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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS TO HIGHER-ORDER NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this article, we consider the nonlinear fractional order three-point boundary-value problem

$$D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u^{(n-2)}(1) = \int_0^{\eta} u(s)ds,$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, $n - 1 < \alpha \le n$, $n \ge 3$. By using a fixed-point theorem in partially ordered sets, we obtain sufficient conditions for the existence and uniqueness of positive and nondecreasing solutions to the above boundary value problem.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so on, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. For an extensive collection of such results, we refer the readers to the monographs by Samko et al [27], Podlubny [25] and Kilbas et al [14]. For the basic theory and recent development of the subject, we refer a text by Lakshmikantham [17]. For more details and examples, see [1, 2, 4, 6, 7, 15, 16, 17, 29] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

Zhang [28] considered the singular fractional differential equation

$$D^{\alpha}_{0+}u(t) + a(t)f(t, u(t), u'(t), \dots, u^{(n-2)}(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0,$$

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where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $n-1<\alpha\leq n$, $n\geq 2$. They used a fixed-point theorem for the mixed monotone operator to show the existence of positive solutions for the above fractional boundary value problem. But the uniqueness is not treated.

In [8], the authors obtained the existence and multiplicity of positive solutions for a class of higher-order nonlinear fractional differential equations with integral boundary conditions by applying Krasnoselskii's fixed-point theorem in cones. But the uniqueness is also not treated.

On the other hand, the study of the existence of solutions of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [13]. Then Gupta [9] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to [10, 12, 20, 21, 22] and the references therein. However, all these papers are concerned with problems with three-point boundary condition restrictions on the slope of the solutions and the solutions themselves, for example,

$$
\begin{gathered}
u(0)=0, \quad \alpha u(\eta)=u(1); \\
u(0)=\beta u(\eta), \quad \alpha u(\eta)=u(1); \\
u'(0)=0, \quad \alpha u(\eta)=u(1); \\
u(0)-\beta u'(0)=0, \quad \alpha u(\eta)=u(1); \\
\alpha u(0)-\beta u'(0)=0, \quad u'(\eta)+u'(1)=0; etc.
\end{gathered}
$$

In this article, we study the higher-order three-point boundary-value problem of fractional differential equation.

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<1, \tag{1.1}$$

$$u(0)=u'(0)=\dots=u^{(n-2)}(0)=0, \quad u^{(n-2)}(1)=\int_0^{\eta}u(s)ds, \tag{1.2}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. $n-1<\alpha\leq n, n\geq 3$, $0<\eta^{\alpha}<\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)$. We will prove the existence and uniqueness of a positive and nondecreasing solution for the boundary value problems (1.1)-(1.2) by using a fixed point theorem in partially ordered sets.

We note that the new three-point boundary conditions are related to the area under the curve of solutions $u(t)$ from $t=0$ to $t=\eta$. Existence of fixed point in partially ordered sets has been considered recently in [5, 11, 23, 24, 26]. This work is motivated by papers [5, 19].

2. Some definitions and fixed point theorems

The following definitions and lemmas will be used for proving our the main results.

Definition 2.1. Let $(E,\|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P\subset E$ is said to be a cone provided the following are satisfied:

- (a) if $y\in P$ and $\lambda\geq 0$, then $\lambda y\in P$;
- (b) if $y\in P$ and $-y\in P$, then $y=0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.2 ([25]). The integral

$$I_{0+}^{s} f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}} dt, \quad x > 0,$$

where $s > 0$, is called Riemann-Liouville fractional integral of order s and $\Gamma(s)$ is the Euler gamma function defined by

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt, \ s > 0.$$

Definition 2.3 ([14]). For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{s} f(x) = \frac{1}{\Gamma(n-s)} (\frac{d}{dx})^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}} dt,$$

where $n = [s] + 1, [s]$ denotes the integer part of number s, is called the Riemann-Liouville fractional derivative of order s.

The following two lemmas can be found in [3, 14] which are crucial in finding an integral representation of fractional boundary value problem (1.1) and (1.2).

Lemma 2.4 ([3, 14]). *Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$. Then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad c_i \in \mathbb{R}, \ i = 0, 1, \dots, n, \ n = [\alpha] + 1$$

as unique solution.

Lemma 2.5 ([3, 14]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \dots, n$, $n = [\alpha] + 1$.

The following fixed-point theorems in partially ordered sets are fundamental and important to the proofs of our main results.

Theorem 2.6 ([11]). *Let $(E, \leq)$ be a partially ordered set and suppose that there exists a metric d in E such that (E, d) is a complete metric space. Assume that E satisfies the following condition:*

$$\text{if } \{x_n\} \text{ is a nondecreasing sequence in } E \text{ such that } x_n \to x, \text{ then } x_n \leq x \text{ for all } n \in \mathbb{N}. \tag{2.1}$$

Let $T : E \to E$ be nondecreasing mapping such that

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)), \quad for \ x \geq y,$$

where $\psi : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0, +\infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in E$ with $x_0 \leq T(x_0)$, then T has a fixed point.

If we consider that $(E, \leq)$ satisfies the condition

$$\text{for } x, y \in E \text{ there exists } z \in E \text{ which is comparable to } x \text{ and } y, \tag{2.2}$$

then we have the following result.

Theorem 2.7 ([23]). *Adding condition* (2.2) *to the hypotheses of Theorem 2.6, we obtain uniqueness of the fixed point.*

3. Related lemmas

The basic space used in this paper is $E = C[0,1]$. Then E is a real Banach space with the norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \Leftrightarrow x(t) \leq y(t), \quad t \in [0,1].$$

In [23] it is proved that $(C[0,1], \leq)$ with the classic metric given by

$$d(x,y) = \sup_{0\leq t\leq 1} \{|x(t) - y(t)|\}$$

satisfied condition (2.1) of Theorem 2.6. Moreover, for $x, y \in C[0,1]$ as the function $\max\{x, y\} \in C[0,1]$, $(C[0,1], \leq)$ satisfies condition (2.2).

Lemma 3.1. *Let $0 < \eta^\alpha < \alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)$. If $h \in C[0,1]$, then the boundary-value problem*

$$D_{0+}^\alpha u(t) + h(t) = 0, \quad 0 < t < 1, \quad n-1 < \alpha \leq n, \tag{3.1}$$

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \quad u^{(n-2)}(1) = \int_0^\eta u(s)ds, \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{3.3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{3.4}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-n+1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.5}$$

$$G_2(t,s) = \frac{\alpha t^{\alpha-1}}{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2) - \eta^\alpha} \int_0^\eta G_1(t,s)dt. \tag{3.6}$$

Proof. By Lemma 2.5, the solution of (3.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds.$$

From (3.2), we know that $c_2 = c_3 = \dots = c_n = 0$ and

$$\begin{aligned} u^{(n-2)}(t) &= c_1(\alpha-1)(\alpha-2)\dots(\alpha-n+2)t^{\alpha-n+1} \\ &\quad - (\alpha-1)(\alpha-2)\dots(\alpha-n+2) \int_0^t \frac{(t-s)^{\alpha-n+1}}{\Gamma(\alpha)} h(s)ds. \end{aligned}$$

Thus, together with $u^{(n-2)}(1) = \int_0^\eta u(s)ds$, we have

$$c_1 = \frac{1}{(\alpha-1)(\alpha-2)\dots(\alpha-n+2)} \int_0^\eta u(s)ds + \int_0^1 \frac{(1-s)^{\alpha-n+1}}{\Gamma(\alpha)} h(s)ds.$$

Therefore, the unique solution of boundary value problem (3.1)-(3.2) is

$$
\begin{aligned}
u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\
&\quad + \frac{t^{\alpha-1}}{(\alpha-1)(\alpha-2)\dots(\alpha-n+2)} \int_0^\eta u(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-n+1}}{\Gamma(\alpha)} h(s)ds \\
&= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds + \frac{t^{\alpha-1}}{(\alpha-1)(\alpha-2)\dots(\alpha-n+2)} \int_0^\eta u(s)dx \\
&\quad + \int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-n+1}}{\Gamma(\alpha)} h(s)ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-n+1}}{\Gamma(\alpha)} h(s)ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}) h(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_t^1 t^{\alpha-1}(1-s)^{\alpha-n+1} h(s)ds \\
&\quad + \frac{t^{\alpha-1}}{(\alpha-1)(\alpha-2)\dots(\alpha-n+2)} \int_0^\eta u(s)ds \\
&= \int_0^1 G_1(t,s)h(s)ds + \frac{t^{\alpha-1}}{(\alpha-1)(\alpha-2)\dots(\alpha-n+2)} \int_0^\eta u(s)ds,
\end{aligned} \tag{3.7}
$$

where $G_1(t,s)$ is defined by (3.5). From (3.7), we have

$$
\int_0^\eta u(t)dt = \int_0^\eta \int_0^1 G_1(t,s)h(s)\,ds\,dt + \frac{\eta^\alpha}{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)} \int_0^\eta u(s)ds.
$$

It follows that

$$
\int_0^\eta u(t)dt = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)}{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)h(s)\,ds\,dt. \tag{3.8}
$$

Substituting (3.8) into (3.7), we obtain

$$
\begin{aligned}
u(t) &= \int_0^1 G_1(t,s)h(s)ds \\
&\quad + \frac{\alpha t^{\alpha-1}}{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)h(s)\,ds\,dt \\
&= \int_0^1 G_1(t,s)h(s)ds + \int_0^1 G_2(t,s)h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds,
\end{aligned}
$$

where $G(t,s)$, $G_1(t,s)$ and $G_2(t,s)$ are defined by (3.4), (3.5), (3.6), respectively. The proof is complete. □

Lemma 3.2. *The function $G_1(t,s)$ defined by (3.5) satisfies*

(i) *G_1 is a continuous function and $G_1(t,s) \ge 0$ for $(t,s) \in [0,1] \times [0,1]$;*

(ii)

$$
\sup_{t\in[0,1]} \int_0^1 G_1(t,s)ds = \frac{n-2}{(\alpha-n+2)\Gamma(\alpha+1)}.
$$

Proof. (i) The continuity of G_1 is easily checked. On the other hand, for $0 \le t \le s \le 1$ it is obvious that

$$G_1(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-n+1}}{\Gamma(\alpha)} \ge 0.$$

In the case $0 \le s \le t \le 1$ ($s \ne 1$), we have

$$\begin{aligned} G_1(t,s) &= \frac{1}{\Gamma(\alpha)}\Big[\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-s)^{n-2}} - (t-s)^{\alpha-1}\Big] \\ &\ge \frac{1}{\Gamma(\alpha)}\big[t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}\big] \\ &= \frac{1}{\Gamma(\alpha)}\big[(t-ts)^{\alpha-1} - (t-s)^{\alpha-1}\big] \ge 0. \end{aligned}$$

Moreover, as $G_1(t,1) = 0$, then we conclude that $G_1(t,s) \ge 0$ for all $(t,s) \in [0,1] \times [0,1]$.

(ii) Since

$$\begin{aligned} \int_0^1 G_1(t,s)ds &= \int_0^t G_1(t,s)ds + \int_t^1 G_1(t,s)ds \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1})ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_t^1 t^{\alpha-1}(1-s)^{\alpha-n+1}ds \\ &= \frac{1}{\Gamma(\alpha)}\Big(\frac{t^{\alpha-1}}{\alpha-n+2} - \frac{1}{\alpha}t^{\alpha}\Big). \end{aligned}$$

On the other hand, let

$$\rho(t) = \int_0^1 G_1(t,s)ds = \frac{1}{\Gamma(\alpha)}\Big(\frac{t^{\alpha-1}}{\alpha-n+2} - \frac{1}{\alpha}t^{\alpha}\Big),$$

then, as $n \ge 3$, we have

$$\rho'(t) = \frac{1}{\Gamma(\alpha)}\Big(\frac{\alpha-1}{\alpha-n+2}t^{\alpha-2} - t^{\alpha-1}\Big) > 0, \quad \text{for } t > 0,$$

the function $\rho(t)$ is strictly increasing and, consequently,

$$\begin{aligned} \sup_{t\in[0,1]}\rho(t) &= \sup_{t\in[0,1]}\int_0^1 G_1(t,s)ds = \rho(1) = \frac{1}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-n+2} - \frac{1}{\alpha}\Big) \\ &= \frac{n-2}{\alpha(\alpha-n+2)\Gamma(\alpha)} = \frac{n-2}{(\alpha-n+2)\Gamma(\alpha+1)}. \end{aligned}$$

The proof is complete. □

Remark 3.3. Obviously, by Lemma 3.1 and 3.2, we have $u(t) \ge 0$ if $h(t) \ge 0$ on $t \in [0,1]$.

Lemma 3.4. *$G_1(t,s)$ is strictly increasing in the first variable.*

Proof. For s fixed, we let

$$g_1(t) = \frac{1}{\Gamma(\alpha)}\big(t^{\alpha-1}(1-s)^{\alpha-n+1} - (t-s)^{\alpha-1}\big) \quad \text{for } s \le t,$$

$$g_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-s)^{\alpha-n+1} \quad \text{for } t \le s.$$

It is easy to check that $g_1(t)$ is strictly increasing on $[s, 1]$ and $g_2(t)$ is strictly increasing on $[0, s]$. Then we have the following cases:

Case 1: $t_1, t_2 \le s$ and $t_1 < t_2$. In this case, we have $g_2(t_1) < g_2(t_2)$, i.e. $G_1(t_1, s) < G_2(t_2, s)$.

Case 2: $s \le t_1, t_2$ and $t_1 < t_2$. In this case, we have $g_1(t_1) < g_1(t_2)$, i.e. $G_1(t_1, s) < G_1(t_2, s)$.

Case 3: $t_1 \le s \le t_2$ and $t_1 < t_2$. In this case, we have $g_2(t_1) \le g_2(s) = g_1(s) \le g_1(t_2)$. We claim that $g_2(t_1) < g_1(t_2)$. In fact, if $g_2(t_1) = g_1(t_2)$, then $g_2(t_1) = g_2(s) = g_1(s) = g_1(t_2)$, from the monotone of g_1 and g_2, we have $t_1 = s = t_2$, which contradicts with $t_1 < t_2$. This fact implies that $G_1(t_1, s) < G_1(t_2, s)$. The proof is complete. □

Remark 3.5. Obviously, by Lemma 3.4, we have

$$\int_0^1 G_2(t,s)ds \le \frac{\eta(n-2)}{\Gamma(\alpha)[\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha](\alpha-n+2)}. \tag{3.9}$$

Proof. In fact, from Lemma 3.4 and (3.6), we have

$$\begin{aligned} G_2(t,s) \le G_2(1,s) &= \frac{\alpha\eta G_1(1,s)}{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha} \\ &= \frac{\alpha\eta((1-s)^{\alpha-n+1}-(1-s)^{\alpha-1})}{\Gamma(\alpha)[\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha]}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^1 G_2(t,s)ds &\le \frac{\alpha\eta\int_0^1((1-s)^{\alpha-n+1}-(1-s)^{\alpha-1})ds}{\Gamma(\alpha)[\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha]} \\ &= \frac{\eta(n-2)}{\Gamma(\alpha)[\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha](\alpha-n+2)}, \end{aligned}$$

for $s, t \in [0,1] \times [0,1]$. □

4. Main Result

For notational convenience, we denote

$$\begin{aligned} L := &\frac{n-2}{(\alpha-n+2)\Gamma(\alpha+1)} \\ &+ \frac{\eta(n-2)}{\Gamma(\alpha)\left[\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+2)-\eta^\alpha\right](\alpha-n+2)} > 0. \end{aligned}$$

The main result of this paper is the following.

Theorem 4.1. *The boundary value problem* (1.1)-(1.2) *has a unique positive and strictly increasing solution $u(t)$ if the following conditions are satisfied:*

(i) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and nondecreasing respect to the second variable and $f(t,u(t)) \not\equiv 0$ for $t \in Z \subset [0,1]$ with $\mu(Z) > 0$ (μ denotes the Lebesgue measure);*

(ii) *There exists $0 < \lambda < \frac{1}{L}$ such that for $u, v \in [0,+\infty)$ with $u \ge v$ and $t \in [0,1]$*

$$f(t,u) - f(t,v) \le \lambda \ln(u-v+1).$$

Proof. Consider the cone

$$K = \{u \in C[0,1] : u(t) \geq 0\}.$$

As K is a closed set of $C[0,1]$, K is a complete metric space with the distance given by $d(u,v) = \sup_{t\in[0,1]} |u(t) - v(t)|$. Now, we consider the operator T defined by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds,$$

where $G(t,s)$ is defined by (3.4). By Lemma 3.2 and condition (i), we have that $T(K) \subset K$.

We now show that all the conditions of Theorem 2.6 and Theorem 2.7 are satisfied.

Firstly, by condition (i), for $u, v \in K$ and $u \geq v$, we have

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \int_0^1 G(t,s)f(s,v(s))ds = Tv(t).$$

This proves that T is a nondecreasing operator.

On the other hand, for $u \geq v$ and by condition (ii) we have

$$\begin{aligned} d(Tu,Tv) &= \sup_{0\leq t\leq 1} |(Tu)(t) - (Tv)(t)| = \sup_{0\leq t\leq 1} ((Tu)(t) - (Tv)(t)) \\ &= \sup_{0\leq t\leq 1} \int_0^1 G(t,s)(f(s,u(s)) - f(s,v(s)))ds \\ &\leq \sup_{0\leq t\leq 1} \int_0^1 G(t,s)\lambda \cdot \ln(u(s) - v(s) + 1)ds. \end{aligned}$$

Since the function $h(x) = \ln(x+1)$ is nondecreasing, by Lemma 3.2 (ii), Remark 3.5 and condition (ii), we have

$$\begin{aligned} & d(Tu,Tv) \\ & \leq \lambda \ln(\|u - v\| + 1)\Big(\sup_{0\leq t\leq 1} \int_0^1 G_1(t,s)ds + \sup_{0\leq t\leq 1} \int_0^1 G_2(t,s)ds\Big) \\ & \leq \lambda \ln(\|u - v\| + 1) \cdot L \\ & \leq \|u - v\| - (\|u - v\| - \ln(\|u - v\| + 1)). \end{aligned}$$

Let $\psi(x) = x - \ln(x+1)$. Obviously $\psi : [0,+\infty) \to [0,+\infty)$ is continuous, nondecreasing, positive in $(0,+\infty)$, $\psi(0) = 0$ and $\lim_{x\to+\infty} \psi(x) = +\infty$. Thus, for $u \geq v$, we have

$$d(Tu,Tv) \leq d(u,v) - \psi(d(u,v)).$$

As $G(t,s) \geq 0$ and $f \geq 0$, $(T0)(t) = \int_0^1 G(t,s)f(s,0)ds \geq 0$ and by Theorem 2.6 we know that problem (1.1)-(1.2) has at least one nonnegative solution. As $(K, \leq)$ satisfies condition (2.2), thus, Theorem 2.7 implies that uniqueness of the solution.

Finally, we will prove that this solution $u(t)$ is strictly increasing function. As $u(0) = \int_0^1 G(0,s)f(s,u(s))ds$ and $G(0,s) = 0$ we have $u(0) = 0$.

Moreover, if we take $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we can consider the following cases.

Case 1: $t_1 = 0$, in this case, $u(t_1) = 0$ and, as $u(t) \geq 0$, suppose that $u(t_2) = 0$. Then

$$0 = u(t_2) = \int_0^1 G(t_2,s)f(s,u(s))ds.$$

This implies that

$$G(t_2, s) \cdot f(s, u(s)) = 0, \quad \text{a.e. } (s)$$

and as $G(t_2, s) \neq 0$ a.e.(s) we get $f(s, u(s)) = 0$ a.e. (s). On the other hand, f is nondecreasing respect to the second variable, then we have

$$f(s, 0) \leq f(s, u(s)) = 0, \quad \text{a.e. } (s)$$

which contradicts the condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1](\mu(Z) \neq 0)$. Thus $u(t_1) = 0 < u(t_2)$.

Case 2: $0 < t_1$. In this case, let us take $t_2, t_1 \in [0, 1]$ with $t_1 < t_2$, then

$$\begin{aligned} u(t_2) - u(t_1) &= (Tu)(t_2) - (Tu)(t_1) \\ &= \int_0^1 (G(t_2, s) - G(t_1, s))f(s, u(s))ds. \end{aligned}$$

Taking into account Lemma 3.4 and the fact that $f \geq 0$, we get $u(t_2) - u(t_1) \geq 0$.

Suppose that $u(t_2) = u(t_1)$ then

$$\int_0^1 (G(t_2, s) - G(t_1, s))f(s, u(s))ds = 0$$

and this implies

$$(G(t_2, s) - G(t_1, s))f(s, u(s)) = 0 \quad \text{a.e. } (s).$$

Again, Lemma 3.4 gives us

$$f(s, u(s)) = 0 \quad \text{a.e. } (s)$$

and using the same reasoning as above we have that this contradicts condition (i) $f(t, 0) \neq 0$ for $t \in Z \subset [0, 1](\mu(Z) \neq 0)$. Thus $u(t_1) = 0 < u(t_2)$. The proof is complete. □

5. Example

The fractional boundary-value problem

$$\begin{gathered} D_{0+}^{7/2} u(t) + (t^2 + 1)\ln(2 + u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \int_0^{1/2} u(s)ds. \end{gathered} \tag{5.1}$$

has a unique and strictly increasing solution.

In this case, $n = 4$, $\alpha = 7/2$, $\eta = 1/2$, $f(t, u) = (t^2 + 1)\ln(2 + u(t))$ for $(t, u) \in [0, 1] \times [0, \infty)$. Note that f is a continuous function and $f(t, u) \neq 0$ for $t \in [0, 1]$. Moreover, f is nondecreasing respect to the second variable since $\frac{\partial f}{\partial u} = \frac{1}{u+2}(t^2 + 1) > 0$. On the other hand, for $u \geq v$ and $t \in [0, 1]$, we have

$$\begin{aligned} f(t, u) - f(t, v) &= (t^2 + 1)\ln(2 + u) - (t^2 + 1)\ln(2 + v) = (t^2 + 1)\ln\Big(\frac{2 + u}{2 + v}\Big) \\ &= (t^2 + 1)\ln\Big(\frac{2 + v + u - v}{2 + v}\Big) = (t^2 + 1)\ln\Big(1 + \frac{u - v}{2 + v}\Big) \\ &\leq (t^2 + 1)\ln(1 + (u - v)) \leq 2\ln(1 + u - v). \end{aligned}$$

In this case, $\lambda = 2$. By simple computation, we have $0 < \lambda < 1/L$. Thus, Theorem 4.1 implies that boundary value problem (1.1)-(1.2) has a unique and strictly increasing solution.

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