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POSITIVE SOLUTIONS FOR NONLINEAR ELLIPTIC SYSTEMS

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Abstract. In this article, we study the existence of positive solutions for the system

$$\Delta u = H(x, u, v),$$
$$\Delta v = K(x, u, v), \text{in } \mathbb{R}^n \ (n \geq 3),$$

where $H, K : \mathbb{R}^n \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous functions satisfying $H(x, u, v) \leq p_1(|x|)F(u + v)$ and $K(x, u, v) \leq q_1(|x|)G(u + v)$. In terms of the growth of the variable potential functions p_1, q_1 and the nonlinearities F and G, we establish some sufficient conditions for the existence of positive continuous solutions for this system and we discuss whether these solutions are bounded or blow up at infinity.

1. Introduction

Semilinear elliptic systems of the form

$$\begin{aligned} \Delta u &= H(x, u, v), \quad \text{in } \mathbb{R}^n \ (n \geq 3), \\ \Delta v &= K(x, u, v), \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

have been studied intensively in the previous few years and various results concerning the existence and nonexistence of positive entire large or bounded solutions have been obtained. We refer the reader to [2, 3, 4, 5, 6, 9, 11, 12, 15, 16, 18, 19, 20, 21] and their references for recent results concerning the existence and qualitative analysis of solutions of (1.1).

The interest in systems of nonlinear stationary equations is motivated by applications to theory of Newtonian fluids and nonlinear optics. More precisely, coupled nonlinear stationary systems arise in the description of several physical phenomena such as the propagation of pulses in birefringent optical fibers and Kerr-like photorefractive media, see [1, 13].

When $H(x, u, v) = p(|x|)v^\alpha$, $K(x, u, v) = q(|x|)u^\beta$, $0 < \alpha \leq \beta$, Lair and Wood in [11] considered the existence and nonexistence of entire positive radial solutions to (1.1) under the conditions of integrability or non integrability of the functions $r \to rp(r)$ and $r \to rq(r)$ on $(0, \infty)$. Their results were extended by Cîrstea and Rădulescu [3], Wang and Wood [18], Ghergu and Rădulescu [6], Peng and Song [15], Ghanmi et al [5], Li et al [12], Zhang [20]. In [21], the authors considered the case

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where $H(x,u,v)=p(x)f(v)$, $K(x,u,v)=q(x)g(u)$ with p,q nontrivial nonnegative continuous on $[0,\infty)$ and f,g continuous , nondecreasing on $[0,\infty)$ that are positive on $(0,\infty)$. In the case where p,q are radial and under the condition

$$\int_1^\infty \Big(\int_0^t (f(s)+g(s))ds\Big)^{-1/2} dt = \infty\,, \tag{1.2}$$

they proved that (1.1) has one positive solution (u,v). Moreover, if

$$\int_0^\infty sp(s)\,ds = \int_0^\infty sq(s)\,ds = \infty\,,$$

then every positive radial entire solution (u,v) of (1.1) is large (i.e. $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} v(x) = \infty$). And if

$$\int_0^\infty sp(s)\,ds < \infty \text{ and } \int_0^\infty sq(s)\,ds < \infty\,,$$

then every positive radial entire solution (u,v) of (1.1) is bounded.

By using the sub-supersolution method, they establish some conditions on p, q in order to prove the existence of positive bounded solutions in the case where p,q are non radial. Their results extend partially those of Zhang [20], where the existence of entire positive radial large solutions or bounded ones for (1.1) was studied under the condition $\int_1^\infty \frac{ds}{f(s)+g(s)} = \infty$.

Their results do not cover the cases where

$$\begin{aligned}
H(x,u,v) &= a_1(x)v^{\alpha_1} + b_1(x)u^{\gamma_1} + c_1(x)(u+v)^{\beta_1} + d_1(x)u^{\delta_1}v^{\lambda_1},\\
K(x,u,v) &= a_2(x)v^{\alpha_2} + b_2(x)u^{\gamma_2} + c_2(x)(u+v)^{\beta_2} + d_2(x)u^{\delta_2}v^{\lambda_2}\,,
\end{aligned}$$

where α_i, β_i, γ_i, δ_i, $\lambda_i \in (0,\infty)$.

Our aim in this paper is to extend the results in [4] for the particular case of the Dirichlet laplacian and to extend those in [21] to a wider class of functions $H(x,u,v)$ and $K(x,u,v)$. More precisely, our results apply in particular to the previous examples of functions H, K and to the case where $H(x,u,v)=p(x)f(u,v)$, $K(x,u,v)=q(x)g(u,v)$ with f and g are nondecreasing with respect to first and the second variables. To this aim, we assume that H and K satisfy the following hypotheses:

- (H1) H, $K : \mathbb{R}^n \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.
- (H2) There exist nonnegative functions p_i, q_i, f_i, g_i, $1 \le i \le 2$, F and G satisfying for each $x \in \mathbb{R}^n$ and $(u,v) \in [0,\infty)\times[0,\infty)$,
$$\begin{aligned}
p_2(|x|)g_1(v)g_2(u) &\le H(x,u,v) \le p_1(|x|)F(u+v),\\
q_2(|x|)f_1(u)f_2(v) &\le K(x,u,v) \le q_1(|x|)G(u+v),
\end{aligned}$$
with f_i, g_i, $1 \le i \le 2$, F, $G : [0,\infty) \to [0,\infty)$ are nondecreasing continuous, positive on $(0,\infty)$ and p_i, $q_i : [0,\infty) \to [0,\infty)$ are continuous.
- (H3) There exist $c > 0$ such that
$$\int_0^t \sqrt{\Lambda(s)}ds < L_c(\infty) := \lim_{r\to\infty} L_c(r),$$

for all $t > 0$, where for $\alpha > 0$,

$$
\begin{gathered}
L_\alpha(t) = \int_\alpha^t \frac{ds}{\sqrt{2(\mathcal{F}(s)+\mathcal{G}(s))}}, \quad t \geq \alpha, \\
\Lambda(r) = \max_{s\in[0,r]} (p_1(s)+q_1(s)), \quad r \geq 0\,, \\
\mathcal{F}(r) = \int_0^r F(s)ds\,, \quad \mathcal{G}(r) = \int_0^r G(s)ds\,, \quad r \geq 0.
\end{gathered} \tag{1.3}
$$

We note that L_α has an inverse function L_α^{-1} from $[\alpha,\infty)$ to $[0, L_\alpha(\infty))$, where $L_\alpha(\infty) = \lim_{r\to\infty} L_\alpha(r) \in (0,\infty]$.

Remark 1.1. If $L_\alpha(\infty) = \infty$, then

$$
\int_\alpha^\infty \frac{ds}{\sqrt{\mathcal{F}(s)}} = \int_\alpha^\infty \frac{ds}{\sqrt{\mathcal{G}(s)}} = \infty.
$$

Remark 1.2. By [9], we see that if $\int_1^\infty \frac{ds}{\sqrt{\mathcal{F}(s)}} < \infty$, then $\int_1^\infty \frac{ds}{F(s)} < \infty$. In other words, if $\int_1^\infty \frac{ds}{F(s)} = \infty$, then $\int_1^\infty \frac{ds}{\sqrt{\mathcal{F}(s)}} = \infty$. Conversely, if $\int_1^\infty \frac{ds}{\sqrt{\mathcal{F}(s)}} = \infty$, then $\int_1^\infty \frac{ds}{F(s)} = \infty$ does not hold. For example, for $\beta > 0$ and $F(t) = 2(1+t)(\ln(t+1))^{2\beta-1}(\ln(t+1)+\beta)$, we have $\mathcal{F}(t) = (t+1)^2(\ln(t+1))^{2\beta}$. So we can see that $\int_1^\infty \frac{ds}{F(s)} = \infty$ if and only if $0 < \beta \leq \frac{1}{2}$ and $\int_1^\infty \frac{ds}{\sqrt{\mathcal{F}(s)}} = \infty$ if and only if $0 < \beta \leq 1$.

To discuss the existence of positive radial solutions to these nonlinear systems, we study the system of nonlinear differential equations

$$
\begin{gathered}
\frac{1}{A}(Ay')' = g(t,y,z), \quad \text{in } (0,\infty), \\
\frac{1}{B}(Bz')' = f(t,y,z), \quad \text{in } (0,\infty), \\
\lim_{t\to 0^+} A(t)y'(t) = \lim_{t\to 0^+} B(t)z'(t) = 0, \\
y(0) = a > 0, \quad z(0) = b > 0,
\end{gathered} \tag{1.4}
$$

where the continuous functions $A, B : [0,\infty) \to [0,\infty)$ are nondecreasing, differentiable and positive on $(0,\infty)$. We assume that f and g satisfy the following hypotheses.

- (A1) f, $g : (0,\infty)\times[0,\infty)\times[0,\infty) \to [0,\infty)$ are continuous, nondecreasing with respect to the second and the third variables.
- (A2) There exist nonnegative functions h_i, k_i, ξ_i, ω_i, $1 \leq i \leq 2$, ϕ and ψ satisfying for each $(t,u,v) \in (0,\infty)\times[0,\infty)\times[0,\infty)$,
$$
\begin{gathered}
h_2(t)\omega_1(v)\omega_2(u) \leq g(t,u,v) \leq h_1(t)\phi(u+v), \\
k_2(t)\xi_1(u)\xi_2(v) \leq f(t,u,v) \leq k_1(t)\psi(u+v),
\end{gathered}
$$
with ξ_i, ω_i, $1 \leq i \leq 2$, ϕ, $\psi : [0,\infty) \to [0,\infty)$ are nondecreasing continuous, positive on $(0,\infty)$ and h_i, $k_i : (0,\infty) \to [0,\infty)$ continuous and satisfying

$$
\int_0^1 \frac{1}{A(t)}\Big(\int_0^t A(s)\,h_i(s)\,ds\Big)dt < \infty, \quad \int_0^1 \frac{1}{B(t)}\Big(\int_0^t B(s)\,k_i(s)\,ds\Big)dt < \infty.
$$

(A3) For all $r > 0$, we suppose that
$$\sqrt{2}\int_0^r \sqrt{\widetilde{\Lambda}(s)}ds < M_{a+b}(\infty)\,,$$
where, for $\alpha > 0$,
$$M_\alpha(t) = \int_\alpha^t \frac{ds}{\sqrt{2(\Phi(s)+\Psi(s))}}\,, \quad t \ge \alpha$$
$$\Phi(r) = \int_0^r \phi(s)ds, \quad \Psi(r) = \int_0^r \psi(s)ds\,, \quad r \ge 0$$
$$\widetilde{\Lambda}(r) = \max_{s\in[0,r]}(h_1(s)+k_1(s)), \quad r \ge 0.$$

For any nonnegative measurable functions φ in $(0,\infty)$, we define
$$S_A\varphi(t) = \int_0^t \frac{1}{A(r)}\Big(\int_0^r A(s)\varphi(s)ds\Big)dr\,, \quad S_B\varphi(t) = \int_0^t \frac{1}{B(r)}\Big(\int_0^r B(s)\varphi(s)ds\Big)dr.$$
Now, we are ready to give our existence result for (1.4).

Theorem 1.3. *Under the hypotheses* (A1)–(A3)*, system* (1.4) *has a positive solution* $(y,z) \in \big(C([0,\infty)) \cap C^1((0,\infty))\big)^2$ *satisfying for each* $t \in [0,\infty)$
$$a + \omega_1(b)\omega_2(a)S_A(h_2)(t) \le y(t) \le M_{a+b}^{-1}\Big(\sqrt{2}\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big),$$
$$b + \xi_1(a)\xi_2(b)S_B(k_2)(t) \le z(t) \le M_{a+b}^{-1}\Big(\sqrt{2}\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big)\,,$$
where M_{a+b}^{-1} *is the inverse function of* M_{a+b} *which is defined from* $[0, M_{a+b}(\infty))$ *to* $[\alpha,\infty)$.

Remark 1.4. In the case $A(t) = B(t)$, the solution (y,z) of system (1.4) satisfies
$$a + \omega_1(b)\omega_2(a)S_A(h_2)(t) \le y(t) \le M_{a+b}^{-1}\Big(\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big),$$
$$b + \xi_1(a)\xi_2(b)S_A(k_2)(t) \le z(t) \le M_{a+b}^{-1}\Big(\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big)\,, \quad \text{for } t \ge 0.$$

Now we give the existence result for system (1.1) under the following hypotheses.

(H4) The function $r \to r^{2(n-1)}\,(p_1(r)+q_1(r))$ is nondecreasing for large r.

(H5) There exists a positive constant ε such that
$$\int_0^\infty r^{1+\varepsilon}(p_1(r)+q_1(r))dr < \infty,$$

Theorem 1.5. *Under assumptions* (H1)–(H5)*, system* (1.1) *has a positive entire bounded continuous solution.*

Example 1.6. Let α_i, β_i, γ_i, δ_i, $\lambda_i \in (0,\infty)$ satisfying $\max(\alpha_i,\beta_i,\gamma_i,\delta_i+\lambda_i) < 1$ for $1 \le i \le 2$ and $2 < \sigma < 2(n-1)$. Put
$$H(x,u,v) = a_1(x)v^{\alpha_1} + b_1(x)u^{\gamma_1} + c_1(x)(u+v)^{\beta_1} + d_1(x)u^{\delta_1}v^{\lambda_1},$$
$$K(x,u,v) = a_2(x)v^{\alpha_2} + b_2(x)u^{\gamma_2} + c_2(x)(u+v)^{\beta_2} + d_2(x)u^{\delta_2}v^{\lambda_2}\,,$$

where and a_i, b_i, c_i, d_i are nontrivial nonnegative continuous function in $\mathbb{R}^n$ satisfying

$$\sum_{i=1}^{2}\Big(\max_{|y|\le s} a_i(y)+\max_{|y|\le s} b_i(y)+\max_{|y|\le s} c_i(y)+\max_{|y|\le s} d_i(y)\Big)\le \frac{1}{1+s^\sigma}.$$

Then problem (1.1) has a positive continuous bounded solution (u,v). Indeed, the hypotheses (H1), (H2), (H4) and (H5) are clearly satisfied. Since $\max(\alpha_i,\beta_i,\gamma_i,\delta_i+\lambda_i)<1$ for $1\le i\le 2$ and

$$\begin{aligned}
H(x,u,v)&\le \frac{1}{1+|x|^\sigma}[(u+v)^{\alpha_1}+(u+v)^{\beta_1}+(u+v)^{\gamma_1}+(u+v)^{\delta_1+\lambda_1}],\\
K(x,u,v)&\le \frac{1}{1+|x|^\sigma}[(u+v)^{\alpha_2}+(u+v)^{\beta_2}+(u+v)^{\gamma_2}+(u+v)^{\delta_2+\lambda_2}]
\end{aligned}$$

we deduce that $L_1(\infty)=\infty$ and so hypothesis (H3) is satisfied.

Example 1.7. Let $2<\sigma<2(n-1)$, and let $\beta_1,\beta_2>0$ be such that $\beta_0=\max(\beta_1,\beta_2)>1$ and

$$1+\frac{2}{\sigma-2}<\frac{1}{2(\beta_0-1)(Log(3))^{\beta_0}}. \tag{1.5}$$

Let p,q be two nontrivial nonnegative continuous functions in $\mathbb{R}^n$ such that

$$\max_{|y|\le s} p(y)+\max_{|y|\le s} q(y)\le \frac{1}{1+s^\sigma}.$$

Then the problem

$$\begin{aligned}
\Delta u&=2p(x)(u+v+2)(Log(u+v+2))^{2\beta_1-1}(Log(u+v+2)+\beta_1)\\
\Delta v&=2q(x)(u+v+2)(Log(u+v+2))^{2\beta_2-1}(Log(u+v+2)+\beta_2)\,,
\end{aligned}$$

in $\mathbb{R}^n$ with $n\ge 3$, has a positive bounded continuous solution (u,v). Indeed, the hypotheses (H1), (H2), (H4) and (H5) are clearly satisfied. Now, (H3) follows from (1.5) and the fact that for each $t\ge 1$,

$$\int_0^t\frac{ds}{\sqrt{1+s^\sigma}}\le\int_0^1\frac{ds}{\sqrt{1+s^\sigma}}+\int_1^t\frac{ds}{\sqrt{1+s^\sigma}}\le 1+\int_1^t\frac{ds}{s^{\frac{\sigma}{2}}}\le 1+\frac{2}{\sigma-2}$$

and

$$\begin{aligned}
\int_1^\infty\frac{ds}{\sqrt{2(s+2)^2((\log(s+2))^{2\beta_1}+(\log(s+2))^{2\beta_2})}}&\ge\int_1^\infty\frac{ds}{2(s+2)(Log(s+2))^{\beta_0}}\\
&=\frac{1}{2(\beta_0-1)(Log(3))^{\beta_0-1}}.
\end{aligned}$$

From Theorem 1.5, we have the following corollaries in the case where $H(x,u,v)=H(|x|,u,v)$ and $K(x,u,v)=K(|x|,u,v)$.

Corollary 1.8. *Under hypotheses* (H1)–(H3)*, problem* (1.1) *has one positive solution. Under the additional hypothesis*

(H6) $\int_0^\infty sp_2(s)\,ds=\int_0^\infty sq_2(s)\,ds=\infty$,

every positive radial entire solution (u,v) *of* (1.1) *is large and satisfies*

$$u(0)+g_1(v(0))g_2(u(0))P_2(r)\le u(r),\quad v(0)+f_1(u(0))f_2(v(0))Q_2(r)\le v(r)\,,$$

for all $r \geq 0$, where

$$P_2(r) = \int_0^r t^{1-n}\Big(\int_0^t s^{n-1}p_2(s)\,ds\Big)\,dt, \quad Q_2(r) = \int_0^r t^{1-n}\Big(\int_0^t s^{n-1}q_2(s)\,ds\Big)\,dt.$$

For the next corollary, we use the assumption

(H7) $L_\alpha(\infty) = \infty$.

Corollary 1.9. *Assume that* (H1), (H2), (H4), (H7) *are satisfied. If* (1.1) *has a nonnegative radial entire large solution, then*

$$\int_0^\infty r^{1+\varepsilon}(p_1(r) + q_1(r))dr = \infty, \quad \forall\, \varepsilon > 0.$$

An other result for the radial case is given under the assumption

(H8) $\sqrt{p_1 + q_1} \in L^1(0, \infty)$.

Theorem 1.10. *Assume that* (H1), (H2), (H4), (H8) *are satisfied. If* (u, v) *is a positive entire large radial solution of* (1.1)*, then F and G satisfy the Keller-Osserman condition*

$$L_1(\infty) = \int_1^\infty \frac{ds}{\sqrt{2(\mathcal{F}(s) + \mathcal{G}(s))}} < \infty.$$

2. Proof of main results

Proof of Theorem 1.3. Let $(y_m)_{m\geq 0}$ and $(z_m)_{m\geq 0}$ be the sequences of positive continuous functions defined on $[0, \infty)$ by

$$\begin{gathered} y_0(t) = a, \quad z_0(t) = b, \\ y_{m+1}(t) = a + \int_0^t \frac{1}{A(r)}\Big(\int_0^r A(s)g(s, y_m(s), z_m(s))ds\Big)dr, \\ z_{m+1}(t) = b + \int_0^t \frac{1}{B(r)}\Big(\int_0^r B(s)f(s, y_m(s), z_m(s))ds\Big)dr. \end{gathered}$$

Clearly $y_m, z_m \in C([0, \infty)) \cap C^1((0, \infty))$ and are positive, so we deduce by (A1) that $(y_m)_{m\geq 0}$ and $(z_m)_{m\geq 0}$ are nondecreasing sequences and for each $m \in \mathbb{N}$, the functions $t \to y_m(t)$ and $t \to z_m(t)$ are nondecreasing. Hence, for each $t \in (0, \infty)$,

$$\begin{gathered} y_m'(t) = \frac{1}{A(t)} \int_0^t A(s)g(s, y_{m-1}(s), z_{m-1}(s))ds, \\ z_m'(t) = \frac{1}{B(t)} \int_0^t B(s)f(s, y_{m-1}(s), z_{m-1}(s))ds \end{gathered}$$

Which implies

$$(A(t)y_m'(t))' = A(t)g(t, y_{m-1}(t), z_{m-1}(t)), \quad \leq A(t)g(t, y_m(t), z_m(t)).$$

Multiplying this expression by $2A(t)y_m'(t)$ and integrate on $[0, t]$, we obtain

$$\begin{aligned} (A(t)y_m'(t))^2 &\leq 2\int_0^t A^2(s)h_1(s)\left(\phi(y_m(s) + z_m(s))\right) y_m'(s)\,ds \\ &\leq 2\widetilde{\Lambda}(t)A^2(t)\int_0^t \left(\phi(y_m(s) + z_m(s))\right)(y_m'(s) + z_m'(s))\,ds \\ &= 2\widetilde{\Lambda}(t)A^2(t)\int_{a+b}^{y_m(t)+z_m(t)} \phi(s)\,ds. \end{aligned}$$

Which implies

$$y_m'(t) \le \sqrt{\widetilde{\Lambda}(t)}\sqrt{2\Phi(y_m(t)+z_m(t))}.$$

Similarly, we have

$$z_m'(t) \le \sqrt{\widetilde{\Lambda}(t)}\sqrt{2\Psi(y_m(t)+z_m(t))}.$$

Then

$$y_m'(t)+z_m'(t) \le \sqrt{2}\sqrt{\widetilde{\Lambda}(t)}\sqrt{2(\Phi(y_m(t)+z_m(t))+\Psi(y_m(t)+z_m(t)))}.$$

Therefore,

$$\begin{aligned}
M_{a+b}(y_m(t)+z_m(t)) &= \int_{a+b}^{y_m(t)+z_m(t)} \frac{ds}{\sqrt{2\left(\Phi(s)+\Psi(s)\right)}}\, ds \\
&= \int_0^t \frac{y_m'(s)+z_m'(s)}{\sqrt{2\left(\Phi(y_m(s)+z_m(s))+\Psi(y_m(s)+z_m(s))\right)}}\, ds \\
&\le \sqrt{2}\int_0^r \sqrt{\widetilde{\Lambda}(s)}ds.
\end{aligned}$$

Since M_{a+b}^{-1} is increasing on $[0, M_{a+b}(\infty))$, we obtain

$$y_m(t)+z_m(t) \le M_{a+b}^{-1}\Big(\sqrt{2}\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big), \quad \forall\, r \ge 0.$$

Therefore, the sequences $(y_m)_{m\ge 0}$ and $(z_m)_{m\ge 0}$ converge to two functions y and z that, for each $t \in [0,\infty)$, satisfy

$$\begin{aligned}
y(t) &= a + \int_0^t \frac{1}{A(r)}\Big(\int_0^r A(s)g(s,y(s),z(s))ds\Big)dr, \\
z(t) &= b + \int_0^t \frac{1}{B(r)}\left(\int_0^r B(s)f(s,y(s),z(s))ds\right)dr.
\end{aligned}$$

Hence, $y,\, z \in C([0,\infty)) \cap C^1((0,\infty))$ and (y,z) is a solution of (1.4) satisfying

$$\begin{aligned}
a + g_1(b)g_2(a)S_A(h_2)(t) \le y(t) \le M_{a+b}^{-1}\Big(\sqrt{2}\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big), \\
b + f_1(a)f_2(b)S_B(k_2)(t) \le z(t) \le M_{a+b}^{-1}\Big(\sqrt{2}\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big).
\end{aligned}$$

In the case where $A(t) = B(t)$, we multiply the inequality

$$(A(t)(y_m'(t)+z_m'(t)))' \le A(t)[g(t,y_m(t),z_m(t))+f(t,y_m(t),z_m(t))]$$

by $2A(t)(y_m'(t)+z_m'(t))$, we integrate on $[0,t]$ and we proceed as below to obtain

$$(A(t)(y_m'(t)+z_m'(t)))^2 \le (A(t))^2\, 2\widetilde{\Lambda}(t)\int_0^{y_m(t)+z_m(t)} \left(\phi(s)+\psi(s)\right)\, ds.$$

Which implies

$$y_m'(t)+z_m'(t) \le \sqrt{\widetilde{\Lambda}(t)}\sqrt{2\Phi(y_m(t)+z_m(t))}.$$

So in this case (y,z) satisfies

$$a + \omega_1(b)\omega_2(a)S_A(h_2)(t) \le y(t) \le M_{a+b}^{-1}\Big(\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\Big),$$

$$b + \xi_1(a)\xi_2(b)S_A(k_2)(t) \le z(t) \le M_{a+b}^{-1}\left(\int_0^t \sqrt{\widetilde{\Lambda}(s)}ds\right), \quad \text{for } t \ge 0.$$

This completes the proof. □

Proof of Theorem 1.5. We will show that (1.1) has a solution by finding a subsolution $(\underline{u}, \underline{v})$ and a supersolution $(\overline{u}, \overline{v})$, such that $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$. To do this, we first prove the existence of a radial subsolution $(\underline{u}, \underline{v})$ to (1.1) by considering the problem

$$\begin{gathered} \triangle \underline{u} = p_1(|x|)F(\underline{u} + \underline{v}), \quad \text{in } \mathbb{R}^n \\ \triangle \underline{v} = q_1(|x|)G(\underline{u} + \underline{v}), \quad \text{in } \mathbb{R}^n, \end{gathered}$$

with $n \ge 3$. That is, $(\underline{u}, \underline{v})$ satisfies

$$\begin{gathered} \frac{1}{r^{n-1}}(r^{n-1}\underline{u}')' = p_1(r)F(\underline{u} + \underline{v}),\ r \in (0, \infty) \\ \frac{1}{r^{n-1}}(r^{n-1}\underline{v}')' = q_1(r)G(\underline{u} + \underline{v})\ r \in (0, \infty). \end{gathered} \tag{2.1}$$

By Theorem 1.3, we conclude that system (2.1) has a positive entire solution $(\underline{u}, \underline{v})$. Now, since

$$\begin{gathered} \triangle \underline{u} = p_1(|x|)F(\underline{u} + \underline{v}) \ge H(x, \underline{u}, \underline{v}), \quad \text{in } \mathbb{R}^n, \\ \triangle \underline{v} = q_1(|x|)G(\underline{u} + \underline{v}) \ge K(x, \underline{u}, \underline{v}), \quad \text{in } \mathbb{R}^n, \end{gathered}$$

we deduce that $(\underline{u}, \underline{v})$ is a subsolution of system (1.1).
Next we prove that $(\underline{u}, \underline{v})$ is bounded. Since $(\underline{u}, \underline{v})$ satisfies

$$\begin{aligned} (r^{n-1}\underline{u}'(r))' &= r^{n-1}p_1(r)F(\underline{u}(r) + \underline{v}(r)), \\ (r^{n-1}\underline{v}'(r))' &= r^{n-1}q_1(r)G(\underline{u}(r) + \underline{v}(r)), \end{aligned}$$

it follows that

$$(r^{n-1}(\underline{u}'(r) + \underline{v}'(r)))' = r^{n-1}[p_1(r)F(\underline{u}(r) + \underline{v}(r)) + q_1(r)G(\underline{u}(r) + \underline{v}(r))]. \tag{2.2}$$

Choose $R > 0$ so that $r^{2(n-1)}\left(p_1(r) + q_1(r)\right)$ is nondecreasing on $[R, \infty)$. Then, after multiplying (2.2) by $r^{n-1}(\underline{u}'(r) + \underline{v}'(r))$ and integrating from R to r, we obtain

$$\begin{aligned} &\left(r^{n-1}(\underline{u}'(r) + \underline{v}'(r))\right)^2 \\ &= C + 2\Big(\int_R^r t^{2(n-1)}[p_1(t)F(\underline{u}(t) + \underline{v}(t)) + q_1(r)\mathcal{G}(\underline{u}(r) + \underline{v}(r))]dt\Big) \\ &\le C + 2r^{2(n-1)}\left(p_1(r) + q_1(r)\right) \\ &\quad \times \Big(\int_R^r [\mathcal{F}'(\underline{u}(t) + \underline{v}(t)) + \mathcal{G}'(\underline{u}(t) + \underline{v}(t))](\underline{u}'(t) + \underline{v}'(t))\, dt\Big) \\ &\le C + 2r^{2(n-1)}(p_1(r) + q_1(r))\left[\mathcal{F}(\underline{u}(r) + \underline{v}(r)) + \mathcal{G}(\underline{u}(r) + \underline{v}(r))\right], \end{aligned}$$

where $C = \left(R^{n-1}(\underline{u}'(R) + \underline{v}'(R))\right)^2$. This yields

$$\begin{aligned} &\frac{\underline{u}'(r) + \underline{v}'(r)}{\sqrt{2}(\mathcal{F}(\underline{u}(r) + \underline{v}(r)) + \mathcal{G}(\underline{u}(r) + \underline{v}(r)))^{\frac{1}{2}}} \\ &\le \frac{\sqrt{C}r^{1-n}}{\sqrt{(\mathcal{F}(\underline{u}(r) + \underline{v}(r)) + \mathcal{G}(\underline{u}(r) + \underline{v}(r)))}} + \sqrt{2}\sqrt{p_1(r) + q_1(r)}. \end{aligned}$$

Integrating the above inequality and using the fact that

$$\mathcal{F}(\underline{u}(r)+\underline{v}(r))+\mathcal{G}(\underline{u}(r)+\underline{v}(r)) \geq \mathcal{F}(\underline{u}(R)+\underline{v}(R))+\mathcal{G}(\underline{u}(R)+\underline{v}(R))=C_1,$$

for all $r \geq R$, and

$$\sqrt{p_1(r)+q_1(r)} \leq 2\sqrt{r^{1+\varepsilon}(p_1(r)+q_1(r))r^{-1-\varepsilon}} \leq r^{1+\varepsilon}(p_1(r)+q_1(r))+r^{-1-\varepsilon}\,,$$

we obtain

$$\begin{aligned} &L_\alpha(\underline{u}(r)+\underline{v}(r)) \\ &\leq \sqrt{2}\int_R^r s^{1+\varepsilon}(p_1(s)+q_1(s))ds+\sqrt{2}\,(\varepsilon R^\varepsilon)^{-1}+\sqrt{\frac{C}{C_1}}(2-n)(R^{2-n})^{-1}\,, \end{aligned}$$

where $\alpha=\underline{u}(R)+\underline{v}(R)$. Letting $r \to \infty$, we deduce from hypothesis (H5) that $(\underline{u},\underline{v})$ is bounded. Thus, since $(\underline{u},\underline{v})$ is nondecreasing, we have

$$\lim_{r\to\infty}\underline{u}(r)=M_1>0, \quad \lim_{r\to\infty}\underline{v}(r)=M_2>0.$$

Now, it is clear that $(\overline{u},\overline{v})=(M_1,M_2)$ is a supersolution for (1.1) and we have for $r \geq 0$,

$$\overline{u}(r) \geq M_1 \geq \underline{u}(r), \quad \overline{v}(r) \geq M_2 \geq \underline{v}(r).$$

Hence the standard sub-supersolution method (see [7, 17]) implies that (1.1) has a bounded solution (u,v) such that $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$. This completes the proof. □

Proof of Theorem 1.10. Let (u,v) be a positive entire large radial solution of (1.1). Then (u,v) satisfies

$$\begin{gathered} (r^{n-1}u'(r))' \leq r^{n-1}p_1(r)F(u(r)+v(r)), \\ (r^{n-1}v'(r))' \leq r^{n-1}q_1(r)G(u(r)+v(r)). \end{gathered}$$

Adding these inequalities, we obtain

$$\begin{aligned} (r^{n-1}(u'(r)+v'(r)))' &\leq r^{n-1}p_1(r)F(u(r)+v(r))+r^{n-1}q_1(r)G(u(r)+v(r)) \\ &\leq r^{n-1}(p_1(r)+q_1(r))\left(F(u(r)+v(r))+G(u(r)+v(r))\right). \end{aligned}$$

Multiplying the above inequality by $2r^{n-1}(u'(r)+v'(r))$ and integrating from 0 to r, and using (H4), we obtain

$$\begin{aligned} \left(r^{n-1}(u'(r)+v'(r))\right)^2 &\leq 2\Big(\int_0^r s^{2(n-1)}(p_1(s)+q_1(s))\Big(F(u(s)+v(s)) \\ &\quad +G(u(s)+v(s))\Big)(u'(s)+v'(s))\,ds\Big) \\ &\leq 2r^{2(n-1)}(p_1(r)+q_1(r))\int_0^r\Big(F(u(s)+v(s)) \\ &\quad +G(u(s)+v(s))\Big)(u'(s)+v'(s))\,ds \\ &\leq 2r^{2(n-1)}(p_1(r)+q_1(r))\left(\mathcal{F}(u(r)+v(r))+\mathcal{G}(u(r)+v(r))\right) \end{aligned}$$

which implies

$$\frac{u'(r)+v'(r)}{\sqrt{2(\mathcal{F}(u(r)+v(r))+\mathcal{G}(u(r)+v(r)))}} \leq \sqrt{p_1(r)+q_1(r)}.$$

By integrating over $[0, r]$, we obtain

$$L_{u(0)+v(0)}(u(r) + (v(r)) \leq \int_0^r \sqrt{p_1(s) + q_1(s)}ds.$$

Letting $r \to \infty$, we obtain that F and G satisfies the Keller-Osserman condition. $\square$

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