Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 213, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

POSITIVE SOLUTIONS TO BOUNDARY-VALUE PROBLEMS OF P-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS WITH A PARAMETER IN THE BOUNDARY

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Abstract. In this article, we consider the following boundary-value problem of nonlinear fractional differential equation with p-Laplacian operator

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha}u(t)))+a(t)f(u)=0,\quad 0<t<1,$$
$$u(0)=\gamma u(\xi)+\lambda,\quad \phi_p(D_{0+}^{\alpha}u(0))=(\phi_p(D_{0+}^{\alpha}u(1)))'=(\phi_p(D_{0+}^{\alpha}u(0)))''=0,$$

where $0<\alpha\leqslant 1$, $2<\beta\leqslant 3$ are real numbers, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Caputo fractional derivatives, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_q$, $1/p+1/q=1$, $0\leqslant\gamma<1$, $0\leqslant\xi\leqslant 1$, $\lambda>0$ is a parameter, $a:(0,1)\to[0,+\infty)$ and $f:[0,+\infty)\to[0,+\infty)$ are continuous. By the properties of Green function and Schauder fixed point theorem, several existence and nonexistence results for positive solutions, in terms of the parameter λ are obtained. The uniqueness of positive solution on the parameter λ is also studied. Some examples are presented to illustrate the main results.

1. Introduction

Fractional differential equations have been of great interest. The motivation for those works stems from both the intensive development of the theory of fractional calculus itself and the applications such as economics, engineering and other fields [1, 11, 17, 18, 19, 20, 24].

Recently, much attention has been focused on the study of the existence and multiplicity of solutions or positive solutions for boundary-value problems of fractional differential equations by the use of techniques of nonlinear analysis (fixed point theorems [2, 3, 4, 8, 21, 23, 25, 26, 28, 29, 30, 32], upper and lower solutions method [13, 15, 22], fixed point index [7, 27], coincidence theory [5], Banach contraction mapping principle [14], etc).

Ma [16] considered the boundary-value problem

$$u''+a(t)f(u)=0,\quad 0<t<1,$$
$$u(0)=0,\quad u(1)-\alpha u(\eta)=b,$$

2000 *Mathematics Subject Classification.* 34A08, 34B18, 35J05.
Key words and phrases. Fractional boundary-value problem; positive solution; cone; Schauder fixed point theorem; uniqueness; p-Laplacian operator.

Submitted September 5, 2012. Published November 27, 2012.

where $b, \alpha > 0$, $\eta \in (0,1)$, $\alpha\eta < 1$ are given. Under some assumptions, it was shown that there exists $b^* > 0$ such that the boundary-value problem has at least one positive solution for $0 < b < b^*$ and no positive solution for $b > b^*$.

Kong et al [12] studied the boundary-value problem with nonhomogeneous three-point boundary condition

$$\begin{gathered}(\phi_p(u'))'' + a(t)f(u) = 0, \quad 0 < t < 1, \\ u(0) = \xi u(\eta) + \lambda, \quad u'(0) = u'(1) = 0,\end{gathered}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$. $0 \leqslant \xi < 1$, $0 \leqslant \eta \leqslant 1$, $\lambda > 0$ is a parameter, $a \in C(0,1)$ and $f \in C([0,+\infty))$ are nonnegative functions. Under some assumptions, several existence, nonexistence, and multiplicity results for positive solutions in terms of different values of the parameter λ are derived.

Zhao et al [31] studied the existence of positive solutions for the boundary-value problem of nonlinear fractional differential equations

$$\begin{gathered}D_{0+}^{\alpha} u(t) + \lambda f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0,\end{gathered}$$

where $2 < \alpha \leqslant 3$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, λ is a positive parameter, and $f : (0,+\infty) \to (0,+\infty)$ is continuous. By the properties of the Green function and Guo-Krasnoselskii fixed point theorem on cones, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary-value problem are established.

Chai [3] investigated the existence and multiplicity of positive solutions for a class of boundary-value problem of fractional differential equation with p-Laplacian operator

$$\begin{gathered}D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha} u(t))) + f(t, u(t), D_{0+}^{\rho} u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) + \sigma D_{0+}^{\gamma} u(1) = 0, \quad D_{0+}^{\alpha} u(0) = 0,\end{gathered}$$

where $1 < \alpha \leq 2, 0 < \gamma \leq 1$, $0 < \gamma \leq 1$, $0 \leq \alpha - \gamma - 1$, σ is a positive constant number, $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\gamma}$ are the standard Riemann-Liouville derivatives. By means of the fixed point theorem on cones, some existence and multiplicity results of positive solutions are obtained.

Although the fractional differential equation boundary-value problems have been studied by several authors, very little is known in the literature on the existence and nonexistence of positive solutions of fractional differential equation boundary-value problems with p-Laplacian operator when a parameter λ is involved in the boundary conditions. We also mention that, there is very little known about the uniqueness of the solution of fractional differential equation boundary-value problems with p-Laplacian operator on the parameter λ. Therefore, to enrich the theoretical knowledge of the above, in this paper, we investigate the following p-Laplacian fractional differential equation boundary-value problem

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha} u(t))) + a(t)f(u) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = \gamma u(\xi) + \lambda, \quad \phi_p(D_{0+}^{\alpha} u(0)) = (\phi_p(D_{0+}^{\alpha} u(1)))' = (\phi_p(D_{0+}^{\alpha} u(0)))'' = 0, \tag{1.2}$$

where $0 < \alpha \leqslant 1$, $2 < \beta \leqslant 3$ are real numbers, $D_{0+}^{\alpha}, D_{0+}^{\beta}$ are the standard Caputo fractional derivatives, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$, $0 \leqslant \gamma < 1$, $0 \leqslant \xi \leqslant 1$, $\lambda > 0$ is a parameter, $a : (0,1) \to [0,+\infty)$ and $f : [0,+\infty) \to [0,+\infty)$ are continuous. By the properties of Green function and Schauder fixed point theorem, several new existence and nonexistence results for positive solutions in terms of different values of the parameter λ are obtained. The uniqueness of positive solution is also obtained for fractional differential equation boundary-value problem (1.1) and (1.2). As applications, examples are presented to illustrate our main results.

The rest of this paper is organized as follows. In Section 2, we shall introduce some definitions and lemmas to prove our main results. In Section 3, we investigate the existence of positive solution for boundary-value problems (1.1) and (1.2). In Section 4, the uniqueness of positive solution on the parameter λ is studied. In Section 5, we consider the nonexistence of positive solution for boundary-value problems (1.1) and (1.2). As applications, examples are presented to illustrate our main results in Section 3, Section 4 and Section 5, respectively.

2. Preliminaries and lemmas

For the convenience of the reader, we give some background material from fractional calculus theory to facilitate analysis of problem (1.1) and (1.2). These results can be found in the recent literature, see [6, 11, 20, 23].

Definition 2.1 ([11]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

provided the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([11]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0, +\infty)$.

Remark 2.3 ([20]). By Definition 2.2, under natural conditions on the function $f(t)$, for $\alpha \to n$ the Caputo derivative becomes a conventional n-th derivative of the function $f(t)$.

Remark 2.4 ([11]). As a basic example,

$$D_{0+}^{\alpha} t^{\mu} = \mu(\mu-1)\dots(\mu-n+1)\frac{\Gamma(1+\mu-n)}{\Gamma(1+\mu-\alpha)} t^{\mu-\alpha}, \quad \text{for } t \in (0, \infty).$$

In particular $D_{0+}^{\alpha} t^{\mu} = 0$, $\mu = 0, 1, \dots, n-1$, where D_{0+}^{α} is the Caputo fractional derivative, n is the smallest integer greater than or equal to α.

From the definition of the Caputo derivative and Remark 2.4, we can obtain the following statement.

Lemma 2.5 ([11]). Let $\alpha > 0$. Then the fractional differential equation

$$D_{0+}^{\alpha} u(t) = 0$$

has a unique solution

$$u(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R},\ i = 0, 1, 2, \dots, n-1,$$

where n is the smallest integer greater than or equal to α.

Lemma 2.6 ([11])**.** Let $\alpha > 0$. Assume that $u \in C^n[0,1]$. Then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.7. *Let $y \in C[0,1]$ and $0 < \alpha \leqslant 1$. Then fractional differential equation boundary-value problem*

$$D_{0+}^{\alpha} u(t) = y(t), \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = \gamma u(\xi) + \lambda \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\gamma}{1-\gamma} \int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\lambda}{1-\gamma}.$$

Proof. We apply Lemma 2.6 to reduce (2.1) to an equivalent integral equation,

$$u(t) = I_{0+}^{\alpha} y(t) + c_0, \quad c_0 \in \mathbb{R}.$$

Consequently, the general solution of (2.1) is

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + c_0, \quad c_0 \in \mathbb{R}.$$

By (2.2), we has

$$c_0 = \frac{\gamma}{1-\gamma} \int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\lambda}{1-\gamma}.$$

Therefore, the unique solution of problem (2.1) and (2.2) is

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\gamma}{1-\gamma} \int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \frac{\lambda}{1-\gamma}.$$

□

Lemma 2.8. *Let $y \in C[0,1]$ and $0 < \alpha \leqslant 1$, $2 < \beta \leqslant 3$. Then fractional differential equation boundary-value problem*

$$D_{0+}^{\beta}(\phi_p(D_{0+}^{\alpha} u(t))) + y(t) = 0, \quad 0 < t < 1, \tag{2.3}$$

$$u(0) = \gamma u(\xi) + \lambda, \quad \phi_p(D_{0+}^{\alpha} u(0)) = (\phi_p(D_{0+}^{\alpha} u(1)))' = (\phi_p(D_{0+}^{\alpha} u(0)))'' = 0 \tag{2.4}$$

has a unique solution

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^1 H(s,\tau) y(\tau) d\tau\Big) ds \\ &+ \frac{\gamma}{1-\gamma} \int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^1 H(s,\tau) y(\tau) d\tau\Big) ds + \frac{\lambda}{1-\gamma}, \end{aligned} \tag{2.5}$$

where

$$H(t,s) = \begin{cases} \frac{(\beta-1)t(1-s)^{\beta-2} - (t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{(\beta-1)t(1-s)^{\beta-2}}{\Gamma(\beta)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.6}$$

Proof. From Lemma 2.6, the boundary-value problem (2.3) and (2.4) is equivalent to the integral equation

$$\phi_p(D_{0+}^{\alpha}u(t)) = -I_{0+}^{\beta}y(t) + c_0 + c_1t + c_2t^2,$$

for some $c_0, c_1, c_2 \in \mathbb{R}$; that is,

$$\phi_p(D_{0+}^{\alpha}u(t)) = -\int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)d\tau + c_0 + c_1t + c_2t^2.$$

By the boundary conditions $\phi_p(D_{0+}^{\alpha}u(0)) = (\phi_p(D_{0+}^{\alpha}u(1)))' = (\phi_p(D_{0+}^{\alpha}u(0)))'' = 0$, we have

$$c_0 = c_2 = 0, \quad c_1 = \int_0^1 \frac{(\beta-1)(1-\tau)^{\beta-2}}{\Gamma(\beta)}y(\tau)d\tau.$$

Therefore, the solution $u(t)$ of fractional differential equation boundary-value problem (2.3) and (2.4) satisfies

$$\begin{aligned}\phi_p(D_{0+}^{\alpha}u(t)) &= -\int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)}y(\tau)d\tau + \int_0^1 \frac{(\beta-1)t(1-\tau)^{\beta-2}}{\Gamma(\beta)}y(\tau)d\tau \\ &= \int_0^1 H(t,\tau)y(\tau)d\tau.\end{aligned}$$

Consequently, $D_{0+}^{\alpha}u(t) = \phi_q\Big(\int_0^1 H(t,\tau)y(\tau)d\tau\Big)$. Thus, fractional differential equation boundary-value problem (2.3) and (2.4) is equivalent to the problem

$$\begin{gathered}D_{0+}^{\alpha}u(t) = \phi_q\Big(\int_0^1 H(t,\tau)y(\tau)d\tau\Big), \quad 0<t<1, \\ u(0) = \gamma u(\xi) + \lambda.\end{gathered}$$

Lemma 2.7 implies that fractional differential equation boundary-value problem (2.3) and (2.4) has a unique solution,

$$\begin{aligned}u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds \\ &\quad + \frac{\gamma}{1-\gamma}\int_0^{\xi} \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds + \frac{\lambda}{1-\gamma}.\end{aligned}$$

The proof is complete. □

Lemma 2.9 ([23])**.** *Let $0 < \alpha \leqslant 1, 2 < \beta \leqslant 3$. The function $H(t,s)$ defined by (2.6) is continuous on $[0,1]\times[0,1]$ and satisfies*

1. *$H(t,s) \geqslant 0, H(t,s) \leqslant H(1,s)$, for $t,s \in [0,1]$;*
2. *$H(t,s) \geqslant t^{\beta-1}H(1,s)$, for $t,s \in (0,1)$.*

Lemma 2.10 (Schauder fixed point theorem [6])**.** *Let (E,d) be a complete metric space, U be a closed convex subset of E, and $A: U \to U$ be a mapping such that the set $\{Au : u \in U\}$ is relatively compact in E. Then A has at least one fixed point.*

To prove our main results, we use the following assumptions.

(H1) $0 < \int_0^1 H(1,\tau)a(\tau)d\tau < +\infty$;

(H2) there exist $0 < \sigma < 1$ and $c > 0$ such that
$$f(x) \leqslant \sigma L \phi_p(x), \quad \text{for } 0 \leqslant x \leqslant c, \tag{2.7}$$
where L satisfies
$$0 < L \leqslant \Big[\phi_p\Big(\frac{1+\gamma(\xi^\alpha - 1)}{\Gamma(\alpha+1)(1-\gamma)}\Big)\int_0^1 H(1,\tau)a(\tau)d\tau\Big]^{-1}; \tag{2.8}$$
(H3) there exist $d > 0$ such that
$$f(x) \leqslant M\phi_p(x), \quad \text{for } d < x < +\infty, \tag{2.9}$$
where M satisfies
$$0 < M < \Big[\phi_p\Big(\frac{1+\gamma(\xi^\alpha - 1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}\Big)\int_0^1 H(1,\tau)a(\tau)d\tau\Big]^{-1}; \tag{2.10}$$
(H4) there exist $0 < \delta < 1$ and $e > 0$ such that
$$f(x) \geqslant N\phi_p(x), \quad \text{for } e < x < +\infty, \tag{2.11}$$
where N satisfies
$$N > \Big[\phi_p\Big(c_\delta \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(s^{\beta-1})ds\Big)\int_\delta^1 H(1,\tau)a(\tau)d\tau\Big]^{-1}; \tag{2.12}$$
with
$$c_\delta = \int_0^\delta \alpha(1-s)^{\alpha-1}\phi_q(s^{\beta-1})ds \in (0,1); \tag{2.13}$$
(H5) $f(x)$ is nondecreasing in x;
(H6) there exist $0 \leqslant \theta < 1$ such that
$$f(kx) \geqslant (\phi_p(k))^\theta f(x), \quad \text{for any } 0 < k < 1 \text{ and } 0 < x < +\infty. \tag{2.14}$$

Remark 2.11. Let
$$f_0 = \lim_{x\to 0^+}\frac{f(x)}{\phi_p(x)}, \quad f_\infty = \lim_{x\to+\infty}\frac{f(x)}{\phi_p(x)}.$$
Then, (H2) holds if $f_0 = 0$, (H3) holds if $f_\infty = 0$, and (H4) holds if $f_\infty = +\infty$.

3. Existence

Theorem 3.1. *Assume that* (H1), (H2) *hold. Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has at least one positive solution for* $0 < \lambda \leqslant (1-\gamma)(1-\phi_q(\sigma))c$.

Proof. Let $c > 0$ be given in (H2). Define
$$K_1 = \{u \in C[0,1] : 0 \leqslant u(t) \leqslant c \text{ on } [0,1]\}$$
and an operator $T_\lambda : K_1 \to C[0,1]$ by
$$\begin{aligned} T_\lambda u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\ &\quad + \frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds + \frac{\lambda}{1-\gamma}. \end{aligned} \tag{3.1}$$
Then, K_1 is a closed convex set. From Lemma 2.8, u is a solution of fractional differential equation boundary-value problem (1.1) and (1.2) if and only if u is a fixed point of T_λ. Moreover, a standard argument can be used to show that T_λ is compact.

For any $u \in K_1$, from (2.7) and (2.8), we obtain

$$f(u(t)) \leqslant \sigma L\phi_p(u(t)) \leqslant \sigma L\phi_p(c), \quad \text{on } [0,1],$$

and

$$\frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q(L)\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)d\tau\Big) \leqslant 1.$$

Let $0 < \lambda \leqslant (1-\gamma)(1-\phi_q(\sigma))c$. Then, from Lemma 2.9 and (3.1), it follows that

$$\begin{aligned}
0 \leqslant T_\lambda u(t) &\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&\quad + \frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)ds + \frac{\lambda}{1-\gamma} \\
&\leqslant \frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big) \\
&\quad + \frac{\gamma\xi^\alpha}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big) + (1-\phi_q(\sigma))c \\
&= \frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big) + (1-\phi_q(\sigma))c \\
&\leqslant \frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q(L)\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)d\tau\Big)\phi_q(\sigma)c + (1-\phi_q(\sigma))c \\
&\leqslant \phi_q(\sigma)c + (1-\phi_q(\sigma))c = c, \quad t \in [0,1].
\end{aligned}$$

Thus, $T_\lambda(K_1) \subseteq K_1$, By Schauder fixed point theorem, T_λ has a fixed point $u \in K_1$; that is, the fractional differential equation boundary-value problem (1.1) and (1.2) has at least one positive solution. The proof is complete. □

Corollary 3.2. *Assume that* (H1) *holds and* $f_0 = 0$. *Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has at least one positive solution for sufficiently small* $\lambda > 0$.

Theorem 3.3. *Assume that* (H1), (H3) *hold. Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has at least one positive solution for all* $\lambda > 0$.

Proof. Let $\lambda > 0$ be fixed and $d > 0$ be given in (H3). Define $D = \max_{0\leqslant x\leqslant d} f(x)$. Then

$$f(x) \leqslant D, \quad \text{for } 0 \leqslant x \leqslant d. \tag{3.2}$$

From (2.10), we have

$$\frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}\phi_q(M)\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)d\tau\Big) < 1.$$

Thus, there exists $d^* > d$ large enough so that

$$\frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}(\phi_q(D)+\phi_q(M)d^*)\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)d\tau\Big) + \frac{\lambda}{1-\gamma} \leqslant d^*. \tag{3.3}$$

Let

$$K_2 = \{u \in C[0,1] : 0 \leqslant u(t) \leqslant d^* \text{ on } [0,1]\}.$$

For $u \in K_2$, define

$$I_1^u = \{t \in [0,1] : 0 \leqslant u(t) \leqslant d\},$$
$$I_2^u = \{t \in [0,1] : d < u(t) \leqslant d^*\}.$$

Then, $I_1^u \cup I_2^u = [0,1], I_1^u \cap I_2^u = \emptyset$, and in view of (2.9), we have

$$f(u(t)) \leqslant M\phi_p(u(t)) \leqslant M\phi_p(d^*), \quad \text{for } t \in I_2^u. \tag{3.4}$$

Let the compact operator T_λ be defined by (3.1). Then from Lemma 2.9, (2.9) and (3.2), we have

$$\begin{aligned}
&0 \leqslant T_\lambda u(t)\\
&\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\\
&\quad + \frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)ds + \frac{\lambda}{1-\gamma}\\
&\leqslant \frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)\\
&\quad + \frac{\gamma\xi^\alpha}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big) + \frac{\lambda}{1-\gamma}\\
&= \frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\Big(\int_{I_1^u} H(1,\tau)a(\tau)f(u(\tau))d\tau + \int_{I_2^u} H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)\\
&\quad + \frac{\lambda}{1-\gamma}\\
&\leqslant \frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q\Big(D\int_{I_1^u} H(1,\tau)a(\tau)d\tau + M\phi_p(d^*)\int_{I_2^u} H(1,\tau)a(\tau)d\tau\Big)\\
&\quad + \frac{\lambda}{1-\gamma}\\
&\leqslant \frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}\phi_q(D+M\phi_p(d^*))\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)d\tau\Big) + \frac{\lambda}{1-\gamma}.
\end{aligned}$$

From (3.3) and the inequality $(a+b)^r \leqslant 2^r(a^r+b^r)$ for any $a,b,r>0$ (see, for example, [10]), we obtain

$$\begin{aligned}
&0 \leqslant T_\lambda u(t)\\
&\leqslant \frac{1+\gamma(\xi^\alpha-1)}{\Gamma(\alpha+1)(1-\gamma)}2^{q-1}(\phi_q(D)+\phi_q(M)d^*)\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)d\tau\Big) + \frac{\lambda}{1-\gamma} \leqslant d^*.
\end{aligned}$$

Thus, $T_\lambda : K_2 \to K_2$. Consequently, by Schauder fixed point theorem, T_λ has a fixed point $u \in K_2$, that is, the fractional differential equation boundary-value problem (1.1) and (1.2) has at least one positive solution. The proof is complete. □

Corollary 3.4. *Assume that* (H1) *holds and* $f_\infty = 0$. *Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has at least one positive solution for all* $\lambda > 0$.

Example 3.5. Consider the boundary-value problem

$$D_{0+}^{5/2}(\phi_p(D_{0+}^{1/2}u(t))) + tu^2 = 0, \quad 0 < t < 1, \tag{3.5}$$

$$u(0)=\frac{1}{2}u(\frac{1}{2})+\lambda,\quad \phi_p(D_{0+}^{1/2}u(0))=(\phi_p(D_{0+}^{1/2}u(1)))'=(\phi_p(D_{0+}^{1/2}u(0)))''=0. \tag{3.6}$$

Let $p=2$. We have $\alpha=1/2$, $\beta=\frac{5}{2}$, $\gamma=\xi=1/2$, $a(t)=t$, $f(u)=u^2$. Clearly, (H1) holds.

By a simple computation, we obtain $0<L\leqslant 2.4155$. Choosing $\sigma=1/2$, $c=1$, $L=2$, (H2) is satisfied. Thus, by Theorem 3.1 the fractional differential equation boundary-value problem (3.5) and (3.6) has at least one positive solution for $0<\lambda\leqslant 1/4$.

Example 3.6. Consider the boundary-value problem

$$D_{0+}^{5/2}(\phi_p(D_{0+}^{1/2}u(t)))+t\sqrt{u}=0,\quad 0<t<1, \tag{3.7}$$

$$u(0)=\frac{1}{2}u(\frac{1}{2})+\lambda,\quad \phi_p(D_{0+}^{1/2}u(0))=(\phi_p(D_{0+}^{1/2}u(1)))'=(\phi_p(D_{0+}^{1/2}u(0)))''=0. \tag{3.8}$$

Let $p=2$. We have $\alpha=1/2$, $\beta=5/2$, $\gamma=\xi=1/2$, $a(t)=t$, $f(u)=\sqrt{u}$. Clearly, (H1) holds. By a simple computation, we obtain $0<M<1.2077$. Choosing $d=1,M=1$, then (H3) is satisfied. Thus, by Theorem 3.3 the fractional differential equation boundary-value problem (3.7) and (3.8) has at least one positive solution for all $\lambda>0$.

4. Uniqueness

Definition 4.1 ([9]). A cone P in a real Banach space X is called solid if its interior P^o is not empty.

Definition 4.2 ([9]). Let P be a solid cone in a real Banach space $X,T:P^o\to P^o$ be an operator, and $0\leqslant\theta<1$. Then T is called a θ-concave operator if

$$T(ku)\geqslant k^\theta Tu\quad\text{for any } 0<k<1 \text{ and } u\in P^o.$$

Lemma 4.3 ([9, Theorem 2.2.6]). *Assume that P is a normal solid cone in a real Banach space X, $0\leqslant\theta<1$, and $T:P^o\to P^o$ is a θ-concave increasing operator. Then T has only one fixed point in P^o.*

Theorem 4.4. *Assume that* (H1), (H5), (H6) *hold. Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has a unique positive solution for any $\lambda>0$.*

Proof. Define $P=\{u\in C[0,1]:u(t)\geqslant 0\text{on } [0,1]\}$. Then P is a normal solid cone in $C[0,1]$ with

$$P^o=\{u\in C[0,1]:u(t)>0\ \text{ on } [0,1]\}.$$

For any fixed $\lambda>0$, let $T_\lambda:P\to C[0,1]$ be defined by (3.1). Define $T:P\to C[0,1]$ by

$$\begin{aligned}Tu(t)=&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\\&+\frac{\gamma}{1-\gamma}\int_0^\xi\frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\end{aligned}$$

Then from (H5), we have T is increasing in $u \in P^o$ and

$$T_\lambda u(t) = Tu(t) + \frac{\lambda}{1-\gamma}.$$

Clearly, $T_\lambda : P^o \to P^o$. Next, we prove that T_λ is a θ-concave increasing operator. In fact, for $u_1, u_2 \in P$ with $u_1(t) \geqslant u_2(t)$ on $[0,1]$, we obtain

$$T_\lambda u_1(t) \geqslant Tu_2(t) + \frac{\lambda}{1-\gamma} = T_\lambda u_2(t);$$

i.e., T_λ is increasing. Moreover, (H6) implies

$$\begin{aligned}
T_\lambda(ku)(t) &\geqslant k^\theta \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&\quad + k^\theta \frac{\gamma}{1-\gamma} \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds + \frac{\lambda}{1-\gamma} \\
&= k^\theta Tu(t) + \frac{\lambda}{1-\gamma} \\
&\geqslant k^\theta (Tu(t) + \frac{\lambda}{1-\gamma}) = k^\theta T_\lambda u(t);
\end{aligned}$$

i.e., T_λ is θ-concave. By Lemma 4.3, T_λ has a unique fixed point u_λ in P^o, that is, the fractional differential equation boundary-value problem (1.1) and (1.2) has a unique positive solution. The proof is complete. □

Example 4.5. Consider the boundary-value problem

$$D_{0+}^{5/2}(\phi_p(D_{0+}^{1/2}u(t))) + t^2\sqrt[3]{u} = 0, \quad 0 < t < 1, \tag{4.1}$$

$$u(0) = \frac{1}{2}u(\frac{1}{2}) + \lambda, \quad \phi_p(D_{0+}^{1/2}u(0)) = (\phi_p(D_{0+}^{1/2}u(1)))' = (\phi_p(D_{0+}^{1/2}u(0)))'' = 0. \tag{4.2}$$

Let $p = 2$. We have $\alpha = 1/2$, $\beta = 5/2$, $\gamma = \xi = 1/2$, $a(t) = t^2$, $f(u) = \sqrt[3]{u}$. Clearly, (H1) and (H_5) hold. Choosing $\theta = 1/2$, then (H6) is satisfied. Thus, by Theorem 4.4 the fractional differential equation boundary-value problem (4.1) and (4.2) has a unique positive solution for any $\lambda > 0$.

5. Nonexistence

In this section, we let the Banach space $C[0,1]$ be endowed with the norm $\|u\| = \max_{0\leqslant t\leqslant 1} |u(t)|$.

Lemma 5.1. *Assume* (H1) *holds and let* $0 < \delta < 1$ *be given in* (H4). *Then the unique solution* $u(t)$ *of fractional differential equation boundary-value problem* (2.3) *and* (2.4) *satisfies*

$$u(t) \geqslant c_\delta \|u\| \quad \text{for } \delta \leqslant t \leqslant 1,$$

where c_δ *is defined by* (2.13).

Proof. In view of Lemma 2.9 and (2.5), we have

$$\begin{aligned}
u(t) &\leqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^1 H(1,\tau)y(\tau)d\tau\Big)ds \\
&\quad + \frac{\gamma}{1-\gamma} \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds + \frac{\lambda}{1-\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\int_0^1 H(1,\tau)y(\tau)d\tau\Big)\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds+\frac{\lambda}{1-\gamma}
\end{aligned}
$$

for $t\in[0,1]$, and

$$
\begin{aligned}
u(t)&\geqslant \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 s^{\beta-1}H(1,\tau)y(\tau)d\tau\Big)ds\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds+\frac{\lambda}{1-\gamma}\\
&=\int_0^t \alpha(t-s)^{\alpha-1}\phi_q(s^{\beta-1})ds\frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\int_0^1 H(1,\tau)y(\tau)d\tau\Big)\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds+\frac{\lambda}{1-\gamma}\\
&\geqslant c_\delta\frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\int_0^1 H(1,\tau)y(\tau)d\tau\Big)\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds+\frac{\lambda}{1-\gamma}\\
&\geqslant c_\delta\Big[\frac{1}{\Gamma(\alpha+1)}\phi_q\Big(\int_0^1 H(1,\tau)y(\tau)d\tau\Big)\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds+\frac{\lambda}{1-\gamma}\Big]
\end{aligned}
$$

for $t\in[\delta,1]$. Therefore, $u(t)\geqslant c_\delta\|u\|$ for $\delta\leqslant t\leqslant 1$. The proof is complete. □

Theorem 5.2. *Assume that* (H1), (H4) *hold. Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has no positive solution for* $\lambda > (1-\gamma)e$.

Proof. Assume, to the contrary, the fractional differential equation boundary-value problem (1.1) and (1.2) has a positive solution $u(t)$ for $\lambda > (1-\gamma)e$. Then by Lemma 2.8, we have

$$
\begin{aligned}
u(t)&=\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\\
&\quad+\frac{\gamma}{1-\gamma}\int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds+\frac{\lambda}{1-\gamma}
\end{aligned}
$$

Therefore, $u(t)>e$ on [0,1]. In view of (2.11) and (2.12), we obtain

$$
\begin{gathered}
f(u(t))\geqslant N\phi_p(u(t))\quad\text{on } [0,1],\\
c_\delta\phi_q(N)\phi_q\Big(\int_\delta^1 H(1,\tau)a(\tau)d\tau\Big)\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(s^{\beta-1})ds>1.
\end{gathered}
$$

Then by Lemmas 2.9 and 5.1, we obtain

$$\begin{aligned}\|u\| = u(1) &> \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\Big(\int_0^1 H(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\\ &\geqslant \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(s^{\beta-1})ds\phi_q\Big(\int_0^1 H(1,\tau)a(\tau)f(u(\tau))d\tau\Big)\\ &\geqslant \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(s^{\beta-1})ds\phi_q(N)\phi_q\Big(\int_\delta^1 H(1,\tau)a(\tau)\phi_p(u(\tau))d\tau\Big)\\ &\geqslant \|u\|c_\delta\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(s^{\beta-1})ds\phi_q(N)\phi_q\Big(\int_\delta^1 H(1,\tau)a(\tau)d\tau\Big)\\ &> \|u\|.\end{aligned}$$

This contradiction completes the proof □

Corollary 5.3. *Assume that* (H1) *holds and* $f_\infty = +\infty$. *Then the fractional differential equation boundary-value problem* (1.1) *and* (1.2) *has no positive solution for sufficiently large* $\lambda > 0$.

Example 5.4. Consider the boundary-value problem

$$D_{0+}^{5/2}(\phi_p(D_{0+}^{1/2}u(t))) + tu^2 = 0, \quad 0 < t < 1, \tag{5.1}$$

$$u(0) = \frac{1}{2}u(\frac{1}{2}) + \lambda, \quad \phi_p(D_{0+}^{1/2}u(0)) = (\phi_p(D_{0+}^{1/2}u(1)))' = (\phi_p(D_{0+}^{1/2}u(0)))'' = 0. \tag{5.2}$$

Let $p = 2$. We have $\alpha = 1/2$, $\beta = 5/2$, $\gamma = \xi = 1/2$, $a(t) = t$, $f(u) = u^2$. Clearly, (H1) holds. Choosing $\delta = 1/2$, by a simple computation, we obtain $c_\delta = 0.04455, N > 222.2104$. Let $N = e = 223$. Then, (H4) is satisfied. Thus, by Theorem 5.2 the fractional differential equation boundary-value problem (5.1) and (5.2) has no positive solution for $\lambda > 111.5$.

Acknowledgments. This research is supported by grants: 11071143, 60904024, 61174217 from the Natural Science Foundation of China; JQ201119 from the Natural Science Outstanding Youth Foundation of Shandong Province; ZR2012AM009, ZR2010AL002 from the Shandong Provincial Natural Science Foundation; J11LA01 from the Natural Science Foundation of Educational Department of Shandong Province.

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