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SOLUTIONS TO FOURTH-ORDER RANDOM DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

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ABSTRACT. Existence of solutions and of extremal random solutions are proved for periodic boundary-value problems of fourth-order ordinary random differential equations. Our investigation is done in the space of continuous real-valued functions defined on closed and bounded intervals. Also we study the applications of the random version of a nonlinear alternative of Leray-Schauder type and an algebraic random fixed point theorem by Dhage.

1. INTRODUCTION

Let $\mathbb{R}$ denote the real line and let $J = [0, 1]$, a closed and bounded interval in $\mathbb{R}$. Let $C^1(J, \mathbb{R})$ denote the class of real-valued functions defined and continuously on J. Given a measurable space $(\Omega, \mathcal{A})$ and a measurable function $x : \Omega \to AC^3(J, \mathbb{R})$, we consider a fourth-order periodic boundary-value problem of ordinary random differential equations (for short PBVP)

$$\begin{gathered} x^{(4)}(t, \omega) = f(t, x(t, \omega), x''(t, \omega), \omega) \quad \text{a.e. } t \in J \\ x^{(i)}(0, \omega) = x^{(i)}(1, \omega), \quad i = 0, 1, 2, 3 \end{gathered} \tag{1.1}$$

for all $\omega \in \Omega$, where $f : J \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$.

By a *random solution* of equation (1.1) we mean a measurable function $x : \Omega \to AC^{(3)}(J, \mathbb{R})$ that satisfies the equation (1.1), where $AC^{(3)}(J, \mathbb{R})$ is the space of real-valued functions whose 3rd derivative exists and is absolutely continuously differentiable on J.

When the random parameter ω is absent, the random (1.1) reduce to the fourth-order ordinary differential equations,

$$\begin{gathered} x^{(4)}(t) = f(t, x(t), x''(t)) \quad \text{a.e. } t \in J \\ x^{(i)}(0) = x^{(i)}(1), \quad i = 0, 1, 2, 3 \end{gathered} \tag{1.2}$$

where, $f : J \times \mathbb{R} \to \mathbb{R}$.

Equation (1.2) has been studied by many authors for different aspects of solutions. See for example [7, 8, 11, 12, 13]. Only a few authors have studied the

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random periodic boundary-value problem, see [5, 1, 14], Dhage [5] studied the periodic boundary-value problems for the random differential equation

$$-x''(t,\omega) = f(t, x(t,\omega), \omega) \quad \text{a.e. } t \in J,$$
$$x(0,\omega) = x(2\pi,\omega), \quad x'(0,\omega) = x'(2\pi,\omega)\,.$$

In this article, we study the existence of solutions and the existence of extremal solutions for the fourth-order random equation (1.1), under suitable conditions. Our work relays on the random versions of fixed point theorems based on the theorems in [2, 3].

2. Existence result

Let E denote a Banach space with the norm $\|\cdot\|$ and let $Q : E \to E$. We further assume that the Banach space E is separable; i.e., E has a countable dense subset and let β_E be the σ-algebra of Borel subsets of E. We say a mapping $x : \Omega \to E$ is measurable if for any $B \in \beta_E$,

$$x^{-1}(B) = \{\omega \in \Omega : x(\omega) \in B\} \in \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a map is jointly measurable, a mapping $x : \Omega \times E \to E$ is called $jointlymeasurable$, if for any $B \in \beta_E$, one has

$$x^{-1}(B) = \{(\omega, x) \in \Omega \times E : x(\omega, x) \in B\} \in \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E respectively.

Let $Q : \Omega \times E \to E$ be a mapping. Then Q is called a random operator if $Q(\omega, x)$ is measurable in ω for all $x \in E$ and it is expressed as $Q(\omega)x = Q(\omega, x)$. A random operator $Q(\omega)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $Q(\omega, x)$ is continuous (resp. compact, totally bounded and completely continuous) in x for all $\omega \in \Omega$. We could get more details of completely continuous random operators on Banach spaces and their properties in Itoh [6]. In this article, we use the following lemma in proving the main result of this paper, that lemma is an immediate corollary to the results in [2, 3].

Lemma 2.1 ([5]). *Let $\mathcal{B}_R(0)$ and $\bar{\mathcal{B}}_R(0)$ be the open and closed balls centered at origin of radius R in the separable Banach space E and let $Q : \Omega \times \bar{\mathcal{B}}_R(0) \to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u \in E$ with $\|u\| = R$ such that $Q(\omega)u = \alpha u$ for all $\alpha \in \Omega$, where $\alpha > 1$. Then the random equation $Q(\omega)x = x$ has a random solution; i.e., there is a measurable function $\xi : \Omega \to \bar{\mathcal{B}}_R(0)$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.*

Lemma 2.2 ([5]). *Let $Q : \Omega \times E \to E$ be a mapping such that $Q(\cdot, x)$ is measurable for all $x \in E$ and $Q(\omega, \cdot)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to Q(\omega, x)$ is jointly measurable.*

We need the following definitions in the sequel.

Definition 2.3. A function $f : J \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if

- the map $(t, \omega) \to f(t, x, y, \omega)$ is jointly measurable for all $(x, y) \in \mathbb{R}^2$, and
- the map $(x, y) \to f(t, x, y, \omega)$ is continuous for almost all $t \in J$ and $\omega \in \Omega$.

Definition 2.4. A function $f : J\times\mathbb{R}\times\mathbb{R}\times\Omega \to \mathbb{R}$ is called random L^1-Carathéodory if

- for each real number $r > 0$ there is a measurable and bounded function $q_r : \Omega \to L^1(J,\mathbb{R})$ such that
$$|f(t,x,y,\omega)| \le q_r(t,\omega) \quad \text{a.e. } t \in J$$
whenever $|x|, |y| \le r$, and for all $\omega \in \Omega$.

Now we seek the random solutions of (1.1) in the Banach space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We equip this space with the supremum norm
$$\|x\| = \sup_{t\in J} |x(t)|.$$
It is know that the Banach space $C(J,\mathbb{R})$ is separable. We use $L^1(J,\mathbb{R})$ denote the space of Lebesque measurable real-valued functions defined on J, and the usual norm in $L^1(J,\mathbb{R})$ defined by
$$\|x\|_{L^1} = \int_0^1 |x(t)|dt.$$
For a given real number $M \in (0, 4\pi^4), h \in C(J,\mathbb{R})$, consider the linear PBVP
$$\begin{gathered} x^{(4)}(t) + Mx(t) = h(t) \quad t \in J \\ x^{(i)}(0) = x^{(i)}(1), \quad i = 0, 1, 2, 3. \end{gathered} \tag{2.1}$$
By the theorem of [10], the unique solution of problem
$$\begin{gathered} x^{(4)}(t) + Mx(t) = 0 \quad t \in J \\ x^{(i)}(0) = x^{(i)}(1), \quad i = 0, 1, 2 \\ x^{(3)}(0) - x^{(3)}(1) = 1 \end{gathered} \tag{2.2}$$
has a unique solution $r(t) \in C^4(J,\mathbb{R})$ satisfying $r(t) > 0$. Then the unique solution of (2.1) is
$$x(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.3}$$
where
$$G(t,s) = \begin{cases} r(t-s), & 0 \le s \le t \le 1; \\ r(1+t-s), & 0 \le t < s \le 1. \end{cases} \tag{2.4}$$
We consider the following set of hypotheses:

(H1) The function f is random Carathéodory on $J \times \mathbb{R} \times \mathbb{R} \times \Omega$.

(H2) There exists a measurable and bounded function $\gamma : \Omega \to L^2(J,\mathbb{R})$ and a continuous and nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$ such that
$$|f(s,x,x'',\omega) + Mx| \le \gamma(t,\omega)\psi(|x|) \quad \text{a.e. } t \in J$$
for all $\omega \in \Omega$ and $x \in \mathbb{R}$.

Our main existence result is as follows.

Theorem 2.5. *Assume that* (H1)–(H2) *hold. Suppose that there exists a real number $R > 0$ such that*
$$R > r_M\|\gamma(\omega)\|_{L^1}\psi(R) \tag{2.5}$$

for all $t \in J$ and $\omega \in \Omega$, where $r_M = \max_{t\in[0,1]} r(t), r(t)$ is in the Green's function (2.4)*. Then* (1.1) *has a random solution defined on J.*

Proof. Set $E = C(J, \mathbb{R})$ and define a mapping $Q : \Omega \times E \to E$ by

$$Q(\omega)x(t) = \int_0^1 G(t,s)(f(s, x(s,\omega), x''(s,\omega), \omega) + Mx(s,\omega))ds \tag{2.6}$$

for all $t \in J$, $\omega \in \Omega$. Then the solutions of (1.1) are fixed points of operator Q.

Define a closed ball $\bar{\mathcal{B}}_R(0)$ in E centered at origin 0 of radius R, where the real number R satisfies the inequality (2.5). We show that Q satisfies all the conditions of Lemma2.1 on $\bar{\mathcal{B}}_R(0)$.

First we show that Q is a random operator in $\bar{\mathcal{B}}_R(0)$, since $f(t, x, x'', \omega)$ is random Carathéodory and $x(t, \omega)$ is measurable, the map $\omega \to f(t, x, x'', \omega) + Mx$ is measurable. Similarly, the production $G(t, s)[f(s, x(s, \omega), x''(s, \omega), \omega) + Mx(s, \omega)]$ of a continuous and measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\omega \mapsto \int_0^1 G(t,s)(f(s, x(s,\omega), x''(s,\omega), \omega) + Mx(s,\omega))ds = Q(\omega)x(t)$$

is measurable. As a result, Q is a random operator on $\Omega \times \bar{\mathcal{B}}_R(0)$ into E.

Next we show that the random operator $Q(\omega)$ is continuous on $\bar{\mathcal{B}}_R(0)$. Let x_n be a sequence of points in $\bar{\mathcal{B}}_R(0)$ converging to the point x in $\bar{\mathcal{B}}_R(0)$. Then it is sufficiente to prove that

$$\lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t) \quad \text{for all } t \in J, \omega \in \Omega.$$

By the dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Q(\omega)x_n(t) &= \lim_{n\to\infty} \int_0^1 G(t,s)(f(s, x_n(s,\omega), x_n''(s,\omega), \omega) + Mx_n(s,\omega))ds \\
&= \int_0^1 G(t,s) \lim_{n\to\infty} [f(s, x_n(x,\omega), x_n''(s,\omega), \omega) + Mx_n(s,\omega)]ds \\
&= \int_0^1 G(t,s)[f(s, x(s,\omega), x''(s,\omega), \omega) + Mx_n(s,\omega)]ds \\
&= Q(\omega)x(t)
\end{aligned}$$

for all $t \in J, \omega \in \Omega$. This shows that $Q(\omega)$ is a continuous random operator on $\bar{\mathcal{B}}_r(0)$.

Now we show that $Q(\omega)$ is compact random operator on $\bar{\mathcal{B}}_R(0)$. To finish it, we should prove that $Q(\omega)(\bar{\mathcal{B}}_r(0))$ is uniformly bounded and equi-continuous set in E for each $\omega \in \Omega$. Since the map $\omega \to \gamma(t, \omega)$ is bounded and $L^2(J, \mathbb{R}) \subset L^1(J, \mathbb{R})$, by ($H_2$), there is a constant c such that $\|\gamma(\omega)\|_{L^1} \le c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed, then for any $x : \Omega \to \bar{\mathcal{B}}_R(0)$, one has

$$\begin{aligned}
|Q(\omega)x(t)| &\le \int_0^1 G(t,s)|(f(s, x(s,\omega), x''(s,\omega), \omega) + Mx(s,\omega))|ds \\
&\le \int_0^1 G(t,s)\gamma(s,\omega)\psi(|x(s,\omega)|)ds \\
&\le r_M c\psi(R) = K
\end{aligned}$$

for all $t \in J$ and each $\omega \in \Omega$. This shows that $Q(\omega)(\bar{\mathcal{B}}_R(0))$ is a uniformly bounded subset of E for each $\omega \in \Omega$.

Next we show $Q(\omega)(\bar{\mathcal{B}}_R(0))$ is an equi-continuous set in E. For any $x \in \bar{\mathcal{B}}_R(0)$, $t_1, t_2 \in J$, we have

$$\begin{aligned}|Q(\omega)x(t_1) - Q(\omega)x(t_2)| &\leq \int_0^1 |(G(t_1,s) - G(t_2,s))|\gamma(s,\omega)\psi(|x(s,\omega)|)ds \\ &\leq \int_0^1 |(G(t_1,s) - G(t_2,s))|\gamma(s,\omega)\psi(R)ds,\end{aligned}$$

by hölder inequality,

$$\begin{aligned}&|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \\ &\leq \Big(\int_0^1 |G(t_1,s) - G(t_2,s)|^2 ds\Big)^{1/2}\Big(\int_0^1 |\gamma(s,\omega)|^2 ds\Big)^{1/2}\psi(R).\end{aligned}$$

Hence for all $t_1, t_2 \in J$,

$$|Q(w)x(t_1) - Q(\omega)x(t_2)| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly for all $x \in \bar{\mathcal{B}}_R(0)$. Therefore, $Q(\omega)\bar{\mathcal{B}}_R(0)$ is an equi-continuous set in E, then we know it is compact by Arzelá-Ascoli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on $\bar{\mathcal{B}}_R(0)$.

Finally, we suppose there exists such an element u in E with $\|u\| = R$ satisfying $Q(\omega)u(t) = \alpha u(t,\omega)$ for some $\omega \in \Omega$, where $\alpha > 1$. Now for this $\omega \in \Omega$, we have

$$\begin{aligned}|u(t,\omega)| &\leq \frac{1}{\alpha}|Q(\omega)u(t)| \\ &\leq \int_0^1 G(t,s)|f(s,u(s,\omega),u''(s,\omega),\omega) + Mu(s,\omega)|ds \\ &\leq r_M \int_0^1 \gamma(s,\omega)\psi(|u(s,\omega)|)ds \\ &\leq r_M \|\gamma(\omega)\|_{L^1}\psi(\|u(\omega)\|) \quad \text{for all } t \in J.\end{aligned}$$

Taking supremum over t in the above inequality yields

$$R = \|u(\omega)\| \leq r_M \|\gamma(\omega)\|_{L^1}\psi(R)$$

for some $\omega \in \Omega$. This contradicts to condition (2.5).

Thus, all the conditions of Lemma2.1 are satisfied. Hence the random equation

$$Q(\omega)x(t) = x(t,\omega)$$

has a random solution in $\bar{\mathcal{B}}_R(0)$; i.e., there is a measurable function $\xi : \Omega \to \bar{\mathcal{B}}_R(0)$ such that $Q(\omega)\xi(t) = \xi(t,\omega)$ for all $t \in J, \omega \in \Omega$. As a result, the random (1.1) has a random solution defined on J. This completes the proof. □

3. Extremal random solutions

It is sometimes desirable to know the realistic behavior of random solutions of a given dynamical system. Therefore, we prove the existence of extremal positive random solution of (1.1) defined on $\Omega \times J$.

We introduce an order relation $\leq$ in $C(J,\mathbb{R})$ with the help of a cone K defined by

$$K = \{x \in C(J,\mathbb{R}) : x(t) \geq 0 \text{ on } J\}.$$

Let $x, y \in X$, then $x \leq y$ if and only if $y - x \in K$. Thus, we have

$$x \leq y \;\Leftrightarrow\; x(t) \leq y(t) \text{ for all } t \in J.$$

It is known that the cone K is normal in $C(J, \mathbb{R})$. For any function $a, b : \Omega \to C(J, \mathbb{R})$ we define a random interval $[a, b]$ in $C(J, \mathbb{R})$ by

$$[a, b] = \{x \in C(J, \mathbb{R}) : a(\omega) \leq x \leq b(\omega)\, \forall \omega \in \Omega\} = \cap_{\omega \in \Omega}[a(\omega), b(\omega)].$$

Definition 3.1. An operator $Q : \Omega \times E \to E$ is called nondecreasing if $Q(\omega)x \leq Q(\omega)y$ for all $\omega \in \Omega$, and for all $x, y \in E$ for which $x \leq y$.

We use the following random fixed point theorem of Dhage in what follows.

Lemma 3.2 (Dhage [2])**.** *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a, b]$ be a random order interval in the separable Banach space E. Let $Q : \Omega \times [a, b] \to [a, b]$ be a completely continuous and nondecreasing random operator. Then Q has a minimal fixed point x_* and a maximal random fixed point y^* in $[a, b]$. Moreover, the sequences $\{Q(\omega)x_n\}$ with $x_0 = a$ and $\{Q(\omega)y_n\}$ with $y_0 = b$ converge to x_* and y^* respectively.*

We need the following definitions in the sequel.

Definition 3.3. A measurable function $\alpha : \Omega \to C(J, \mathbb{R})$ is called a lower random solution of (1.1) if

$$\begin{gathered}
\alpha^{(4)}(t, \omega) \leq f(t, \alpha(t, \omega), \alpha(t, \omega), \omega) \quad \text{a.e. } t \in J. \\
\alpha^{(i)}(0, \omega) = \alpha^{(i)}(1, \omega), \quad i = 0, 1, 2. \\
\alpha^{(3)}(0, \omega) \leq \alpha^{(3)}(1, \omega)
\end{gathered}$$

for all $\omega \in \Omega$. Similarly, a measurable function $\beta : \Omega \to C(J, \mathbb{R})$ is called an upper random solution of (1.1) if

$$\begin{gathered}
\beta^{(4)}(t, \omega) \geq f(t, \alpha(t, \omega), \alpha(t, \omega), \omega) \quad \text{a.e. } t \in J. \\
\beta^{(i)}(0, \omega) = \beta^{(i)}(1, \omega), \quad i = 0, 1, 2. \\
\beta^{(3)}(0, \omega) \geq \beta^{(3)}(1, \omega)
\end{gathered}$$

for all $t \in J$ and $\omega \in \Omega$.

Definition 3.4. A random solution θ of (1.1) is called maximal if for all random solutions of (1.1), one has $x(t, \omega) \leq \theta(t, \omega)$ for all $t \in J$ and $\omega \in \Omega$.

A minimal random solution of (1.1) on J is defined similarly,

We consider the following set of assumptions:

(H3) Problem (1.1) has a lower random solution α and upper random solution β with $\alpha \leq \beta$ on J.

(H4) For any $u_2, u_1 \in [\alpha, \beta]$ and $u_2 > u_1$

$$f(t, u_2, v, \omega) - f(t, u_1, v, \omega) \geq -M(u_1 - u_2)$$

for a.e. $t \in [0, 1]$ and $\omega \in \Omega$.

(H5) The function $q : J \times \Omega \to \mathbb{R}_+$ defined by

$$q(t, \omega) = |f(t, \alpha(t, \omega), \alpha''(t, \omega), \omega) + M\alpha(t, \omega)| + |f(t, \beta(t, \omega), \beta''(t, \omega), \omega) + M\beta(t, \omega)|$$

is Lebesgue integrable in t for all $\omega \in \Omega$.

Hypotheses (H3) holds, in particular, when there exist measurable functions $u, v : \Omega \to C(J, \mathbb{R})$ such that for each $\omega \in \Omega$,

$$u(t, \omega) \le f(t, x, y, \omega) + Mx \le v(t, \omega)$$

for all $t \in J$ and $x \in \mathbb{R}$. In this case, the lower and upper random solutions of (1.1) are given by

$$\alpha(t, \omega) = \int_0^1 G(t, s)u(s, \omega)ds$$

and

$$\beta(t, \omega) = \int_0^1 G(t, s)v(s, \omega)ds$$

respectively. The details about the lower and upper random solutions for different types of random differential equations could be found in [9]. Hypotheses (H4) is natural and used in several research papers. Finally, if f is L^1-Carathéodory on $\mathbb{R} \times \Omega$, then (H5) remains valid.

Theorem 3.5. *Assume that* (H), (H3)–(H5) *hold, then* (1.1) *has a minimal random solution $x_*(\omega)$ and a maximal random solution $y^*(\omega)$ defined on J. Moreover,*

$$x_*(t, \omega) = \lim_{n\to\infty} x_n(t, \omega), \quad y^*(t, \omega) = \lim_{n\to\infty} y_n(t, \omega)$$

for all $t \in J$ and $\omega \in \Omega$, where the random sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given by

$$x_{n+1}(t, \omega) = \int_0^1 G(t, s)(f(s, x_n(s, \omega), x_n''(s, \omega), \omega) + Mx_n(s, \omega))ds$$

for $n \ge 0$ with $x_0 = \alpha$, and

$$y_{n+1}(t, \omega) = \int_0^1 G(t, s)(f(s, y_n(s, \omega), y_n''(s, \omega), \omega) + My_n(s, \omega))ds$$

for $n \ge 0$ with $y_0 = \beta$, for all $t \in J$ and $\omega \in \Omega$.

Proof. We Set $E = C(J, \mathbb{R})$ and define an operator $Q : \Omega \times [\alpha, \beta] \to E$ by (2.6). We show that Q satisfies all the conditions of Lemma3.1 on $[\alpha, \beta]$.

It can be shown as in the proof of Theorem 2.1 that Q is a random operator on $\Omega \times [\alpha, \beta]$. We show that it is nondecreasing random operator on $[\alpha, \beta]$. Let $x, y : \Omega \to [\alpha, \beta]$ be arbitrary such that $x \le y$ on Ω. Then

$$\begin{aligned}
&Q(\omega)y(t) - Q(\omega)x(t) \\
&= \int_0^1 G(t, s)\Big[\big(f(s, y(s, \omega), y''(s, \omega), \omega) - f(s, x(s, \omega), x''(s, \omega), \omega)\big) \\
&\quad + M(y(s, \omega) - x(s, \omega))\Big]ds \\
&\ge \int_0^1 G(t, s)[(-M(y(s, \omega) - x(s, \omega)) + M(y(s, \omega) - x(s, \omega)]ds = 0
\end{aligned}$$

for all $t \in J$ and $\omega \in \Omega$. As a result, $Q(\omega)x \le Q(\omega)y$ for all $\omega \in \Omega$ and that Q is nondecreasing random operator on $[\alpha, \beta]$.

Now, by (H4),

$$\alpha(t, \omega) \le Q(\omega)\alpha(t)$$

$$
\begin{aligned}
&= \int_0^1 G(t,s)[f(\alpha(s,\alpha'(s,\omega),\alpha''(s,\omega),\omega),\omega) + M\alpha(s,\omega)]ds \\
&\le \int_0^1 G(t,s) f(s,x'(s,\omega),x''(s,\omega),\omega) + Mx(s,\omega)ds \\
&= Q(\omega)x(t) \\
&\le Q(\omega)\beta(t) \\
&= \int_0^1 G(t,s)[f(\beta(s,\beta'(s,\omega),\beta''(s,\omega),\omega),\omega) + M\beta(s,\omega)]ds \\
&\le \beta(t,\omega)
\end{aligned}
$$

for all $t \in J$ and $\omega \in \Omega$. As a result Q defines a random operator $Q : \Omega \times [\alpha, \beta] \to [\alpha, \beta]$.

Then, since (H5) holds, we replace $\gamma(t,\omega)$ and $\psi(r)$ with $\gamma(t,\omega) = q(t,\omega)$ for all $(t,\omega) \in J \times \Omega$ and $\psi(R) = 1$ for all real number $R \ge 0$. Now it can be show as in the proof of Theorem 2.1 that the random operator $Q(\omega)$ satisfies all the conditions of Lemma 3.1 and so the random operator equation $Q(\omega)x = x(\omega)$ has a least and a greatest random solution in $[\alpha, \beta]$. Consequently, (1.1) has a minimal and a maximal random solution defined on J. The proof is complete. □

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