to time (which is well posed for initial data $U_0 \in D(A)$); that is, $\tilde{E}_4(t) = E_4(U_t(t))$, where $E_4(U(t)) = E_4(t)$ and E_4 is defined by (6.10). A simple calculation (as for (6.32)) implies that

$$\tilde{E}_4'(t) = \frac{1}{2}g'\circ\psi_{xt} - k_5\int_0^L q_t^2dx \le 0. \tag{6.33}$$

Now, let $N_1, N_2, N_3 > 0$ and

$$I_{11}(t) = N_1(E_4(t) + \tilde{E}_4(t)) + N_2I_1(t) + N_3I_8(t) + I_7(t), \tag{6.34}$$

where I_1, I_7 and I_8 are defined in Lemmas 3.3, 3.8 and 3.10, respectively. Then (as for I_{10} in (6.14)), by combining (6.32) and (6.33), and using the same computations as in Section 3 (we keep the terms depending on θ_x), we obtain (instead of (3.17) and with $g_0 = \int_0^{+\infty} g(s)ds$ and $\delta = \frac{k_1}{8N_2}$ in (3.5))

$$
\begin{aligned}
I_{11}'(t)n \le & -(\frac{lN_3}{2} - c)\int_0^L \psi_x^2 dx - (\frac{\rho_1}{16} - \epsilon N_3)\int_0^L \varphi_t^2 dx \\
& - (N_2\rho_2 g_0 - \frac{cN_3}{\epsilon} - c)\int_0^L \psi_t^2 dx - \frac{k_1}{8}\int_0^L (\varphi_x + \psi)^2 dx \\
& + (\frac{N_1}{2} - c_{N_2}) g' \circ \psi_x + \frac{N_1}{2} g' \circ \psi_{xt} + c_{N_2,N_3} g \circ \psi_x \\
& + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t \psi_{xt} dx - k_5 N_1 \int_0^L (q^2 + q_t^2) dx \\
& + k_4 \int_0^L \theta_x \Big((\frac{1}{8} - N_3)\psi - (\varphi_x + \psi) \\
& + N_2 \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s)) ds \\
& - \frac{1}{2\epsilon} m(x) \Big(k_2 \psi_x - \int_0^{+\infty} g(s)\psi_x(t-s) ds \Big)\Big) dx.
\end{aligned} \tag{6.35}
$$

Using Young's inequality, (3.2), (6.6) (for ψ) and (6.10), for any $\epsilon_1 > 0$, we have

$$
\begin{aligned}
& k_4 \int_0^L \theta_x \Big((\frac{1}{8} - N_3)\psi - (\varphi_x + \psi) + N_2 \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s)) ds \\
& - \frac{1}{2\epsilon} m(x) \Big(k_2 \psi_x - \int_0^{+\infty} g(s)\psi_x(t-s) ds \Big)\Big) dx \\
& \le \epsilon_1 E_4(t) + c_{N_2,N_3,\epsilon,\epsilon_1} \int_0^L \theta_x^2 dx.
\end{aligned}
$$

We choose N_3, ϵ and N_2 as for (3.17) (to get negative coefficients of the first three integrals of (6.35)) and using (6.6) for $\tilde{\theta}$ (note also that $\tilde{\theta}_x = \theta_x$), we obtain

$$
\begin{aligned}
I_{11}'(t) \le & -c\Big(g \circ \psi_x + \int_0^L \Big(\varphi_t^2 + \psi_t^2 + \tilde{\theta}^2 + q^2 + (\varphi_x + \psi)^2 + \psi_x^2 \Big) dx \Big) \\
& + \epsilon_1 E_4(t) + c g \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t \psi_{xt} dx \\
& + c_{\epsilon_1} \int_0^L \theta_x^2 dx + (\frac{N_1}{2} - c) g' \circ \psi_x + \frac{N_1}{2} g' \circ \psi_{xt} \\
& - (k_5 N_1 - c)\int_0^L q^2 dx - k_5 N_1 \int_0^L q_t^2 dx.
\end{aligned} \tag{6.36}
$$

On the other hand, the fourth equation of (6.4) implies that

$$
\int_0^L \theta_x^2 dx \le c \int_0^L (q^2 + q_t^2) dx. \tag{6.37}
$$

Therefore, by combining (6.36) and (6.37), using (6.10) and choosing ϵ_1 small enough,

$$\begin{aligned} I_{11}'(t) \leq & -cE_4(t) + cg \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \\ & - (N_1 k_5 - c) \int_0^L (q^2 + q_t^2) dx + (\frac{N_1}{2} - c) g' \circ \psi_x + \frac{N_1}{2} g' \circ \psi_{xt}. \end{aligned} \tag{6.38}$$

Now, exactly as in Lemma 4.1, we have, for any $\epsilon > 0$,

$$(\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \leq \epsilon E_4(t) + c_\epsilon (g \circ \psi_{xt} - g' \circ \psi_x). \tag{6.39}$$

Then, by combining (6.38) and (6.39), and choosing ϵ small enough,

$$\begin{aligned} I_{11}'(t) \leq & -cE_4(t) + c(g \circ \psi_x + g \circ \psi_{xt}) + \frac{N_1}{2} g' \circ \psi_{xt} \\ & - (N_1 k_5 - c) \int_0^L (q^2 + q_t^2) dx + (\frac{N_1}{2} - c) g' \circ \psi_x. \end{aligned} \tag{6.40}$$

Then, choosing N_1 large enough so that $N_1 k_5 - c \geq 0$ and $\frac{N_1}{2} - c \geq 0$, we find

$$I_{11}'(t) \leq -cE_4(t) + c(g \circ \psi_x + g \circ \psi_{xt}). \tag{6.41}$$

Now, similarly to the case $g \circ \psi_x$ and $g \circ \psi_{xt}$ in Sections 3 and 4 (estimates (3.19) and (4.6)) and using (2.10), (6.32) and (6.33), we obtain the following two estimates:

$$G'(\epsilon_0 E_4(t)) g \circ \psi_x \leq -cE_4'(t) + c\epsilon_0 E_4(t) G'(\epsilon_0 E_4(t)), \quad \forall \epsilon_0 > 0, \tag{6.42}$$

$$G'(\epsilon_0 E_4(t)) g \circ \psi_{xt} \leq -c\tilde{E}_4'(t) + c\epsilon_0 E_4(t) G'(\epsilon_0 E_4(t)), \quad \forall \epsilon_0 > 0. \tag{6.43}$$

Multiplying (6.41) by $G'(\epsilon_0 E_4(t))$, inserting (6.42) and (6.43), and choosing ϵ_0 small enough, we deduce that

$$\begin{aligned} E_4(t) G'(\epsilon_0 E_4(t)) \leq & -c\Big(G'(\epsilon_0 E_4(t)) I_{11}(t) + c\epsilon_0 E_4'(t) G''(\epsilon_0 E_4(t)) I_{11}(t) \\ & - c(E_4'(t) + \tilde{E}_4'(t)) \Big)'. \end{aligned} \tag{6.44}$$

Then, by integrating (6.44) over $[0, T]$ and using the fact that $I_{11} \leq c(E_4 + \tilde{E}_4)$ (thanks to (6.34)) and $E_4'(t) G''(\epsilon_0 E_4(t)) \leq 0$ (thanks to (6.32) and (H3)),

$$\int_0^T G_0(E_4(t)) dt \leq c(G'(\epsilon_0 E_4(0)) + 1)(E_4(0) + \tilde{E}_4(0)), \quad \forall T \in \mathbb{R}_+, \tag{6.45}$$

where G_0 is defined by (2.12). The fact that $G_0(E_4)$ is non-increasing and (6.45) imply (2.11) for E_4 with $C = c(G'(\epsilon_0 E_4(0)) + 1)(E_4(0) + \tilde{E}_4(0))$. This completes the proof of Theorem 6.5.

7. Timoshenko-heat: Green and Naghdi's theory

In this section, we consider coupled Timoshenko-thermoelasticity systems of type III on $]0, L[$ with an infinite history acting on the second equation:

$$\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + k_4\theta_{xt} + \int_0^{+\infty} g(s)\psi_{xx}(x, t-s)ds = 0, \\
\rho_3\theta_{tt} - k_3\theta_{xx} + k_4\psi_{xt} - k_5\theta_{xxt} = 0, \\
\varphi(0,t) = \psi(0,t) = \theta_x(0,t) = \varphi(L,t) = \psi(L,t) = \theta_x(L,t) = 0, \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\
\psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x),\, \theta_t(x,0) = \theta_1(x),
\end{gathered} \tag{7.1}$$

and with an infinite history acting on the first equation:

$$\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x + \int_0^{+\infty} g(s)\varphi_{xx}(x, t-s)ds = 0, \\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + k_4\theta_{xt} = 0, \\
\rho_3\theta_{tt} - k_3\theta_{xx} + k_4\psi_{xt} - k_5\theta_{xxt} = 0, \\
\varphi(0,t) = \psi(0,t) = \theta_x(0,t) = \varphi(L,t) = \psi(L,t) = \theta_x(L,t) = 0, \\
\varphi(x,-t) = \varphi_0(x,t), \quad \varphi_t(x,0) = \varphi_1(x), \\
\psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x).
\end{gathered} \tag{7.2}$$

Systems (7.1) and (7.2) model the transverse vibration of a thick beam, taking in account the heat conduction given by Green and Naghdi's theory [9, 9, 11].

We prove the stability of (7.1) and (7.2) under (H2) and the following hypothesis:

- (H4) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function satisfying $g(0) > 0$ such that (2.1) and (6.5) hold in cases of (7.1) and (7.2), respectively.

7.1. **Well-posedness.** We discuss briefly here the well-posedness of (7.1) and (7.2) under (H4). As in Section 5 and following the idea of [6], we consider

$$\eta(x,t,s) = \begin{cases} \psi(x,t) - \psi(x,t-s) & \text{in case (7.1)} \\ \varphi(x,t) - \varphi(x,t-s) & \text{in case (7.2)} \end{cases}$$

$\eta_0(x,s) = \eta(x,0,s)$, $U = (\varphi, \psi, \theta, \varphi_t, \psi_t, \theta_t, \eta)^T$, and

$$U_0 = \begin{cases} (\varphi_0, \psi_0(\cdot,0), \theta_0, \varphi_1, \psi_1, \theta_1, \eta_0)^T & \text{in case (7.1)} \\ (\varphi_0(\cdot,0), \psi_0, \theta_0, \varphi_1, \psi_1, \theta_1, \eta_0)^T & \text{in case (7.2)} \end{cases}$$

Then (7.1) and (7.2) can be formulated in the form (5.8), where A is the linear operator given by $A(v_1, v_2, v_3, v_4, v_5, v_6, v_7) = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$, where

$$\begin{gathered}
f_1 = -v_4, \quad f_2 = -v_5, \; f_3 = -v_6, \quad f_4 = -\frac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2), \\
f_5 = -\frac{1}{\rho_2}\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_{xx}v_2 + \frac{k_1}{\rho_2}(\partial_x v_1 + v_2) \\
+ \frac{k_4}{\rho_2}\partial_x v_6 - \frac{1}{\rho_2}\int_0^{+\infty} g(s)\partial_{xx}v_7(s),
\end{gathered}$$

$$f_6 = -\frac{k_3}{\rho_3}\partial_{xx}v_3 + \frac{k_4}{\rho_3}\partial_x v_5 - \frac{k_5}{\rho_3}\partial_{xx}v_6, \quad f_7 = -v_5 + \partial_s v_7,$$

in case (7.1), and

$$f_1 = -v_4, \quad f_2 = -v_5, \quad f_3 = -v_6,$$
$$f_4 = -\frac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \frac{1}{\rho_1}\Big(\int_0^{+\infty} g(s)ds\Big)\partial_{xx}v_1 - \frac{1}{\rho_1}\int_0^{+\infty} g(s)\partial_{xx}v_7(s),$$
$$f_5 = -\frac{k_2}{\rho_2}\partial_{xx}v_2 + \frac{k_1}{\rho_2}(\partial_x v_1 + v_2) + \frac{k_4}{\rho_2}\partial_x v_6,$$
$$f_6 = -\frac{k_3}{\rho_3}\partial_{xx}v_3 + \frac{k_4}{\rho_3}\partial_x v_5 - \frac{k_5}{\rho_3}\partial_{xx}v_6, \quad f_7 = -v_5 + \partial_s v_7,$$

in case (7.2). Let

$$\mathcal{H} = \big(H_0^1(]0,L[)\big)^2 \times H_*^1(]0,L[) \times \big(L^2(]0,L[)\big)^2 \times L_*^2(]0,L[) \times L_g,$$

where L_g is defined by (5.5) and endowed with the inner product (5.6),

$$H_*^1(]0,L[) = \{v \in H^1(]0,L[), \int_0^L v dx = 0\},$$
$$L_*^2(]0,L[) = \{v \in L^2(]0,L[), \int_0^L v dx = 0\}.$$

The set $\mathcal{H}$, together with the inner product defined, for $V, W \in \mathcal{H}$, by

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} &= \langle v_7, w_7\rangle_{L_g} + \int_0^L \Big(k_1(\partial_x v_1 + v_2)(\partial_x w_1 + w_2) + k_3\partial_x v_3 \partial_x w_3\Big)dx \\ &\quad + \int_0^L \Big(\Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\partial_x v_2 \partial_x w_2 + \rho_1 v_4 w_4 + \rho_2 v_5 w_5 + \rho_3 v_6 w_6\Big)dx\end{aligned}$$

in case (7.1), and

$$\begin{aligned}\langle V, W\rangle_{\mathcal{H}} &= \langle v_7, w_7\rangle_{L_g} + \int_0^L \Big(k_1(\partial_x v_1 + v_2)(\partial_x w_1 + w_2) + k_3\partial_x v_3 \partial_x w_3\Big)dx \\ &\quad + \int_0^L \Big(-\Big(\int_0^{+\infty} g(s)ds\Big)\partial_x v_1 \partial_x w_1 + k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_4 w_4 \\ &\quad + \rho_2 v_5 w_5 + \rho_3 v_6 w_6\Big)dx\end{aligned}$$

in case (7.2), is a Hilbert space. The domain $D(A)$ of A endowed with the norm (5.9) is given by $D(A) = \{V \in \mathcal{H},\ AV \in \mathcal{H} \text{ and } v_7(0) = 0\}$ and A is a maximal monotone operator (the proof is similar to the one of Section 5), and then A is an infinitesimal generator of a contraction semigroup in $\mathcal{H}$, which implies the well-posedness results of Theorem 5.1 for (7.1) and (7.2).

Stability. We introduce the energy functionals in (7.1) and (7.2), respectively, as

$$\begin{aligned}E_5(t) &= \frac{1}{2}\int_0^L \Big(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k_1(\varphi_x + \psi)^2 + \Big(k_2 - \int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big)dx \\ &\quad + \frac{1}{2}\int_0^L (\rho_3\tilde{\theta}_t^2 + k_3\tilde{\theta}_x^2)dx + \frac{1}{2} g \circ \psi_x\end{aligned} \tag{7.3}$$

and

$$\begin{aligned} E_6(t) = &\frac{1}{2}\int_0^L \Big(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k_1(\varphi_x+\psi)^2 - \Big(\int_0^{+\infty} g(s)ds\Big)\varphi_x^2\Big)dx \\ &+ \frac{1}{2}\int_0^L (k_2\psi_x^2 + \rho_3\tilde{\theta}_t^2 + k_3\tilde{\theta}_x^2)dx + \frac{1}{2} g\circ\varphi_x, \end{aligned} \tag{7.4}$$

where

$$\tilde{\theta}(x,t) = \theta(x,t) - \frac{t}{L}\int_0^L \theta_1(y)dy - \frac{1}{L}\int_0^L \theta_0(y)dy. \tag{7.5}$$

Remark 7.1. Using the third equation in (7.1) and (7.2) and the boundary conditions, we easily verify that

$$\partial_{tt}\Big(\int_0^L \theta(x,t)dx\Big) = \int_0^L \theta_{tt}(x,t)dx = \frac{1}{\rho_3}\Big[k_3\theta_x - k_4\psi_t + k_5\theta_{xt}\Big]_{x=0}^{x=L} = 0, \quad \forall t\in\mathbb{R}_+,$$

which implies that, using the initial data of θ,

$$\int_0^L \theta(x,t)dx = t\int_0^L \theta_1(x)dx + \int_0^L \theta_0(x)dx, \quad \forall t\in\mathbb{R}_+. \tag{7.6}$$

Therefore, (7.5) and (7.6) imply that $\int_0^L \tilde{\theta}(x,t)dx = 0$, and then Poincaré's inequality (6.6) is applicable also for $\tilde{\theta}$ and, in addition, (7.1) and (7.2) are satisfied with $\tilde{\theta}$, $\theta_0 - \frac{1}{L}\int_0^L \theta_0(y)dy$ and $\theta_1 - \frac{1}{L}\int_0^L \theta_1(y)dy$ instead of θ, θ_0 and θ_1, respectively. In the sequel, we work with $\tilde{\theta}$ instead of θ.

For the stability of (7.1), we distinguish two cases depending on (1.2).

Theorem 7.2 ((1.2) holds)**.** . *Assume that* (H2), (H4) *and* (1.2) *hold, and let* $U_0\in\mathcal{H}$ *satisfying* (2.5)*. Then there exist positive constants* c', c'', ϵ_0 *(depending continuously on* $E_5(0)$*) for which* E_5 *satisfies* (2.6)*.*

Theorem 7.3 ((1.2) does not hold)**.** *Assume that (H2) and (H4) hold, and let* $U_0\in D(A)$ *satisfying* (2.10)*. Then there exist positive constants* C *and* ϵ_0 *(depending continuously on* $\|U_0\|_{D(A)}$*) such that* E_5 *satisfies* (2.11)*.*

Concerning (7.2), the estimate (2.6) holds for E_6 independently of (1.2).

Theorem 7.4. *Assume that* (H2) *and* (H4) *hold, and let* $U_0\in\mathcal{H}$ *satisfying* (2.5)*. Then there exist positive constants* c', c'', ϵ_0 *(depending continuously on* $E_6(0)$*) for which* E_6 *satisfies* (2.6)*.*

7.2. **Proof of Theorems 7.2 and 7.3.** As for $E_1 - E_4$ and by multiplying the first equation in (7.1) by φ_t, the second one by ψ_t and the third one (with $\tilde{\theta}$ instead of θ) by $\tilde{\theta}_t$, integrating over $]0,L[$ and using the boundary conditions, we obtain

$$E_5'(t) = \frac{1}{2} g'\circ\psi_x - k_5\int_0^L \tilde{\theta}_{xt}^2 dx \le 0. \tag{7.7}$$

Now, we consider the functionals I_1–I_8 defined in Section 3 and (as in (6.13))

$$I_{12}(t) = N_1E_5(t) + N_2I_1(t) + N_3I_8(t) + I_7(t). \tag{7.8}$$

Using (7.7) and the same computations as in Section 3 (for (7.1) with $\tilde\theta$ instead of θ), we obtain (instead of (3.17) and with $g_0=\int_0^{+\infty}g(s)ds$ and $\delta=\frac{k_1}{8N_2}$ in (3.5)),

$$
\begin{aligned}
&I_{12}'(t)\\
&\le -(\frac{lN_3}{2}-c)\int_0^L\psi_x^2dx-(\frac{\rho_1}{16}-\epsilon N_3)\int_0^L\varphi_t^2dx-\frac{k_1}{8}\int_0^L(\varphi_x+\psi)^2dx\\
&\quad-\Big(N_2\rho_2g_0-\frac{cN_3}{\epsilon}-c\Big)\int_0^L\psi_t^2dx\\
&\quad-\frac{1}{2\epsilon}m(x)\Big(k_2\psi_x-\int_0^{+\infty}g(s)\psi_x(t-s)ds\Big)\Big)dx-N_1k_5\int_0^L\tilde\theta_{xt}^2dx\\
&\quad+(\frac{N_1}{2}-c_{N_2})g'\circ\psi_x+c_{N_2,N_3}g\circ\psi_x+(\frac{\rho_1k_2}{k_1}-\rho_2)\int_0^L\varphi_t\psi_{xt}dx\\
&\quad+k_4\int_0^L\tilde\theta_{xt}\Big((\frac18-N_3)\psi-(\varphi_x+\psi)+N_2\int_0^{+\infty}g(s)(\psi(t)-\psi(t-s))ds\,.
\end{aligned}\tag{7.9}
$$

Using Young's inequality, (3.2), (6.6) (for ψ) and (7.3), for any $\epsilon_1>0$, we have

$$
\begin{aligned}
&k_4\int_0^L\tilde\theta_{xt}\Big((\frac18-N_3)\psi-(\varphi_x+\psi)+N_2\int_0^{+\infty}g(s)(\psi(t)-\psi(t-s))ds\\
&-\frac{1}{2\epsilon}m(x)\Big(k_2\psi_x-\int_0^{+\infty}g(s)\psi_x(t-s)ds\Big)\Big)dx\\
&\le\epsilon_1E_5(t)+c_{N_2,N_3,\epsilon,\epsilon_1}\int_0^L\tilde\theta_{xt}^2dx.
\end{aligned}
$$

Then, with the same choice of N_3, ϵ and N_2 as in Section 3, we obtain, from (7.9),

$$
\begin{aligned}
I_{12}'(t)&\le -c\Big(g\circ\psi_x+\int_0^L(\varphi_t^2+\psi_t^2+\tilde\theta_t^2+(\varphi_x+\psi)^2+\psi_x^2)dx\Big)\\
&\quad+\epsilon_1E_5(t)+cg\circ\psi_x+(\frac{\rho_1k_2}{k_1}-\rho_2)\int_0^L\varphi_t\psi_{xt}dx+c\int_0^L\tilde\theta_t^2dx\\
&\quad-(N_1k_5-c_{\epsilon_1})\int_0^L\tilde\theta_{xt}^2dx+(\frac{N_1}{2}-c)g'\circ\psi_x.
\end{aligned}\tag{7.10}
$$

On the other hand, let us set

$$
R_1(t)=\int_0^L(\rho_3\tilde\theta_t\tilde\theta+k_4\psi_x\tilde\theta+\frac{k_5}{2}\tilde\theta_x^2)dx.\tag{7.11}
$$

By differentiating R_1 and using the third equation in (7.1) (with $\tilde\theta$ instead of θ), we obtain

$$
R_1'(t)=-k_3\int_0^L\tilde\theta_x^2dx+\rho_3\int_0^L\tilde\theta_t^2dx+k_4\int_0^L\psi_x\tilde\theta_tdx.
$$

Young's inequality and the definition of E_5 then yield, for any $\epsilon_1>0$,

$$
R_1'(t)\le -k_3\int_0^L\tilde\theta_x^2dx+c_{\epsilon_1}\int_0^L\tilde\theta_t^2dx+\epsilon_1E_5(t).\tag{7.12}
$$

Let $R_2 = I_{12} + R_1$. Then, using the definition of E_5 and applying Poincaré's inequality (6.6) for $\tilde{\theta}_t$, we deduce from (7.10) and (7.12) that

$$\begin{aligned} R_2'(t) \leq -(c-2\epsilon_1)E_5(t) + cg \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \\ - (N_1 k_5 - c_{\epsilon_1}) \int_0^L \tilde{\theta}_{xt}^2 dx + (\frac{N_1}{2} - c) g' \circ \psi_x. \end{aligned} \tag{7.13}$$

Then, by choosing ϵ_1 small enough so that $c - 2\epsilon_1 > 0$, and N_1 large enough so that $\frac{N_1}{2} - c \geq 0$, $N_1 k_5 - c_{\epsilon_1} \geq 0$ and $R_2 \sim E_5$ (which is possible thanks to the definition of E_5, I_{12}, R_1 and R_2), we deduce that

$$R_2'(t) \leq -cE_5(t) + cg \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \tag{7.14}$$

which is similar to (3.18). Then the proof of Theorems 7.2 and 7.3 can be completed as in Sections 3 and 4, respectively.

7.3. **Proof of Theorem 7.4.** First, as for E_5, we have

$$E_6'(t) = \frac{1}{2} g' \circ \varphi_x - k_5 \int_0^L \tilde{\theta}_{xt}^2 dx \leq 0. \tag{7.15}$$

Now, we consider the functionals $T_1 = J_1$ and $T_2 = J_2$, where J_1 and J_2 are defined in Lemmas 6.8 and 6.9, respectively. Then, exactly as for (6.18) and (6.19) (where we used the first two equations in (6.2), which are the same as in (7.2) with θ_{xt} instead of θ_x), we have, for any $\delta > 0$,

$$\begin{aligned} T_1'(t) \leq -\rho_1 \Big(\int_0^{+\infty} g(s) ds - \delta \Big) \int_0^L \varphi_t^2 \, dx \\ + \delta \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx + c_\delta g \circ \varphi_x - c_\delta g' \circ \varphi_x, \end{aligned} \tag{7.16}$$

and for some positive constants c and $\tilde{c}$,

$$\begin{aligned} T_2'(t) \leq \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx \\ - c \int_0^L ((\varphi_x + \psi)^2 + \psi_x^2) dx + \tilde{c} \Big(\int_0^L \tilde{\theta}_t^2 dx + g \circ \varphi_x \Big). \end{aligned} \tag{7.17}$$

On the other hand, we we have the following lemma.

Lemma 7.5. *The functional*

$$T_3(t) = \rho_2 \rho_3 \int_0^L \psi_t \Big(\int_0^x \tilde{\theta}_t(y,t) dy \Big) dx - \rho_2 k_3 \int_0^L \tilde{\theta}_x \psi dx$$

satisfies, for any $\epsilon > 0$,

$$T_3'(t) \leq -(\rho_2 k_4 - \epsilon) \int_0^L \psi_t^2 + \epsilon \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx + c_\epsilon \int_0^L \tilde{\theta}_{xt}^2 dx. \tag{7.18}$$

Proof. By using the second and the third equations in (7.2) and integrating over $]0, L[$, we obtain (note that $\int_0^L \tilde{\theta}(x,t) dx = 0$, $\tilde{\theta}_{tt} = \theta_{tt}$ and $\tilde{\theta}_x = \theta_x$)

$$T_3'(t) = \rho_2 \int_0^L \psi_t \Big(\int_0^x (k_3 \tilde{\theta}_{xx}(y,t) - k_4 \psi_{xt}(y,t) + k_5 \tilde{\theta}_{xxt}(y,t)) dy \Big) dx$$

$$
\begin{aligned}
&+ \rho_3 \int_0^L (k_2\psi_{xx} - k_1(\varphi_x + \psi) - k_4\tilde{\theta}_{xt}) \Big(\int_0^x \tilde{\theta}_t(y,t)dy \Big) dx \\
&- \rho_2 k_3 \int_0^L (\tilde{\theta}_x \psi_t + \tilde{\theta}_{xt}\psi) dx \\
= &-\rho_2 k_4 \int_0^L \psi_t^2 dx + \rho_2 k_5 \int_0^L \psi_t \tilde{\theta}_{xt} dx + \rho_3 \int_0^L (-k_2\psi_x + k_4\tilde{\theta}_t)\tilde{\theta}_t dx \\
&- \rho_3 k_1 \int_0^L (\varphi_x + \psi) \Big(\int_0^x \tilde{\theta}_t(y,t)dy \Big) dx - \rho_2 k_3 \int_0^L \tilde{\theta}_{xt}\psi dx.
\end{aligned}
$$

Using Young's and Hölder's inequalities for $\int_0^x \tilde{\theta}_t(y,t)dy$ and Poincaré's inequality for $\tilde{\theta}_t$ and ψ to estimate the last four integrals, we obtain (7.18). □

Now, we come back to the proof of Theorem 7.4. Let $N_1, N_2, N_3, N_4 > 0$ and

$$
T_5(t) = N_1 E_6(t) + N_2 T_1(t) + N_3 T_2(t) + N_4 T_3(t) + T_4(t), \tag{7.19}
$$

where $T_4 = R_1$ and R_1 is defined by (7.11). Note that, exactly as for (7.12) (where we used the third equation and the boundary conditions in (7.1), which are the same as in (7.2)), T_4 satisfies, for any $\epsilon_1 > 0$,

$$
T_4'(t) \le -k_3 \int_0^L \tilde{\theta}_x^2 dx + c_{\epsilon_1} \int_0^L \tilde{\theta}_t^2 dx + \epsilon_1 E_6(t). \tag{7.20}
$$

Then, using Poincaré's inequality (6.6) for $\tilde{\theta}_t$, combining (7.15)-(7.18) and (7.20), and choosing $\delta = \frac{1}{N_2}$ and $\epsilon = \frac{1}{N_4}$ in (7.16) and (7.18), respectively, we find

$$
\begin{aligned}
T_5'(t) \le &-(cN_3 - 2) \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx + \epsilon_1 E_6(t) \\
&- (N_4\rho_2 k_4 - \rho_2 N_3 - 1) \int_0^L \psi_t^2 dx - k_3 \int_0^L \tilde{\theta}_x^2 dx \\
&- (N_2\rho_1 g_0 - \rho_1 N_3 - \rho_1) \int_0^L \varphi_t^2 dx + c_{N_2,N_3} g \circ \varphi_x \\
&- (N_1 k_5 - c_{N_3,N_4,\epsilon_1}) \int_0^L \tilde{\theta}_{xt}^2 dx + (\frac{N_1}{2} - c_{N_2}) g' \circ \varphi_x.
\end{aligned} \tag{7.21}
$$

So, we choose N_3 large enough so that $cN_3 - 2 > 0$, then N_2 and N_4 large enough so that $N_2\rho_1 g_0 - \rho_1 N_3 - \rho_1 > 0$ and $N_4\rho_2 k_4 - \rho_2 N_3 - 1 > 0$. Consequently, using again Poincaré's inequality (6.6) for $\tilde{\theta}_t$, from (7.21), we obtain

$$
\begin{aligned}
T_5'(t) \le &-c\Big(g \circ \varphi_x + \int_0^L \Big(\varphi_t^2 + \psi_t^2 + \tilde{\theta}_t^2 + (\varphi_x + \psi)^2 + \psi_x^2 + \tilde{\theta}_x^2 \Big) dx \Big) \\
&+ \epsilon_1 E_6(t) + c g \circ \varphi_x - (N_1 k_5 - c_{\epsilon_1}) \int_0^L \tilde{\theta}_{xt}^2 dx + (\frac{N_1}{2} - c) g' \circ \varphi_x.
\end{aligned}
$$

Therefore, using the definition of E_6,

$$
T_5'(t) \le -(c - \epsilon_1) E_6(t) + c g \circ \varphi_x - (N_1 k_5 - c_{\epsilon_1}) \int_0^L \tilde{\theta}_{xt}^2 dx + (\frac{N_1}{2} - c) g' \circ \varphi_x.
$$

Now, as in previous sections, choosing $\epsilon_1 < c$ and N_1 large enough so that $\frac{N_1}{2} - c \ge 0$, $N_1 k_5 - c_{\epsilon_1} \ge 0$ and $T_5 \sim E_6$, we deduce that

$$
T_5'(t) \le -c E_6(t) + c g \circ \varphi_x, \tag{7.22}
$$

which is similar to (6.23). Then the proof can be ended exactly as for (6.23) in Section 6, which shows (2.6) for E_6.

8. General comments

Comment 1. Our results hold if we consider the following Neumann boundary conditions (instead of the corresponding Dirichlet ones):

$$\varphi_x(0,t) = \varphi_x(L,t) = 0 \quad \text{for (1.1), (6.1), (6.4), (7.1)} \tag{8.1}$$

and

$$\theta_x(0,t) = \theta_x(L,t) = 0 \quad \text{for (6.1)}. \tag{8.2}$$

The energy is defined with $\tilde{\varphi}$ instead of φ in case (8.1), and with $\tilde{\theta}$ instead of θ in case (8.2), where $\tilde{\theta}$ is defined by (6.11) and

$$\tilde{\varphi}(x,t) = \varphi(x,t) - \frac{t}{L}\int_0^L \varphi_1(y)dy - \frac{1}{L}\int_0^L \varphi_0(y)dy. \tag{8.3}$$

Thanks to (6.11) and (8.3), we have $\int_0^L \tilde{\varphi}(x,t)dx = \int_0^L \tilde{\theta}(x,t)dx = 0$, and then Poincaré's inequality (6.6) is still applicable for $\tilde{\varphi}$ and $\tilde{\theta}$.

In case (6.2) and (7.2), the following boundary conditions:

$$\varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = \theta(0,t) = \theta(L,t) = 0 \tag{8.4}$$

can be considered. The energy functionals E_2 and E_6 are now defined with $\tilde{\psi}$ and θ instead of ψ and $\tilde{\theta}$, respectively, where

$$\tilde{\psi}(x,t) = \psi(x,t) - \frac{1}{L}\sqrt{\frac{\rho_2}{k_1}}\Big(\int_0^L \psi_1(y)dy\Big)\sin(\sqrt{\frac{k_1}{\rho_2}}t) - \frac{1}{L}\Big(\int_0^L \psi_0(y)dy\Big)\cos(\sqrt{\frac{k_1}{\rho_2}}t).$$

Note that systems (6.2) and (7.2) are satisfied with $\tilde{\psi}$, $\psi_0 - \frac{1}{L}\int_0^L \psi_0(y)dy$ and $\psi_1 - \frac{1}{L}\int_0^L \psi_1(y)dy$ instead of ψ, ψ_0 and ψ_1, respectively. On the other hand, by integrating the second equation of (6.2) and (7.2) over $]0,L[$ and using (8.4), we obtain that $\int_0^L \tilde{\psi}(x,t)dx = 0$, and then (6.6) is applicable for $\tilde{\psi}$. In this case (8.4), and as in [32], the functionals J_3 (Lemma 6.10) and T_3 (Lemma 7.5) are now defined by

$$J_3(t) = -\rho_2\rho_3\int_0^L \theta\Big(\int_0^x \tilde{\psi}_t(y,t)dy\Big)dx$$

and

$$T_3(t) = -\rho_2\rho_3\int_0^L \theta_t\Big(\int_0^x \tilde{\psi}_t(y,t)dy\Big)dx - \rho_2 k_3\int_0^L \theta_x\tilde{\psi}dx.$$

Comment 2. Our results remain true if we consider variable coefficients $\rho_i(x)$ $(i = 1,\dots,4)$ and $k_i(x)$ $(\ i = 1,\dots,5)$ satisfying some smallness and smoothness conditions. On the other hand, our approach can be adapted to different kind of systems with an infinite history to get their stability with kernels satisfying the weaker hypothesis (H2) (see [13]).

Comment 3. Our stability results concerning (6.1) (Theorems 6.2 and 6.3) hold for this porous thermoelastic system with an infinite history

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + k_4\theta_x = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) - k_5\theta + \int_0^{+\infty} g(s)\psi_{xx}(t-s)ds = 0,\\
\rho_3\theta_t - k_3\theta_{xx} + k_4\varphi_{xt} + k_5\psi_t = 0,\\
\varphi(0,t)=\psi(0,t)=\theta(0,t)=\varphi(L,t)=\psi(L,t)=\theta(L,t)=0,\\
\varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\quad \theta(0,t)=\theta_0(x),\\
\psi(x,-t)=\psi_0(x,t),\quad \psi_t(x,0)=\psi_1(x).
\end{gathered}
\tag{8.5}
$$

This system was considered in [22, 39] under condition (1.2) and with a finite history (that is the infinite integral in (8.5) is replaced by the finite one $\int_0^t$), and exponential and polynomial decay estimates were proved in [39] under condition (1.4), and a general decay estimate was proved in [22] under condition (1.6).

Comment 4. Our stability results hold if we consider a finite history of type $\int_0^t$ (instead of $\int_0^{+\infty}$) with $\psi_0=0$ in case (1.1), (6.1), (6.4) and (7.1), and $\varphi_0=0$ in case (6.2), (6.3) and (7.2). In this case, the restrictions (2.5) and (2.10) are automatically satisfied. In [16], the stability of (1.1) with a finite history was proved under condition (1.7) but independently of $\psi_0=0$ and (1.2). The arguments of [16] can be applied to (6.1), (6.4) and (7.1) with a finite history to get their stability under (1.7) even if $\psi_0\neq 0$ and (1.2) does not hold.

Comment 5. To the best of our knowledge, getting a decay estimate for the solution of (1.1) with (finite or infinite) history (or even with a frictional damping) acting only on the first equation is an open problem. But if we consider an infinite history on both first and second equations of (1.1), then the energy functional of (1.1) satisfies (2.6) without the restriction (1.2). Let us consider this situation

$$
\begin{gathered}
\rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + \int_0^{+\infty} g_1(s)\varphi_{xx}(t-s)ds = 0,\\
\rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) + \int_0^{+\infty} g_2(s)\psi_{xx}(t-s)ds = 0,\\
\varphi(0,t)=\psi(0,t)=\varphi(L,t)=\psi(L,t)=0,\\
\varphi(x,-t)=\varphi_0(x,t),\quad \varphi_t(x,0)=\varphi_1(x),\\
\psi(x,-t)=\psi_0(x,t),\quad \psi_t(x,0)=\psi_1(x)
\end{gathered}
\tag{8.6}
$$

under the following hypothesis:

(H5) $g_i:\mathbb{R}_+\to\mathbb{R}_+$ $(i=1,2)$ is differentiable non-increasing function such that $g_i(0)>0$,

$$
l_2 = k_2 - \int_0^{+\infty} g_2(s)ds > 0, \tag{8.7}
$$

$$
l_1 = k_1 - \int_0^{+\infty} g_1(s)ds > \frac{k_0k_1}{l_2}\int_0^{+\infty} g_1(s)ds, \tag{8.8}
$$

where k_0 is defined in (6.6).

As for (1.1) in Section 5 and following the idea of [6], we can prove that, under (H5), (8.6) is well-posed according to Theorem 5.1 by considering

$$\begin{gathered}\eta^1(x,t,s)=\varphi(x,t)-\varphi(x,t-s),\quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+,\\ \eta^2(x,t,s)=\psi(x,t)-\psi(x,t-s),\quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+,\\ \eta_0^i(x,s)=\eta^i(x,0,s),\quad U=(\varphi,\psi,\varphi_t,\psi_t,\eta^1,\eta^2)^T,\\ U_0=(\varphi_0(\cdot,0),\psi_0(\cdot,0),\varphi_1,\psi_1,\eta_0^1,\eta_0^2)^T\end{gathered}$$

and

$$\mathcal{H}=\big(H_0^1(]0,L[)\big)^2\times\big(L^2(]0,L[)\big)^2\times L_{g_1}\times L_{g_2},$$

where L_{g_i} is defined by (5.5) and endowed with the inner product (5.6) (with g_i instead of g). Then (8.6) is equivalent to (5.8), where $A(v_1,v_2,v_3,v_4,v_5,v_6)=(f_1,f_2,f_3,f_4,f_5,f_6)^T$ and

$$\begin{gathered}f_1=-v_3,\quad f_2=-v_4,\\ f_3=-\frac{k_1}{\rho_1}\partial_x(\partial_x v_1+v_2)+\frac{1}{\rho_1}\Big(\int_0^{+\infty}g_1(s)ds\Big)\partial_{xx}v_1-\frac{1}{\rho_1}\int_0^{+\infty}g_1(s)\partial_{xx}v_5(s)ds,\\ f_4=-\frac{1}{\rho_2}\Big(k_2-\int_0^{+\infty}g_2(s)ds\Big)\partial_{xx}v_2-\frac{1}{\rho_2}\int_0^{+\infty}g_2(s)\partial_{xx}v_6(s)ds+\frac{k_1}{\rho_2}(\partial_x v_1+v_2),\\ f_5=-v_3+\partial_s v_5,\quad f_6=-v_4+\partial_s v_6.\end{gathered}$$

The domain $D(A)$ of A (endowed with the graph norm (5.9)) is given by $D(A)=\{V\in\mathcal{H},\ AV\in\mathcal{H}$ and $v_5(0)=v_6(0)=0\}$. The proof of Theorem 5.1 for (8.6) can be completed as in Section 5 by proving that A is a maximal monotone operator.

Now, the energy functional associated with (8.6) is defined by

$$\begin{aligned}E_8(t)=\frac12\int_0^L\Big(&\rho_1\varphi_t^2+\rho_2\psi_t^2-\Big(\int_0^{+\infty}g_1(s)ds\Big)\varphi_x^2\\ &+\Big(k_2-\int_0^{+\infty}g_2(s)ds\Big)\psi_x^2+k_1(\varphi_x+\psi)^2\Big)dx+\frac12(g_1\circ\varphi_x+g_2\circ\psi_x).\end{aligned}\tag{8.9}$$

Remark 8.1. Similarly to Remark 6.1, we mention here that (6.6) implies

$$k_1\int_0^L(\varphi_x+\psi)^2dx\ge k_1(1-\epsilon)\int_0^L\varphi_x^2dx+k_0k_1(1-\frac{1}{\epsilon})\int_0^L\psi_x^2dx,$$

for any $0<\epsilon<1$. Then, thanks to (8.7) and (8.8), we can choose $\frac{k_0k_1}{l_2+k_0k_1}<\epsilon<\frac{l_1}{k_1}$ and obtain

$$\begin{aligned}\hat c\int_0^L(\varphi_x^2+\psi_x^2)dx\le\int_0^L\Big(&-\Big(\int_0^{+\infty}g_1(s)ds\Big)\varphi_x^2\\ &+\Big(k_2-\int_0^{+\infty}g_2(s)ds\Big)\psi_x^2+k_1(\varphi_x+\psi)^2\Big)dx,\end{aligned}\tag{8.10}$$

where $\hat c=\min\{l_1-\epsilon k_1,l_2+(1-\frac{1}{\epsilon})k_0k_1\}>0$. This implies that the expression

$$\int_0^L\Big(-\Big(\int_0^{+\infty}g_1(s)ds\Big)\varphi_x^2+\Big(k_2-\int_0^{+\infty}g_2(s)ds\Big)\psi_x^2+k_1(\varphi_x+\psi)^2\Big)dx$$

defines a norm on $\big(H_0^1(]0,L[)\big)^2$, for (φ,ψ), equivalent to the one induced by $\big(H^1(]0,L[)\big)^2$. Consequently, the energy E_8 defines a norm on $\mathcal{H}$ for U, and therefore, $\mathcal{H}$ equipped with the inner product that induces this energy norm is a Hilbert space.

Theorem 8.2. *Assume that* (H5) *holds and g_i ($i=1,2$) satisfies* (H2) *(instead of g). Let $U_0 \in \mathcal{H}$ such that*

$$\exists M_i \geq 0: \ \|\eta_{0x}^i(s)\|_{L^2(]0,L[)} < M_i, \quad \forall s > 0 \quad (i=1,2). \tag{8.11}$$

Then there exist positive constants c', c'', ϵ_0 (depending continuously on $E_8(0)$) for which E_8 satisfies (2.6).

Proof. First, as for (1.1), we have that E_8 satisfies

$$E_8'(t) = \frac{1}{2} g_1' \circ \varphi_x + \frac{1}{2} g_2' \circ \psi_x \leq 0.$$

Second, we consider the functionals

$$\begin{gathered}
D_1(t) = -\rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)(\varphi(t) - \varphi(t-s))\,ds\,dx, \\
D_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)(\psi(t) - \psi(t-s))\,ds\,dx, \\
D_3(t) = \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) dx.
\end{gathered}$$

As in the previous sections, we can prove that, for any positive constant δ, $D_1 - D_3$ satisfy

$$\begin{aligned}
D_1'(t) \leq \delta \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx - \rho_1 \Big(\int_0^{+\infty} g_1(s) ds - \delta \Big) \int_0^L \varphi_t^2 dx \\
+ c_\delta (g_1 \circ \varphi_x - g_1' \circ \varphi_x),
\end{aligned} \tag{8.12}$$

$$\begin{aligned}
D_2'(t) \leq \delta \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx - \rho_2 \Big(\int_0^{+\infty} g_2(s) ds - \delta \Big) \int_0^L \psi_t^2 dx \\
+ c_\delta (g_2 \circ \psi_x - g_2' \circ \psi_x)
\end{aligned} \tag{8.13}$$

and, for some positive constants δ_1 and δ_2,

$$\begin{aligned}
D_3'(t) \leq \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) dx - \delta_1 \int_0^L (\psi_x^2 + (\varphi_x + \psi)^2) dx \\
+ \delta_2 (g_1 \circ \varphi_x + g_2 \circ \psi_x).
\end{aligned} \tag{8.14}$$

Now, let $g_0 = \min\big\{ \int_0^{+\infty} g_1(s) ds, \int_0^{+\infty} g_2(s) ds \big\}$, $N_1, N_2 > 0$ and

$$D_4 = N_1 E_8 + N_2 (D_1 + D_2) + D_3.$$

By combining (8)–(8.14) and taking $\delta = \frac{1}{N_2^2}$ in (8.12) and (8.13), we obtain

$$\begin{aligned}
D_4'(t) &\leq -(\delta_1 - \frac{2}{N_2}) \int_0^L \psi_x^2 dx - (\delta_1 - \frac{2}{N_2}) \int_0^L (\varphi_x + \psi)^2 dx \\
&\quad - \rho_1 \Big(N_2 g_0 - \frac{1}{N_2} - 1 \Big) \int_0^L \varphi_t^2 dx - \rho_2 \Big(N_2 g_0 - \frac{1}{N_2} - 1 \Big) \int_0^L \psi_t^2 dx \\
&\quad + (\frac{N_1}{2} - c_{N_2})(g_1' \circ \varphi_x + g_2' \circ \psi_x) + c_{N_2} (g_1 \circ \varphi_x + g_2 \circ \psi_x).
\end{aligned}$$

At this point, we choose N_2 large enough so that

$$\min\{\delta_1 - \frac{2}{N_2}, N_2 g_0 - \frac{1}{N_2} - 1\} > 0.$$

Using (6.6) (for φ and ψ) and (8.10), we can find that there exists a positive constant M_{N_2} (depending on N_2) such that

$$(N_1 - M_{N_2})E_8 \le D_4 \le (N_1 + M_{N_2})E_8. \tag{8.15}$$

Thus, choosing N_1 large enough so that $\frac{N_1}{2} - c_{N_2} \ge 0$ and $N_1 > M_{N_2}$,

$$D_4'(t) \le -c\int_0^L \Big(\varphi_t^2 + \psi_t^2 + \psi_x^2 + (\varphi_x + \psi)^2\Big)dx + c(g_1 \circ \varphi_x + g_2 \circ \psi_x). \tag{8.16}$$

Then, by using (8.9), Equation (8.16) implies

$$D_4'(t) \le -cE_8(t) + c(g_1 \circ \varphi_x + g_2 \circ \psi_x). \tag{8.17}$$

Using (H2) and (8.11), we have (as for (3.19) and (6.24))

$$\begin{gathered}
G'(\epsilon_0 E_8(t))g_1 \circ \varphi_x \le -cE_8'(t) + c\epsilon_0 E_8(t)G'(\epsilon_0 E_8(t)), \quad \forall t \in \mathbb{R}_+,\ \forall \epsilon_0 > 0, \\
G'(\epsilon_0 E_8(t))g_2 \circ \psi_x \le -cE_8'(t) + c\epsilon_0 E_8(t)G'(\epsilon_0 E_8(t)), \quad \forall t \in \mathbb{R}_+,\ \forall \epsilon_0 > 0.
\end{gathered}$$

The proof of Theorem 8.2 can be finalized as in Section 3. □

Comment 6. For (1.1), (6.1) and (7.1) when (1.2) does not hold, and for (6.4), the estimate (2.11) proved in Theorems 2.4, 6.3, 6.5 and 7.3 can be generalized by giving a relationship between the smoothness of the initial data and the decay rate of the energy. Indeed, let us consider the case (1.1). We have the following result.

Theorem 8.3. *Assume that* (H1) *and* (H2) *hold and let $n \in \mathbb{N}^*$ and $U_0 \in D(A^n)$ satisfying*

$$\exists M_0 \ge 0 : \max_{m \in \{0,\dots,n\}} \{\|\partial_s^{(m)} \eta_{0x}(s)\|_{L^2(]0,L[)}\} \le M_0, \quad \forall s > 0. \tag{8.18}$$

Then there exist positive constants c_n and ϵ_0 (depending continuously on $\|U_0\|_{D(A^n)}$) such that

$$E(t) \le \phi_n(\frac{c_n}{t}), \quad \forall t > 0, \tag{8.19}$$

where $\phi_1 = G_0^{-1}$, G_0 is defined by (2.12) *and $\phi_m(s) = G_0^{-1}(s\phi_{m-1}(s))$, for $m = 2, 3, \dots, n$ and $s \in \mathbb{R}_+$.*

Remark 8.4. 1. Under the hypotheses of Theorem 8.3, E_1, E_4 and E_5 satisfy (8.19). The proof is exactly the same one given below.

2. The estimate (8.19) is weaker than

$$E(t) \le \frac{c_n}{t^n}, \quad \forall t > 0. \tag{8.20}$$

The estimate (8.19) coincides with (8.20) when $G = Id$, and (8.19) generalizes (8.20) proved in [30] (under (1.3)) and the one proved in [28, 29] (under (1.4) and $n = 1$).

Example 8.5. Let us consider here a simple example to illustrate how the smoothness of U_0 improves the decay rate in (8.19). Let $g(t) = d/(1+t)^q$, for $q > 1$, and $d > 0$ small enough so that (2.1) is satisfied. The classical condition (1.4) is not satisfied if $1 < q \le 2$, while (H2) holds with $G(t) = t^p$, for any $q > 1$ and $p > \frac{q+1}{q-1}$.

Then $\phi_n(s) = cs^{r_n}$, where c is some positive constant and $r_n = \sum_{m=1}^{n} \frac{1}{p^m}$. Therefore, (8.19) takes the form

$$E(t) \le \frac{c_n}{t^{r_n}}, \quad \forall t > 0,\ \forall p > \frac{q+1}{q-1}. \tag{8.21}$$

The decay rate r_n increases when n increases or p decreases, and it converges to n (which is the decay rate in (8.20)) when p converges to 1 (that is, when q converges to $+\infty$).

Proof of Theorem 8.3. We prove (8.19) by induction on n. For $n = 1$, condition (8.18) coincides with (2.10), and (8.19) is exactly (2.11) given in Theorem 2.4 and proved in Section 4.

Now, suppose that (8.19) holds and let $U_0 \in D(A^{n+1})$ satisfying (8.18), for $n+1$ instead of n. We have $U_t(0) \in D(A^n)$ (thanks to Theorem 5.1), $U_t(0)$ satisfies (8.18) (because U_0 satisfies (8.18), for $n+1$) and U_t satisfies the first two equations and the boundary conditions of (1.1), and then the energy $\tilde{E}$ of (4.1) (defined in Section 4) also satisfies

$$\tilde{E}(t) \le \phi_n\big(\frac{\tilde{c}_n}{t}\big), \quad \forall t > 0, \tag{8.22}$$

where $\tilde{c}_n$ is a positive constant depending continuously on $\|U_0\|_{D(A^{n+1})}$. Now, integrating (4.7) over $[T, 2T]$, for $T \in \mathbb{R}_+$, and using the fact that $F \sim E$ and $G_0(E)$ is non-increasing, we deduce that

$$G_0(E(2T))T \le \int_T^{2T} E(t)G'(\epsilon_0 E(t))dt \le c(E(T) + \tilde{E}(T)). \tag{8.23}$$

By combining (8.19), (8.22) and (8.23), we obtain that for all $T > 0$,

$$E(2T) \le G_0^{-1}\Big(\frac{2c}{2T}\Big(\phi_n\big(\frac{2c_n}{2T}\big) + \phi_n\big(\frac{2\tilde{c}_n}{2T}\big)\Big)\Big),$$

which implies, for $t = 2T$ and $c_{n+1} = \max\{2c_n, 2\tilde{c}_n, 4c\}$ (note that G_0^{-1} and ϕ_n are non-decreasing),

$$E(t) \le G_0^{-1}\Big(\frac{c_{n+1}}{t}\phi_n\big(\frac{c_{n+1}}{t}\big)\Big) = \phi_{n+1}\big(\frac{c_{n+1}}{t}\big), \quad \forall t > 0.$$

This proves (8.19), for $n + 1$. The proof of Theorem 8.3 is complete □

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