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LIMIT CYCLES FOR FOURTH-ORDER AUTONOMOUS DIFFERENTIAL EQUATIONS

JAUME LLIBRE, AMAR MAKHLOUF

Abstract. We provide sufficient conditions for the existence of periodic solutions of the fourth-order differential equation

$$\ddot{\ddot{x}} - (\lambda+\mu)\dddot{x} + (1+\lambda\mu)\ddot{x} - (\lambda+\mu)\dot{x} + \lambda\mu x = \varepsilon F(x,\dot{x},\ddot{x},\dddot{x}),$$

where λ, μ and ε are real parameters, ε is small and F is a nonlinear function.

1. Introduction and statement of the main results

The objective of this paper is to study the periodic solutions of the fourth-order differential equation

$$\ddot{\ddot{x}} - (\lambda+\mu)\dddot{x} + (1+\lambda\mu)\ddot{x} - (\lambda+\mu)\dot{x} + \lambda\mu x = \varepsilon F(x,\dot{x},\ddot{x},\dddot{x}), \tag{1.1}$$

where λ, μ and ε are real parameters, ε is small and F is a nonlinear function. The dot denotes derivative with respect to an independent variable t.

There are many papers studying the periodic orbits of fourth–order differential equations, see for instance in [3, 4, 5, 6, 7, 11, 12, 13, 14, 15]. But our main tool for studying the periodic orbits of equation (1.1) is completely different to the tools of the mentioned papers, and consequently the results obtained are distinct and new. We shall use the averaging theory, more precisely Theorem 2.1. Many of the quoted papers dealing with the periodic orbits of four-order differential equations use Schauder's or Leray-Schauder's fixed point theorem, or the nonlocal reduction method, or variational methods.

In general to obtain analytically periodic solutions of a differential system is a very difficult task, usually impossible. Here with the averaging theory this difficult problem for the differential equations (1.1) is reduced to find the zeros of a nonlinear function. We must say that the averaging theory for finding periodic solutions in general does not provide all the periodic solutions of the system. For more information about the averaging theory see section 2 and the references quoted there.

Llibre, Makhlouf and Sellami [8] studied equation (1.1) with the nonlinear function $F(x,\dot{x},\ddot{x},\dddot{x},t)$ which depends explicitly on the independent variable t. Here we study the autonomous case using a different approach.

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We recall that a simple zero r_0^* of a real function $\mathcal{F}(r_0)$ is defined by $\mathcal{F}(r_0^*) = 0$ and $(d\mathcal{F}/dr_0)(r_0^*) \neq 0$.

Our main results on the periodic solutions of this fourth-order differential equation (1.1) are the following.

Theorem 1.1. *Assume that $\lambda \neq \mu$ and $\lambda\mu \neq 0$. For every positive simple zero r_0^* of the function*

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\, d\theta,$$

where

$$\begin{gathered}
\mathcal{A} = \frac{((\lambda+\mu)\cos\theta \ + (\lambda\mu - 1)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)},\\
\mathcal{B} = \frac{((\lambda\mu-1)\cos\theta \ - (\lambda+\mu)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)},\\
\mathcal{C} = -\frac{((\lambda+\mu)\cos\theta \ + (\lambda\mu - 1)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)},\\
\mathcal{D} = \frac{((1-\lambda\mu)\cos\theta \ + (\lambda+\mu)\sin\theta)r_0}{(1+\lambda^2)(1+\mu^2)},
\end{gathered}$$

the differential equation (1.1) *has a periodic solution $x(t,\varepsilon)$ tending to the periodic solution*

$$x(t,\varepsilon) \to \frac{r_0^*((\lambda+\mu)\cos t + (-1+\lambda\mu)\sin t)}{(1+\lambda^2)(1+\mu^2)} \tag{1.2}$$

of $\ddddot{x} - (\lambda+\mu)\dddot{x} + (1+\lambda\mu)\ddot{x} - (\lambda+\mu)\dot{x} + \lambda\mu x = 0$ when $\varepsilon \to 0$.

Theorem 1.1 is proved in section 3. Its proof is based on the averaging theory for computing periodic orbits, see section 2. Two easy applications of Theorem 1.1 are given in the following two corollaries. They are proved in section 4.

Corollary 1.2. *Assume $\lambda \neq \mu$, $\lambda\mu \neq 0$ and $\lambda\mu \neq 1$. If $F(x,\dot{x},\ddot{x},\dddot{x}) = \dot{x} - \dot{x}^3$, then the differential equation* (1.1) *has a periodic solution $x(t,\varepsilon)$ tending to the periodic solution*

$$x(t,\varepsilon) \to \frac{r_0^*((\lambda+\mu)\cos t + (\lambda\mu - 1)\sin t)}{(1+\lambda^2)(1+\mu^2)} \tag{1.3}$$

of $\ddddot{x} - (\lambda+\mu)\dddot{x} + (1+\lambda\mu)\ddot{x} - (\lambda+\mu)\dot{x} + \lambda\mu x = 0$ when $\varepsilon \to 0$, where

$$r_0^* = \frac{2\sqrt{1+\lambda^2+\mu^2+\lambda^2\mu^2}}{\sqrt{3}}.$$

Corollary 1.3. *Assume $\mu = -\lambda \neq 0$. If $F(x,\dot{x},\ddot{x},\dddot{x}) = \sin\dot{x}$, then for every positive integer m there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the differential equation* (1.1) *has at least m periodic solutions.*

Theorem 1.4. *Assume that $\lambda = \mu \neq 0$. For every positive simple zero r_0^* of the function*

$$\mathcal{F}(r_0) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\, d\theta,$$

where

$$\mathcal{A} = \frac{(2\mu\cos\theta \ + (\mu^2-1)\sin\theta)r_0}{(1+\mu^2)^2},$$

$$\mathcal{B} = \frac{((\mu^2-1)\cos\theta \ -2\mu\sin\theta)r_0}{(1+\mu^2)^2},$$
$$\mathcal{C} = \frac{(-2\mu\cos\theta+(1-\mu^2)\sin\theta)r_0}{(1+\mu^2)^2},$$
$$\mathcal{D} = \frac{((1-\mu^2)\cos\theta+2\mu\sin\theta)r_0}{(1+\mu^2)^2},$$

the differential equation (1.1) *has a periodic solution* $x(t,\varepsilon)$ *tending to the periodic solution*

$$x(t,\varepsilon) \to \frac{r_0^*(2\mu\cos t+(\mu^2-1)\sin t)}{(1+\mu^2)^2} \tag{1.4}$$

of $\ddot{\ddot{x}} - 2\mu\dddot{x} + (1+\mu^2)\ddot{x} - 2\mu\dot{x} + \mu^2 x = 0$ *when* $\varepsilon \to 0$.

Theorem 1.4 is proved in section 5. Two easy applications of Theorem 1.4 are given in the following two corollaries. They are proved in section 6.

Corollary 1.5. *Assume* $\lambda = \mu \notin \{-1,0,1\}$. *If* $F(x,\dot{x},\ddot{x},\dddot{x}) = \dot{x} - \dot{x}^3$, *then the differential equation* (1.1) *has a periodic solution tending to the periodic solution*

$$x(t,\varepsilon) \to \frac{r_0^*(2\mu\cos t+(\mu^2-1)\sin t)}{(1+\mu^2)^2} \tag{1.5}$$

of $\ddot{\ddot{x}} - 2\mu\dddot{x} + (1+\mu^2)\ddot{x} - 2\mu\dot{x} + \mu^2 x = 0$ *when* $\varepsilon \to 0$, *where*

$$r_0^* = \frac{2\sqrt{1+2\mu^2+\mu^4}}{\sqrt{3}}.$$

Corollary 1.6. *Assume* $\lambda = \mu = 1$.*If* $F(x,\dot{x},\ddot{x},\dddot{x}) = \sin x$, *then for every positive integer* m *there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0,\varepsilon_0)$ *the differential equation* (1.1) *has at least* m *periodic solutions.*

Theorem 1.7. *Assume* $\lambda \neq \mu = 0$. *For every* (r_0^*, V_0^*) *solution of the system*

$$.\mathcal{F}_1(r_0,V_0) = 0, \quad \mathcal{F}_2(r_0,V_0) = 0, \tag{1.6}$$

satisfying

$$\det\Big(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(r_0,V_0)}\Big|_{(r_0,V_0)=(r_0^*,V_0^*)}\Big) \neq 0, \tag{1.7}$$

with

$$\mathcal{F}_1(r_0,V_0) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\,d\theta,$$
$$\mathcal{F}_2(r_0,V_0) = \frac{1}{2\pi}\int_0^{2\pi} F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\,d\theta,$$

when

$$\mathcal{A} = -\frac{(1+\lambda^2)V_0+(\lambda\sin\theta-\lambda^2\cos\theta)r_0}{\lambda+\lambda^3},$$
$$\mathcal{B} = -\frac{(\cos\theta \ +\lambda\sin\theta)r_0}{1+\lambda^2},$$
$$\mathcal{C} = \frac{(-\lambda\cos\theta+ \ \sin\theta)r_0}{1+\lambda^2},$$
$$\mathcal{D} = \frac{(\cos\theta \ +\lambda\sin\theta)r_0}{1+\lambda^2},$$

the differential equation (1.1) *has a periodic solution* $x(t,\varepsilon)$ *tending to the periodic solution*

$$x(t,\varepsilon) \to -\frac{(1+\lambda^2)V_0^* + (\lambda \sin t - \lambda^2 \cos t) r_0^*}{\lambda + \lambda^3} \tag{1.8}$$

of $\dddot{x} - \lambda \dddot{x} + \ddot{x} - \lambda \dot{x} = 0$ *when* $\varepsilon \to 0$.

Theorem 1.7 is proved in section 7. We remark that the case $\mu \neq 0$ and $\lambda = 0$ can be studied as the case $\lambda \neq 0$ and $\mu = 0$. One application of Theorem 1.7 is given in the following corollary. It is proved in section 8.

Corollary 1.8. *Assume* $\lambda \neq \mu = 0$. *If* $F(x,\dot{x},\ddot{x},\dddot{x}) = x - x^3$, *then the differential equation* (1.1) *has three periodic solutions* $x(t,\varepsilon)$ *tending to the periodic solution*

$$x(t,\varepsilon) \to -\frac{V_0^* + V_0^* \lambda^2 - \lambda^2 \cos t \ r_0^* + \lambda \sin t \ r_0^*}{\lambda + \lambda^3} \tag{1.9}$$

of $\ddddot{x} - \lambda \dddot{x} + \ddot{x} - \lambda \dot{x} = 0$ *when* $\varepsilon \to 0$, *where* $(r_0^*, V_0^*) = \Big(2\sqrt{2/15}\sqrt{1+\lambda^2}, -\frac{\lambda}{\sqrt{5}}\Big)$, $\Big(2\sqrt{2/15}\sqrt{1+\lambda^2}, \frac{\lambda}{\sqrt{5}}\Big)$ *and* $\Big(\frac{2\sqrt{1+\lambda^2}}{\sqrt{3}}, 0\Big)$.

Theorem 1.9. *Assume that* $\lambda = \mu = 0$. *Then the averaging theorem used in this paper cannot be applied to the differential equation* $\ddddot{x} + \ddot{x} = \varepsilon F(x,\dot{x},\ddot{x},\dddot{x})$.

2. Basic results on averaging theory

In this section we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T-periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{2.1}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}), \tag{2.2}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of the system (2.2) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{2.3}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (2.3), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1,\dots,x_n) = (x_1,\dots,x_k)$.

We assume that there exists a k–dimensional submanifold $\mathcal{Z}$ of Ω filled with T-periodic solutions of (2.2). Then an answer to the problem of bifurcation of T-periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (2.1) is given in the following result.

Theorem 2.1. *Let* W *be an open and bounded subset of* $\mathbb{R}^k$, *and let* $\beta : Cl(W) \to \mathbb{R}^{n-k}$ *be a* $\mathcal{C}^2$ *function. We assume that*

(i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)),\ \alpha \in Cl(W)\} \subset \Omega$ *and that for each* $\mathbf{z}_\alpha \in \mathcal{Z}$ *the solution* $\mathbf{x}(t,\mathbf{z}_\alpha)$ *of* (2.2) *is* T*-periodic;*

(ii) *for each* $\mathbf{z}_\alpha \in \mathcal{Z}$ *there is a fundamental matrix* $M_{\mathbf{z}_\alpha}(t)$ *of* (2.3) *such that the matrix* $M^{-1}_{\mathbf{z}_\alpha}(0) - M^{-1}_{\mathbf{z}_\alpha}(T)$ *has in the upper right corner the* $k \times (n-k)$ *zero matrix, and in the lower right corner a* $(n-k) \times (n-k)$ *matrix* Δ_α *with* $\det(\Delta_\alpha) \neq 0$.

We consider the function $\mathcal{F} : Cl(W) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\Big(\frac{1}{T}\int_0^T M^{-1}_{\mathbf{z}_\alpha}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\Big). \tag{2.4}$$

If there exists $a \in W$ *with* $\mathcal{F}(a) = 0$ *and* $\det\big((d\mathcal{F}/d\alpha)(a)\big) \neq 0$, *then there is a* T*-periodic solution* $\varphi(t, \varepsilon)$ *of system* (2.1) *such that* $\varphi(0, \varepsilon) \to \mathbf{z}_a$ *as* $\varepsilon \to 0$.

Theorem 2.1 goes back to Malkin [9] and Roseau [10], for a shorter proof see [2]. Theorem 2.1 will be used for proving our theorems.

3. Proof of Theorem 1.1

Introducing the variables $(x, y, z, v) = (x, \dot{x}, \ddot{x}, \dddot{x})$ we write the fourth–order differential equation (1.1) as a first–order differential system defined in an open subset Ω of $\mathbb{R}^4$. Thus we have the differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= v, \\ \dot{v} &= -\lambda\mu x + (\lambda + \mu)y - (1 + \lambda\mu)z + (\lambda + \mu)v + \varepsilon F(x, y, z, v). \end{aligned} \tag{3.1}$$

Of course as before the dot denotes derivative with respect to the independent variable t. System (3.1) with $\varepsilon = 0$ will be called the *unperturbed system*, otherwise we have the *perturbed system*. The unperturbed system has a unique singular point at the origin with eigenvalues $\pm i$, λ and μ. We shall write system (3.1) in such a way that the linear part at the origin will be in its real Jordan normal form. Then, doing the change of variables $(x, y, z, v) \to (X, Y, Z, V)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \\ V \end{pmatrix} = \begin{pmatrix} 0 & \lambda\mu & -\lambda-\mu & 1 \\ \lambda\mu & -\lambda-\mu & 1 & 0 \\ 1 & -\frac{1}{\mu} & 1 & -\frac{1}{\mu} \\ -\lambda & 1 & -\lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix},$$

the differential system (3.1) becomes

$$\begin{aligned} \dot{X} &= -Y + \varepsilon G(X, Y, Z, V), \\ \dot{Y} &= X, \\ .\dot{Z} &= \lambda Z - \frac{\varepsilon}{\mu} G(X, Y, Z, V), \\ \dot{V} &= \mu V + \varepsilon G(X, Y, Z, V), \end{aligned} \tag{3.2}$$

where $G(X, Y, Z, V) = F(A, B, C, D)$ with

$$A = \frac{-V(1+\lambda^2) + Y(\lambda-\mu)(\lambda\mu-1) + X(\lambda^2-\mu^2) - Z\mu(1+\mu^2)}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)},$$

$$B = \frac{X(\lambda-\mu)(\lambda\mu-1) + Y(-\lambda^2+\mu^2) - \mu(V(1+\lambda^2) + Z\lambda(1+\mu^2))}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)},$$

$$
\begin{aligned}
C &= \frac{-Y(\lambda-\mu)(\lambda\mu-1)+X(-\lambda^2+\mu^2)-\mu(V\mu+\lambda^2(Z+V\mu+Z\mu^2))}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)},\\
D &= \frac{-X(\lambda-\mu)(\lambda\mu-1)+Y(\lambda^2-\mu^2)-\mu(V(1+\lambda^2)\mu^2+Z\lambda^3(1+\mu^2))}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)}.
\end{aligned}
$$

Note that the linear part of the differential system (3.2) at the origin is in its real normal form of Jordan, and that A, B, C and D are well defined because $\lambda \neq \mu$.

Now we pass from the cartesian variables (X, Y, Z, V) to the cylindrical variables (r, θ, Z, V) of $\mathbb{R}^4$, where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables the differential system (3.2) can be written as

$$
\begin{aligned}
\dot{r} &= \varepsilon\cos\theta\, H(r,\theta,Z,V),\\
\dot{\theta} &= 1-\varepsilon\frac{\sin\theta}{r}H(r,\theta,Z,V),\\
\dot{Z} &= \lambda Z-\varepsilon\frac{1}{\mu}H(r,\theta,Z,V),\\
\dot{V} &= V\mu+\varepsilon H(r,\theta,Z,V),
\end{aligned}
\tag{3.3}
$$

where $H(r,\theta,Z,V) = F(a,b,c,d)$ with

$$
\begin{aligned}
a &= -\frac{V+V\lambda^2+Z\mu+Z\mu^3-r(\lambda-\mu)((\lambda+\mu)\cos\theta+(\lambda\mu-1)\sin\theta)}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)},\\
b &= \frac{-\mu(V(1+\lambda^2)+Z\lambda(1+\mu^2))+r(\lambda-\mu)((\lambda\mu-1)\cos\theta-(\lambda+\mu)\sin\theta)}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)},\\
c &= -\frac{\mu(\mu V+\lambda^2(Z+\mu V+\mu^2 Z)+r(\lambda-\mu)((\lambda+\mu)\cos\theta+(\lambda\mu-1)\sin\theta)}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)},\\
d &= -\frac{V(1+\lambda^2)\mu^3+Z\lambda^3\mu(1+\mu^2)+r(\lambda-\mu)((\lambda\mu-1)\cos\theta-(\lambda+\mu)\sin\theta)}{(1+\lambda^2)(\lambda-\mu)(1+\mu^2)}.
\end{aligned}
$$

Now we change the independent variable from t to θ, and denoting the derivative with respect to θ by a prime the differential system (3.3) becomes

$$
\begin{aligned}
r' &= \varepsilon\,\cos\theta\, H+O(\varepsilon^2),\\
Z' &= \lambda Z+\varepsilon\frac{\lambda\mu Z\sin\theta-r}{\mu r}H+O(\varepsilon^2),\\
V' &= \mu V+\varepsilon\frac{\mu V\sin\theta+r}{r}H+O(\varepsilon^2),
\end{aligned}
\tag{3.4}
$$

where $H = H(r,\theta,Z,V)$.

We shall apply Theorem 2.1 to the differential system (3.4). We note that system (3.4) can be written as system (2.1) taking

$$
\mathbf{x} = \begin{pmatrix} r\\ Z\\ V\end{pmatrix},\quad t=\theta,\quad F_0(\theta,\mathbf{x}) = \begin{pmatrix} 0\\ \lambda\, Z\\ \mu\, V\end{pmatrix},
$$

$$
F_1(\theta,\mathbf{x}) = \begin{pmatrix} \cos\theta\, H\\ \frac{\lambda\mu\sin\theta\, Z-r}{\mu r}H\\ \frac{\mu\sin\theta\, V+r}{r}H\end{pmatrix}.
$$

We shall study the periodic solutions of system (2.2) in our case, i.e. the periodic solutions of system (3.4) with $\varepsilon = 0$. Clearly these periodic solutions are

$$(r(\theta), Z(\theta), V(\theta)) = (r_0, 0, 0),$$

for any $r_0 > 0$; i.e. are all the circles of the plane $Z = V = 0$ of system (3.3). Of course all these periodic solutions in the coordinates (r, Z, V) have period 2π in the variable θ.

We shall describe the different elements which appear in the statement of Theorem 2.1 in the particular case of the differential system (3.4). Thus we have that $k = 1$ and $n = 3$. Let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. Then we take the open bounded subset W of $\mathbb{R}$ as $W = (r_1, r_2)$, $\alpha = r_0$ and $\beta : [r_1, r_2] \to \mathbb{R}^2$ defined as $\beta(r_0) = (0, 0)$. The set $\mathcal{Z}$ is

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, 0, 0),\ r_0 \in [r_1, r_2]\}\,.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider that the solution $\mathbf{x}(\theta) = \mathbf{z}_\alpha = (r_0, 0, 0)$ is 2π-periodic.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear differential system (3.4) with $\varepsilon = 0$ associated to the 2π-periodic solution $\mathbf{z}_\alpha = (r_0, 0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^3$, we obtain

$$M(\theta) = M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\lambda\,\theta} & 0 \\ 0 & 0 & e^{\mu\,\theta} \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\theta)$ does not depend of the particular periodic orbit $\mathbf{z}_\alpha$. Since the matrix

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - e^{-2\pi\lambda} & 0 \\ 0 & 0 & 1 - e^{-2\pi\mu} \end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 2.1 because λ and μ are not zero, we can apply it to system (3.4).

Now $\xi : \mathbb{R}^3 \to \mathbb{R}$ is $\xi(r, Z, V) = r$. We calculate the function

$$\begin{aligned} \mathcal{F}(r_0) = \mathcal{F}(\alpha) &= \xi\Big(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\Big), \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\, d\theta, \end{aligned}$$

where the expressions of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are the ones given in the statement of Theorem 1.1. Then by Theorem 2.1 we have that for every simple zero $r_0^* \in [r_1, r_2]$ of the function $\mathcal{F}(r_0)$ we have a periodic solution $(r, Z, V)(\theta, \varepsilon)$ of system (3.4) such that

$$(r, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0) \quad \text{as } \varepsilon \to 0.$$

Going back through the change of coordinates we obtain a periodic solution $(r, \theta, Z, V)(t, \varepsilon)$ of system (3.3) such that

$$(r, \theta, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0, 0) \quad \text{as } \varepsilon \to 0.$$

Consequently we obtain a periodic solution $(X, Y, Z, V)(t, \varepsilon)$ of system (3.2) such that

$$(X, Y, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0, 0) \quad \text{as } \varepsilon \to 0.$$

We have a periodic solution $(x,y,z,v)(t,\varepsilon)$ of system (3.1) such that
$$x(t,\varepsilon) \to \frac{r_0^*((\lambda+\mu)\cos t + (-1+\lambda\mu)\sin t)}{(1+\lambda^2)(1+\mu^2)} \quad \text{as } \varepsilon \to 0.$$
Of course, it is easy to check that the previous expression provides a periodic solution of the linear differential equation $\ddddot{x} - (\lambda+\mu)\dddot{x} + (1+\lambda\mu)\ddot{x} - (\lambda+\mu)\dot{x} + \lambda\mu x = 0$. Hence Theorem 1.1 is proved.

4. Proof of corollaries 1.2 and 1.3

Proof of Corollary 1.2. If $F(x,\dot{x},\ddot{x},\dddot{x}) = \dot{x} - \dot{x}^3$, then the function $\mathcal{F}(r_0)$ of the statement of Theorem 1.1 is
$$\mathcal{F}(r_0) = \frac{r_0(-1+\lambda\mu)(-3r_0^2 + 4(1+\lambda^2)(1+\mu^2))}{8(1+\lambda^2)^2(1+\mu^2)^2}.$$
The function $\mathcal{F}(r_0)$, has the positive zero
$$r_0^* = \frac{2\sqrt{1+\lambda^2+\mu^2+\lambda^2\mu^2}}{\sqrt{3}}.$$
The derivative
$$\mathcal{F}'(r_0^*) = \frac{1-\lambda\mu}{(1+\lambda^2)(1+\mu^2)} \neq 0.$$
The corollary follows from Theorem 1.1. □

Proof of Corollary 1.3. If $F(x,\dot{x},\ddot{x},\dddot{x}) = \sin\dot{x}$, since $\mu = -\lambda$ it is not difficult to show that
$$\mathcal{F}(r_0) = J_1\Big(\frac{r_0}{1+\lambda^2}\Big),$$
where $J_1(z)$ is the Bessel function of first kind. This function has infinitely many simple zeros when $r_0 \to \infty$, see for more details [1]. In this case the differential system has as many periodic orbits as we want taking ε sufficiently small. Hence the corollary is proved. □

5. Proof of Theorem 1.4

We have the differential system
$$\begin{gathered}
\dot{x} = y,\\
\dot{y} = z,\\
\dot{z} = v,\\
\dot{v} = -\mu^2 x + 2\mu y - (1+\mu^2)z + 2\mu v + \varepsilon F(x,y,z,v).
\end{gathered} \tag{5.1}$$
The unperturbed system has a unique singular point at the origin with eigenvalues $\pm i, \mu, \mu$. We shall write system (5.1) in such a way that the linear part at the origin will be in its real Jordan normal form. Then doing the change of variables $(x,y,z,v) \to (X,Y,Z,V)$ given by
$$\begin{pmatrix} X\\ Y\\ Z\\ V\end{pmatrix} = \begin{pmatrix} 0 & \mu^2 & -2\mu & 1\\ \mu^2 & -2\mu & 1 & 0\\ 1 & 0 & 1 & 0\\ -\mu & 1 & -\mu & 1\end{pmatrix} \begin{pmatrix} x\\ y\\ z\\ v\end{pmatrix},$$

the differential system (5.1), becomes

$$\begin{aligned}\dot{X} &= -Y + \varepsilon G(X,Y,Z,V),\\ \dot{Y} &= X,\\ \dot{Z} &= \mu Z + V,\\ \dot{V} &= \mu V + \varepsilon G(X,Y,Z,V),\end{aligned} \tag{5.2}$$

where $G(X,Y,Z,V) = F(A,B,C,D)$ with

$$\begin{gathered}A = \frac{-Y + Z + 2(-V + X)\mu + (Y+Z)\mu^2}{(1+\mu^2)^2},\\ B = \frac{V - \mu^2 V + (-1+\mu^2)X + \mu(-2Y + Z + \mu^2 Z)}{(1+\mu^2)^2},\\ C = \frac{Y - \mu^2 Y + \mu(-2X + 2V + \mu Z(1+\mu^2))}{(1+\mu^2)^2},\\ D = \frac{X - \mu^2 X + \mu(2Y + \mu(\mu Z(1+\mu^2) + (3+\mu^2)V))}{(1+\mu^2)^2}.\end{gathered}$$

Now we pass from the cartesian variables (X,Y,Z,V) to the cylindrical coordinates (r,θ,Z,V) of $\mathbb{R}^4$ where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables the differential system (5.2) can be written as

$$\begin{aligned}\dot{r} &= \varepsilon\cos\theta\, H(r,\theta,Z,V),\\ \dot{\theta} &= 1 - \varepsilon\frac{\sin\theta}{r}H(r,\theta,Z,V),\\ \dot{Z} &= \mu Z + V,\\ \dot{V} &= \mu V + \varepsilon H(r,\theta,Z,V),\end{aligned} \tag{5.3}$$

where $H(r,\theta,Z,V) = F(a,b,c,d)$ and

$$\begin{gathered}a = \frac{Z + \mu^2 Z - 2\mu V + 2\mu r\cos\theta + r(\mu^2-1)\sin\theta}{(1+\mu^2)^2},\\ b = \frac{V - \mu^2 V + \mu Z(1+\mu^2) + (\mu^2-1)r\cos\theta - 2\mu r\sin\theta}{(1+\mu^2)^2},\\ c = \frac{\mu[\mu(1+\mu^2)Z + 2V - 2r\cos\theta] + (1-\mu^2)r\sin\theta}{(1+\mu^2)^2},\\ d = \frac{(1-\mu^2)r\cos\theta + \mu[\mu(\mu(1+\mu^2)Z + (3+\mu^2)V) + 2r\sin\theta]}{(1+\mu^2)^2}.\end{gathered}$$

Now we change the independent variable from t to θ, and denoting the derivative with respect to θ by a prime the differential system (5.3) becomes

$$\begin{aligned}r' &= \varepsilon\,\cos\theta\, H + O(\varepsilon^2),\\ Z' &= \mu\, Z + V + \varepsilon\frac{(\mu Z + V)\sin\theta}{r}H + O(\varepsilon^2),\\ V' &= \mu V + \varepsilon\frac{r + \mu V\sin\theta}{r}H + O(\varepsilon^2),\end{aligned} \tag{5.4}$$

where $H = H(r,\theta,Z,V)$.

We shall apply Theorem 2.1 to the differential system (5.4). We note that system (5.4) can be written as system (2.1) taking

$$\mathbf{x}=\begin{pmatrix} r\\ Z\\ V\end{pmatrix},\quad t=\theta,\quad F_0(\theta,\mathbf{x})=\begin{pmatrix}0\\ \mu\, Z+V\\ \mu\, V\end{pmatrix},$$

$$F_1(\theta,\mathbf{x})=\begin{pmatrix}\cos\theta\, H\\ \frac{(\mu Z+V)\sin\theta}{r}H\\ \frac{r+\mu V\sin\theta}{r}H\end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case; i.e., the periodic solutions of system (5.4) with $\varepsilon=0$. Clearly these periodic solutions are

$$(r(\theta),Z(\theta),V(\theta))=(r_0,0,0),$$

for any $r_0>0$; i.e. are all the circles of the plane $Z=V=0$ of system (5.3). Of course all these periodic solutions in the coordinates (r,Z,V) have period 2π in the variable θ.

We shall describe the different elements which appear in the statement of Theorem 2.1 in the particular case of the differential system (5.4). Thus we have that $k=1$ and $n=3$. Let $r_1>0$ be arbitrarily small and let $r_2>0$ be arbitrarily large. Then we take the open bounded subset W of $\mathbb{R}$ as $W=(r_1,r_2)$, $\alpha=r_0$ and $\beta:[r_1,r_2]\to\mathbb{R}^2$ defined as $\beta(r_0)=(0,0)$. The set $\mathcal{Z}$ is

$$\mathcal{Z}=\{\mathbf{z}_\alpha=(r_0,0,0),\ r_0\in[r_1,r_2]\}\,.$$

Clearly for each $\mathbf{z}_\alpha\in\mathcal{Z}$ we can consider that the solution $\mathbf{x}(\theta)=\mathbf{z}_\alpha=(r_0,0,0)$ is 2π-periodic.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear differential system (5.4) with $\varepsilon=0$ associated to the 2π-periodic solution $\mathbf{z}_\alpha=(r_0,0,0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^3$, we obtain

$$M(\theta)=M_{\mathbf{z}_\alpha}(\theta)=\begin{pmatrix}1&0&0\\ 0&e^{\mu\,\theta}&\theta e^{\mu\,\theta}\\ 0&0&e^{\mu\,\theta}\end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\theta)$ does not depend of the particular periodic orbit $\mathbf{z}_\alpha$. Since the matrix

$$M^{-1}(0)-M^{-1}(2\pi)=\begin{pmatrix}0&0&0\\ 0&1-e^{-2\pi\mu}&2\pi e^{-2\pi\mu}\\ 0&0&1-e^{-2\pi\mu}\end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 2.1 we can apply it to system (5.4).

Now $\xi:\mathbb{R}^3\to\mathbb{R}$ is $\xi(r,Z,V)=r$. We calculate the function

$$\begin{aligned}\mathcal{F}(r_0)=\mathcal{F}(\alpha)&=\xi\Big(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha))dt\Big)\\ &=\frac{1}{2\pi}\int_0^{2\pi}\cos\theta\,F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\,d\theta,\end{aligned}$$

where the expressions of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are the ones given in the statement of Theorem 1.4. Then by Theorem 2.1 we have that for every simple zero $r_0^*\in[r_1,r_2]$

of the function $\mathcal{F}(r_0)$ we have a periodic solution $(r, Z, V)(\theta, \varepsilon)$ of system (5.4) such that

$$(r, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0) \quad \text{as } \varepsilon \to 0.$$

Going back through the changes of coordinates we obtain a periodic solution $(r, \theta, Z, V)(t, \varepsilon)$ of system (5.3) such that

$$(r, \theta, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0, 0) \quad \text{as } \varepsilon \to 0.$$

Consequently we obtain a periodic solution $(X, Y, Z, V)(t, \varepsilon)$ of system (5.2) such that

$$(X, Y, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0, 0) \quad \text{as } \varepsilon \to 0.$$

We have a periodic solution $(x, y, z, v)(t, \varepsilon)$ of system (5.1) such that

$$x(t, \varepsilon) \to \frac{r_0^*(2\mu \cos t + (\mu^2 - 1)\sin t)}{(1 + \mu^2)^2} \quad \text{as } \varepsilon \to 0.$$

Of course, it is easy to check that the previous expression provides a periodic solution of the linear differential equation $\ddot{\ddot{x}} - 2\mu \dddot{x} + (1 + \mu^2)\ddot{x} - 2\mu\dot{x} + \mu^2 x = 0$. Hence Theorem 1.4 is proved.

6. Proof of corollaries 1.5 and 1.6

Proof of Corollary 1.5. If $F(x, \dot{x}, \ddot{x}, \dddot{x}) = \dot{x} - \dot{x}^3$, then the function $\mathcal{F}(r_0)$ of the statement of Theorem 1.4 is

$$\mathcal{F}(r_0) = \frac{r_0(\mu^2 - 1)(4(1 + \mu^2)^2 - 3r_0^2)}{8(1 + \mu^2)^4}.$$

The function $\mathcal{F}(r_0)$ has the positive zero

$$r_0^* = \frac{2\sqrt{1 + 2\mu^2 + \mu^4}}{\sqrt{3}}.$$

The derivative

$$\mathcal{F}'(r_0^*) = \frac{1 - \mu^2}{(1 + \mu^2)^2} \neq 0.$$

The corollary follows from Theorem 1.4. ∎

Proof of Corollary 1.6. If $F(x, \dot{x}, \ddot{x}, \dddot{x}) = \sin x$, it is not difficult to show that

$$\mathcal{F}(r_0) = J_1\Big(\frac{r_0}{2}\Big),$$

where $J_1(z)$ is the Bessel function of first kind,when $\mu = 1$. This function has infinitely many simple zeros when $r_0 \to \infty$, see for more details [1]. In this case the differential system has as many periodic orbits as we want taking ε sufficiently small. Hence the corollary is proved. ∎

7. Proof of Theorem 1.7

We have the differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= v, \\ \dot{v} &= \lambda y + \lambda v - z + \varepsilon F(x,y,z,v). \end{aligned} \tag{7.1}$$

The unperturbed system has a unique singular point at the origin with eigenvalues $\pm i, 0, \lambda$. We shall write system (7.1) in such a way that the linear part at the origin will be in its real Jordan normal form. Then doing the change of variables $(x,y,z,v) \to (X,Y,Z,V)$ given by

$$\begin{pmatrix} X \\ Y \\ Z \\ V \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\lambda & 1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\lambda & 1 & -\lambda & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ v \end{pmatrix},$$

the differential system (7.1), becomes

$$\begin{aligned} \dot{X} &= -Y + \varepsilon G(X,Y,Z,V), \\ \dot{Y} &= X, \\ \dot{Z} &= \lambda Z + \varepsilon G(X,Y,Z,V), \\ \dot{V} &= \varepsilon G(X,Y,Z,V), \end{aligned} \tag{7.2}$$

where $G(X,Y,Z,V) = F(A,B,C,D)$ with

$$\begin{aligned} A &= \frac{Z + \lambda(-Y + \lambda X) - (1+\lambda^2)V}{\lambda + \lambda^3}, \\ B &= -\frac{X - Z + \lambda Y}{1+\lambda^2}, \\ C &= \frac{Y - \lambda X + \lambda Z}{1+\lambda^2}, \\ D &= \frac{X + \lambda(Y + \lambda Z)}{1+\lambda^2}. \end{aligned}$$

Note that λ cannot be zero. Now we pass from the cartesian variables (X,Y,Z,V) to the cylindrical ones (r,θ,Z,V) of $\mathbb{R}^4$, where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables the differential system (7.2) can be written as

$$\begin{aligned} \dot{r} &= \varepsilon \cos\theta\, H(r,\theta,Z,V), \\ \dot{\theta} &= 1 - \varepsilon \frac{\sin\theta}{r} H(r,\theta,Z,V), \\ \dot{Z} &= \lambda Z + \varepsilon H(r,\theta,Z,V), \\ \dot{V} &= \varepsilon H(r,\theta,Z,V), \end{aligned} \tag{7.3}$$

where $H(r,\theta,Z,V) = F(a,b,c,d)$ with

$$\begin{aligned} a &= \frac{Z - (1+\lambda^2)V + \lambda(\lambda\cos\theta - \sin\theta)r}{\lambda + \lambda^3}, \\ b &= \frac{-(\cos\theta + \lambda\sin\theta)r + Z}{1+\lambda^2}, \end{aligned}$$

$$
\begin{aligned}
c &= \frac{(-\lambda\cos\theta + \sin\theta)r + \lambda Z}{1+\lambda^2},\\
d &= \frac{(\cos\theta + \lambda\sin\theta)r + \lambda^2 Z}{1+\lambda^2}.
\end{aligned}
$$

Now we change the independent variable from t to θ, and denoting the derivative with respect to θ by a prime the differential system (7.3) becomes

$$
\begin{aligned}
r' &= \varepsilon\,\cos\theta\, H + O(\varepsilon^2),\\
Z' &= \lambda Z + \varepsilon\frac{r+\lambda Z\sin\theta}{r}H + O(\varepsilon^2),\\
V' &= \varepsilon H + O(\varepsilon^2),
\end{aligned}
\tag{7.4}
$$

where $H = H(r,\theta,Z,V)$.

We shall apply Theorem 2.1 to the differential system (7.4). We note that system (7.4) can be written as system (2.1) taking

$$
\mathbf{x} = \begin{pmatrix} r\\ Z\\ V\end{pmatrix}, \quad t=\theta, \quad F_0(\theta,\mathbf{x}) = \begin{pmatrix} 0\\ \lambda\, Z\\ 0\end{pmatrix},
$$

$$
F_1(\theta,\mathbf{x}) = \begin{pmatrix} \cos\theta\, H\\ \frac{r+\lambda\sin\theta\, Z}{r}H\\ H\end{pmatrix}.
$$

We shall study the periodic solutions of system (2.2) in our case; i.e., the periodic solutions of system (7.4) with $\varepsilon = 0$. Clearly these periodic solutions are

$$
(r(\theta), Z(\theta), V(\theta)) = (r_0, 0, V_0),
$$

for any $r_0 > 0$. These are all the circles in the plane $Z = 0$, $V = V_0$ of system (7.3). Of course all these periodic solutions in the coordinates (r, Z, V) have period 2π in the variable θ.

We shall describe the different elements which appear in the statement of Theorem 2.1 in the particular case of the differential system (7.4). Thus we have that $k = 2$ and $n = 3$. We take the open bounded subset W of $\mathbb{R}^2$ as

$$
W = \{(r_0, V_0) : 0 < r_0^2 + V_0^2 < R^2\},
$$

with $R > 0$ arbitrarily large. Here $\alpha = (r_0, V_0)$ and $\beta : W \to \mathbb{R}$, $\beta(r_0, V_0) = 0$. The set $\mathcal{Z}$ is

$$
\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, V_0, 0),\ (r_0, V_0) \in W\}.
$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider that the solution $\mathbf{x}(\theta) = \mathbf{z}_\alpha = (r_0, V_0, 0)$ is 2π-periodic.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear differential system (7.4) with $\varepsilon = 0$ associated to the 2π-periodic solution $\mathbf{z}_\alpha = (r_0, V_0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^3$, we obtain

$$
M(\theta) = M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{\lambda\theta}\end{pmatrix}.
$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\theta)$ does not depend of the particular periodic orbit $\mathbf{z}_\alpha$. Since the matrix

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-e^{-2\pi\lambda} \end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 2.1, for $\lambda \neq 0$, we can apply it to system (7.4).

Now $\xi : \mathbb{R}^3 \to \mathbb{R}^2$ is $\xi(r, Z, V) = (r, V)$. We calculate the function

$$\begin{aligned} \mathcal{F}(r_0, V_0) = \mathcal{F}(\alpha) &= \xi\Big(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\Big) \\ &= \begin{pmatrix} \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\, F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\ d\theta \\ \frac{1}{2\pi}\int_0^{2\pi} F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\ d\theta \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1(r_0, V_0) \\ \mathcal{F}_2(r_0, V_0) \end{pmatrix}, \end{aligned}$$

where the expression of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are the ones given in the statement of Theorem 1.7. Then, by Theorem 2.1 we have that for every simple zero $(r_0^*, V_0^*) \in W$ of the function $\mathcal{F}(r_0, V_0)$ we have a periodic solution $(r, Z, V)(\theta, \varepsilon)$ of system (7.4) such that

$$(r, Z, V)(0, \varepsilon) \to (r_0^*, 0, V_0^*) \quad \text{as } \varepsilon \to 0.$$

Going back through the changes of coordinates we obtain a periodic solution $(r, \theta, Z, V)(t, \varepsilon)$ of system (7.3) such that

$$(r, \theta, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0, V_0^*) \quad \text{as } \varepsilon \to 0.$$

Consequently we obtain a periodic solution $(X, Y, Z, V)(t, \varepsilon)$ of system (7.2) such that

$$(X, Y, Z, V)(0, \varepsilon) \to (r_0^*, 0, 0, V_0^*) \quad \text{as } \varepsilon \to 0.$$

We have a periodic solution $(x, y, z, v)(t, \varepsilon)$ of system (7.1) such that

$$x(t, \varepsilon) \to -\frac{(1+\lambda^2)V_0^* + (\lambda\sin t - \lambda^2\cos t)r_0^*}{\lambda + \lambda^3} \quad \text{as } \varepsilon \to 0.$$

Of course, it is easy to check that the previous expression provides a periodic solution of the linear differential equation $\dddot{x} - \lambda\dddot{x} + \ddot{x} - \lambda\dot{x} = 0$ Hence Theorem 1.7 is proved.

8. Proof of corollary 1.8

If $F(x, \dot{x}, \ddot{x}, \dddot{x}) = x - x^3$, then the function $\mathcal{F}(r_0, V_0)$ of the statement of Theorem 1.7 provides the system

$$\mathcal{F}_1(r_0, V_0) = -\frac{r_0(12V_0^2(1+\lambda^2) + \lambda^2(3r_0^2 - 4(1+\lambda^2)))}{8\lambda(1+\lambda^2)^2} = 0,$$

$$\mathcal{F}_2(r_0, V_0) = \frac{V_0(2V_0^2(1+\lambda^2) + \lambda^2(3r_0^2 - 2(1+\lambda^2)))}{2\lambda^3(1+\lambda^2)} = 0.$$

This system has the three solutions (r_0, V_0) with $r_0 > 0$: $\Big(2\sqrt{\frac{2(1+\lambda^2)}{15}}, -\frac{\lambda}{\sqrt{5}}\Big)$, $\Big(2\sqrt{\frac{2(1+\lambda^2)}{15}}, \frac{\lambda}{\sqrt{5}}\Big)$ and $\Big(\frac{2\sqrt{1+\lambda^2}}{\sqrt{3}}, 0\Big)$. The corresponding determinants of the Jacobian matrix are $\frac{4}{5+5\lambda^2}$, $\frac{4}{5+5\lambda^2}$, $-\frac{1}{1+\lambda^2}$, respectively. The corollary follows from Theorem 1.7.

9. Proof of Theorem 1.9.

We have the differential system

$$
\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= v,\\
\dot{v} &= -z + \varepsilon F(x, y, z, v).
\end{aligned} \tag{9.1}
$$

The unperturbed system has a unique singular point at the origin with eigenvalues $\pm i, 0, 0$. We shall write system (9.1) in such a way that the linear part at the origin will be in its real Jordan normal form. Then doing the change of variables $(x, y, z, v) \to (X, Y, Z, V)$ given by

$$
\begin{pmatrix} X\\ Y\\ Z\\ V \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x\\ y\\ z\\ v \end{pmatrix},
$$

the differential system (9.1), becomes

$$
\begin{aligned}
\dot{X} &= -Y + \varepsilon G(X, Y, Z, V),\\
\dot{Y} &= X,\\
\dot{Z} &= \varepsilon G(X, Y, Z, V),\\
\dot{V} &= Z,
\end{aligned} \tag{9.2}
$$

where $G(X, Y, Z, V) = F(A, B, C, D)$ with

$$
\begin{aligned}
A &= V - Y,\\
B &= -X + Z,\\
C &= Y,\\
D &= X.
\end{aligned}
$$

Now we pass from the cartesian variables (X, Y, Z, V) to the cylindrical variables (r, θ, Z, V) of $\mathbb{R}^4$, where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables the differential system (9.2) can be written as

$$
\begin{aligned}
\dot{r} &= \varepsilon \cos\theta\, H(r, \theta, Z, V),\\
\dot{\theta} &= 1 - \varepsilon \frac{\sin\theta}{r} H(r, \theta, Z, V),\\
\dot{Z} &= \varepsilon H(r, \theta, Z, V),\\
\dot{V} &= Z,
\end{aligned} \tag{9.3}
$$

where $H(r, \theta, Z, V) = F(a, b, c, d)$ with

$$
\begin{aligned}
a &= V - r\sin\theta,\\
b &= Z - r\cos\theta,\\
c &= r\sin\theta,\\
d &= r\cos\theta.
\end{aligned}
$$

Now we change the independent variable from t to θ, and denoting the derivative with respect to θ by a prime the differential system (9.3) becomes

$$\begin{aligned} r' &= \varepsilon \cos\theta\, H + O(\varepsilon^2), \\ Z' &= \varepsilon H + O(\varepsilon^2), \\ V' &= Z + \varepsilon \frac{Z\sin\theta}{r} H + O(\varepsilon^2), \end{aligned} \tag{9.4}$$

where $H = H(r, \theta, Z, V)$.

We shall apply Theorem 2.1 to the differential system (9.4). We note that system (9.4) can be written as system (2.1) taking

$$\mathbf{x} = \begin{pmatrix} r \\ Z \\ V \end{pmatrix}, \quad t = \theta, \quad F_0(\theta, \mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ Z \end{pmatrix},$$

$$F_1(\theta, \mathbf{x}) = \begin{pmatrix} \cos\theta\, H \\ H \\ \frac{Z\sin\theta}{r} H. \end{pmatrix}.$$

We shall study the periodic solutions of system (2.2) in our case, i.e. the periodic solutions of the system (9.4) with $\varepsilon = 0$. Clearly these periodic solutions are

$$(r(\theta), Z(\theta), V(\theta)) = (r_0, 0, V_0),$$

for any $r_0 > 0$. There are all the circles in the plane $Z = 0$, $V = V_0$ of system (9.3). Of course all these periodic solutions in the coordinates (r, Z, V) have period 2π in the variable θ.

We shall describe the different elements which appear in the statement of Theorem 2.1 in the particular case of the differential system (9.4). Thus we have that $k = 2$ and $n = 3$. We take the open bounded subset W of $\mathbb{R}^2$ as

$$W = \{(r_0, V_0) : 0 < r_0^2 + V_0^2 < R^2\},$$

where $R > 0$ is arbitrarily large. Here $\alpha = (r_0, V_0)$ and $\beta : W \to \mathbb{R}$ with $\beta(r_0, V_0) = 0$. The set $\mathcal{Z}$ is

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, V_0, 0),\ (r_0, V_0) \in W\}.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider that the solution $\mathbf{x}(\theta) = \mathbf{z}_\alpha = (r_0, V_0, 0)$ is 2π-periodic.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\theta)$ of the linear differential system (9.4) with $\varepsilon = 0$ associated to the 2π-periodic solution $\mathbf{z}_\alpha = (r_0, V_0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^3$, we obtain

$$M(\theta) = M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\theta)$ does not depend of the particular periodic orbit $\mathbf{z}_\alpha$. Since the matrix

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\pi \\ 0 & 0 & 0 \end{pmatrix},$$

This matrix does not verify the assumption of statement (ii) of Theorem 2.1. Therefore we cannot apply it to system (9.4).

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