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MULTIPLE POSITIVE SOLUTIONS FOR INTEGRO-DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS AND SIGN CHANGING NONLINEARITIES

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Abstract. In this article, we show the existence of multiple positive solutions for integro-differential equations with one-dimensional p-Laplacian operator, sign changing nonlinearities, and integral boundary conditions. By using the Schauder fixed point theorem and the Krasnosel'skii fixed point theorem, we obtain sufficient conditions for the existence of at least two positive solutions.

1. Introduction

In this article, we study the existence of positive solutions for the following integro-differential equation with integral boundary conditions, and sign changing nonlinearities:

$$\begin{gathered}(\varphi_p(u'(t)))' + w(t)f(t,u(t),Au(t),Bu(t)) = 0, \quad 0 < t < 1, \\ u(0) = Au(\xi), \quad u(1) = -Bu(\eta),\end{gathered} \tag{1.1}$$

where $Au(t) = \int_0^t g(t,s)u'(s)\,\mathrm{d}s$, $Bu(t) = \int_t^1 h(t,s)u'(s)\,\mathrm{d}s$, $0 < \xi \le \eta < 1$, $\varphi_p(u) = |u|^{p-2}u$ is the one-dimensional p-Laplacian operator wiht $p > 1$, $\varphi_q = (\varphi_p)^{-1}$, and $\frac{1}{q} + \frac{1}{p} = 1$. By using the Schauder fixed point theorem and the Krasnosel'skii fixed point theorem, we obtain sufficient conditions for the existence of at least two positive solutions under suitable conditions assumed on the nonlinear terms f and w.

The theory of boundary-value problems for integro-differential equations arises in different areas of applied mathematics, fluid dynamics, plasma physics, biological sciences and chemical kinetics (for details, see [4, 3, 22] and the references therein). Since boundary-value problems with integral boundary conditions include two,

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three, multi-point and nonlocal boundary-value problems as special cases, the existence and multiplicity of positive solutions for such problems have been put emphasis on continuously (see [8, 20, 23, 24, 25, 14, 17, 13, 18] and references therein). Because of the wide mathematical and physical background, the existence of positive solutions for nonlinear boundary-value problems with p-Laplacian has also received wide attention. For details, we can refer to see [5, 16, 7, 6, 19, 21, 15, 17, 13, 12, 9]. The main tools for such problems are various kinds of fixed-point theorem in cones (see [5, 16, 7, 21, 15, 17, 13, 12]), the monotone iterative technique (see [19]) and the fixed point index theory (see [6, 12]). If the nonlinear term is nonnegative, we can apply the concavity of solutions in the proofs. Under the assumption that the nonlinear term is nonnegative, authors obtained the existence of at least one positive solutions or multiple positive solutions, see [5, 16, 7, 19, 21, 15, 13].

By using the upper and lower solution approach and the growth restriction approach, in [1] the author presented some general existence theorems second-order boundary-value problems with sign changing nonlinearities:

$$\begin{gathered} y'' + q(t)f(t,y) = 0, \quad 0 < t < 1, \\ y(0) = 0 = y(1), \end{gathered} \tag{1.2}$$

and

$$\begin{gathered} y'' + q(t)f(t,y) = 0, \quad 0 < t < 1, \\ y(0) = 0,\ \theta(y(1)) + y(1) = 0, \end{gathered}$$

where the nonlinear term f is allowed to change sign and θ may be nonlinear. Moreover, in [2], the authors discussed the singular Dirichlet boundary-value problem (1.2) and established existence results, where nonlinearity f is allowed to change sign and may be singular at $y = 0$.

Guo [11] established a new fixed point theorem in double cones and discussed the existence of positive solutions for the second-order three-point boundary-value problem

$$\begin{gathered} x'' + f(t,x) = 0, \quad 0 \le t \le 1, \\ x(0) - \beta x'(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered}$$

where f is allowed to change sign. Sufficient conditions of the existence of at least two positive solutions for the boundary-value problems above are obtained by imposing growth conditions on f. By applications of fixed point index theory, Cheung and Ren [6] proved the existence of two positive solutions for the problem

$$(\Phi_p(u'))' + h(t)f(t,u) = 0, \quad 0 < t < 1,$$

with each of the following two sets of boundary conditions

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i);$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0,$$

where $h : [0,1] \to \mathbb{R}^+$ and $f : [0,1] \times [0,\infty) \to \mathbb{R}$ are continuous functions. Ji [12] studied the existence of positive solutions for the one-dimensional p-Laplacian equation

$$
\begin{gathered}
(\Phi_p(u'))' + f(t,u,u') = 0, \quad 0 < t < 1, \\
u'(0) = \sum_{i=1}^{m-2} \alpha_i u'(\xi_i), \quad u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),
\end{gathered}
$$

where f may change sign. They show that it has at least one or two positive solutions under some assumptions by applying the fixed point theorem and fixed point index theory. Liu, Jia and Tian [17] studied the existence of positive solutions for the boundary-value problem, with integral boundary conditions and sign changing nonlinearities of one-dimensional p-Laplacian

$$
\begin{gathered}
(\Phi_p(u'))' + f(t,u) = 0, \quad 0 < t < 1, \\
au(0) - bu'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad u(1) = \int_0^1 g(s)u(s)\,\mathrm{d}s,
\end{gathered}
$$

where $a, b \in [0,+\infty)$, $a_i \in (0,+\infty)$, $i = 1, 2, \dots, m$, $0 < \xi_1 < \xi_2 \dots < \xi_{m-2} < 1$, $m \geq 3$. The sufficient conditions for the existence of at least two positive solutions were obtained by using a fixed point theorem in double cones given in [11].

Recently, by using the expansion and compression fixed point theorem of norm in cone under suitable conditions imposed on the nonlinear term f and w, Jia and Wang [13] established sufficient conditions for the existence of at least one positive solutions for (1.1), where the nonlinear term f and w are nonnegative. However, there are a few works devoted to the integro-differential boundary-value problems with integral boundary conditions, one-dimensional p-Laplacian operator and sign changing nonlinearities.

Motivated by the above, we obtain some meaningful conclusions by considering the existence of multiply positive solutions for (1.1), with integral boundary conditions and sign changing nonlinearities of one-dimensional p-Laplacian.

The following hypotheses will be assumed throughout this paper:

- (H1) $f : [0,1] \times [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous;
- (H2) $f(t,0,\cdot,\cdot) \geq 0$, $w \in L^1[0,1]$, $f(t,0,\cdot,\cdot) \not\equiv 0$, $w(t) \geq 0$ and $w \not\equiv 0$ a.e. on [0,1];
- (H3) g, $h \in C([0,1] \times [0,1], [0,+\infty))$, $g(\xi,s)$ is monotone decreasing with respect to $s \in [0,1]$ and $h(\eta,s)$ is monotone increasing with respect to $s \in [0,1]$.

2. Preliminaries

For any $y \in L^1[0,1]$, $y(t) \geq 0$ and $y(t) \not\equiv 0$ for $t \in [0,1]$, we denote

$$
\begin{aligned}
H(C) = &\int_0^\xi g(\xi,s)\varphi_q\Big(C - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&+ \int_\eta^1 h(\eta,s)\varphi_q\Big(C - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_0^1 \varphi_q\Big(C - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.
\end{aligned}
$$

Lemma 2.1. *Suppose that* (H3) *holds. Then for each $y \in L^1[0,1]$, $y(t) \geq 0$ and $y(t) \not\equiv 0$ for $t \in [0,1]$, the equation $H(C) = 0$ has a unique solution in $(-\infty,+\infty)$ and the solution $C_y \in (0, \int_0^1 y(\tau)\,\mathrm{d}\tau)$. Moreover, there exists $\sigma \in (0,1)$ such that $C_y = \int_0^\sigma y(\tau)\,\mathrm{d}\tau$.*

The proof of Lemma 2.1 is similar to that of [13, Lemma 2.2]. In this article, we take

$$\sigma_y = \inf\big\{\sigma \in (0,1)\ :\ C_y = \int_0^\sigma y(\tau)\,\mathrm{d}\tau\big\}. \tag{2.1}$$

For the convenience, we recall the following results (see [13]).

Lemma 2.2 ([13, Lemma 2.3]). *Assume* (H3) *holds. Then for each $y \in L^1[0,1]$, $y(t) \ge 0$ and $y(t) \not\equiv 0$ for $t \in [0,1]$, the boundary-value problem*

$$\begin{gathered} (\varphi_p(u'(t)))' + y(t) = 0, \quad 0 < t < 1, \\ u(0) = Au(\xi), \quad u(1) = -Bu(\eta), \end{gathered} \tag{2.2}$$

has a unique solution of the form

$$u(t) = \int_0^\xi g(\xi,s)\varphi_q\Big(C_y - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_0^t \varphi_q\Big(C_y - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \tag{2.3}$$

or

$$u(t) = -\int_\eta^1 h(\eta,s)\varphi_q\Big(C_y - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s - \int_t^1 \varphi_q\Big(C_y - \int_0^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \tag{2.4}$$

where C_y satisfies $H(C_y) = 0$.

Remark 2.3. By Lemma 2.1 and Lemma 2.2, (2.3) and (2.4) can be changed into (2.5) and (2.6), respectively.

$$u(t) = \int_0^\xi g(\xi,s)\varphi_q\Big(\int_s^{\sigma_y} y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_0^t \varphi_q\Big(\int_s^{\sigma_y} y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \tag{2.5}$$

and

$$u(t) = \int_\eta^1 h(\eta,s)\varphi_q\Big(\int_{\sigma_y}^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_t^1 \varphi_q\Big(\int_{\sigma_y}^s y(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{2.6}$$

Lemma 2.4 ([13, Lemma 2.4]). *Suppose* (H3) *holds. If $y \in L^1[0,1]$, $y(t) \ge 0$ and $y(t) \not\equiv 0$ for $t \in [0,1]$. Then the solution of boundary-value problem* (2.2) *has the following properties:*

1. *$u(t)$ is a concave function;*
2. *$u(t) \ge 0$, $t \in [0,1]$;*
3. *$u(\sigma_y) = \max_{0\le t\le 1} u(t)$ and $u'(\sigma_y) = 0$, where σ_y is defined in* (2.1).

Lemma 2.5 ([9]). *If $u \in C[0,1]$ and $u(t) \ge 0$ is a concave function. Then for each $\gamma \in (0, \frac{1}{2})$, we have*

$$\min_{\gamma\le t\le 1-\gamma} u(t) \ge \gamma\|u\|.$$

Let $X = C[0,1]$ and $\|u\| = \max_{0\le t\le 1}|u(t)|$, take $0 < \delta < \min\{\frac{1}{2}, \xi, \eta\}$ and denote

$$P = \{u \in X : u(t) \ge 0, t \in [0,1]\}$$

and

$$K = \{u \in P : u(t) \text{ is a concave function on } [0,1] \text{ and } \min_{\delta\le t\le 1-\delta} u(t) \ge \delta\|u\|\}.$$

Obviously, $P,\ K \subset X$ are two cones of X with $K \subset P$.

Denote $B^+ = \max\{B, 0\}$. For $u \in K$, we define $T^* : K \to K$ by

$$T^*u(t) = \begin{cases} \int_0^\xi g(\xi,s)\varphi_q\Big(\int_s^{\sigma_u} w(\tau)f^+(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ +\int_0^t \varphi_q\Big(\int_s^{\sigma_u} w(\tau)f^+(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, & 0 \le t \le \sigma_u, \\ \int_\eta^1 h(\eta,s)\varphi_q\Big(\int_{\sigma_u}^s w(\tau)f^+(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ +\int_t^1 \varphi_q\Big(\int_{\sigma_u}^s w(\tau)f^+(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, & \sigma_u \le t \le 1, \end{cases}$$

where σ_u is defined in (2.1). It follows T^* is well definition from (H1)–(H3), Lemma 2.4 and Lemma 2.5. Define $T : P \to P$ by

$$Tu(t) = \begin{cases} \Big[\int_0^\xi g(\xi,s)\varphi_q\Big(\int_s^{\sigma_u} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ +\int_0^t \varphi_q\Big(\int_s^{\sigma_u} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big]^+, & 0 \le t \le \sigma_u, \\ \Big[\int_\eta^1 h(\eta,s)\varphi_q\Big(\int_{\sigma_u}^s w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ +\int_t^1 \varphi_q\Big(\int_{\sigma_u}^s w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big]^+, & \sigma_u \le t \le 1\,. \end{cases}$$

Define $S : P \to X$ by

$$Su(t) = \begin{cases} \int_0^\xi g(\xi,s)\varphi_q\Big(\int_s^{\sigma_u} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ +\int_0^t \varphi_q\Big(\int_s^{\sigma_u} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, & 0 \le t \le \sigma_u, \\ \int_\eta^1 h(\eta,s)\varphi_q\Big(\int_{\sigma_u}^s w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\ +\int_t^1 \varphi_q\Big(\int_{\sigma_u}^s w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s, & \sigma_u \le t \le 1\,. \end{cases}$$

From Lemma 2.2, we have the following result.

Lemma 2.6. *Suppose that* (H1)–(H3) *hold. Then a function $u(t)$ is a solution of boundary-value problem* (1.1) *if and only if $u(t)$ is a fixed point of the operator S.*

We can easily prove that the following lemma holds.

Lemma 2.7. *Suppose that* (H1)–(H3) *hold. Then $T^* : K \to K$ is completely continuous.*

For $u \in X$, denote $\theta : X \to P$ by $(\theta u)(t) = \max\{u(t), 0\}$, then $T = \theta \circ S$.

Lemma 2.8 ([6, Lemma 2.2])**.** *If $S : K \to X$ is completely continuous, then $T = \theta \circ S : K \to K$ is also completely continuous.*

Lemma 2.9. *Suppose* (H1)–(H3) *hold. If u is a fixed point of operator T, then u is also a fixed point of operator S.*

Proof. Let u be a fixed point of the operator T, if we prove $Su(t) \ge 0$ for $t \in [0,1]$, then $u(t)$ is a fixed point of operator S.

Suppose $Su(t) \ge 0$ for $t \in [0,1]$ is not true, then there exists a $t_0 \in (0,1)$ such that $u(t_0) = 0 > Su(t_0)$. Let (t_1, t_2) be the maximal interval which contains t_0 and such that $Su(t) < 0$, $t \in (t_1,t_2)$. It follows $[t_1,t_2] \ne [0,1]$ from (H2).

Case 1: If $t_2 < 1$, we have $u(t) = 0$ for $t \in [t_1,t_2]$, $Su(t) < 0$ for $t \in (t_1,t_2)$ and $Su(t_2) = 0$. Thus $(Su)'(t_2) \ge 0$. From (H2), we know $[\varphi_p((Su)'(t))]' = -w(t)f(t,0,Au(t),Bu(t)) \le 0$ for $t \in [t_1,t_2]$ and we can get $(Su)'(t)$ is monotone decreasing on $[t_1,t_2]$. So $t_1 = 0$, $Su(t_1) < 0$ and

$$(Su)'(t) \ge (Su)'(t_2) \ge 0,\ t \in [0,t_2].$$

On the other hand, if $\xi \le t_2$, we have

$$0 > Su(0) = \int_0^{\xi} g(\xi,s)(Su)'(s)\,\mathrm{d}s \ge 0,$$

which is a contradiction.

If $\xi > t_2$, by using mean value theorem of integral, we have

$$\begin{aligned}
Su(0) &= \int_0^{t_2} g(\xi,s)(Su)'(s)\,\mathrm{d}s + \int_{t_2}^{\xi} g(\xi,s)(Su)'(s)\,\mathrm{d}s\\
&= g(\xi,\xi_1)\int_0^{t_2}(Su)'(s)\,\mathrm{d}s + g(\xi,t_2)\int_{t_2}^{\xi_2}(Su)'(s)\,\mathrm{d}s\\
&= g(\xi,\xi_1)(Su(t_2)-Su(0)) + g(\xi,t_2)(Su(\xi_2)-Su(t_2))\\
&= g(\xi,\xi_1)(-Su(0)) + g(\xi,t_2)Su(\xi_2),
\end{aligned}$$

and

$$0 > (1+g(\xi,\xi_1))Su(0) = g(\xi,t_2)Su(\xi_2), \tag{2.7}$$

where $\xi_1 \in [0,t_2]$ and $\xi_2 \in [t_2,\xi]$.

It follows that (2.7) is a contradiction if $Su(\xi_2) \ge 0$.

If $Su(\xi_2) < 0$, let (t_3,t_4) be the maximal interval which contains ξ_2 and such that $Su(t) < 0$, $t \in (t_3,t_4)$. It is obvious that $[t_3,t_4] \subset [t_2,1]$. If $t_4 < 1$, we have $u(t) = 0$ for $t \in [t_3,t_4]$, $Su(t) < 0$ for $t \in (t_3,t_4)$ and $Su(t_3) = 0$. Thus $(Su)'(t_3) \le 0$. From (H2), we know $[\varphi_p((Su)'(t))]' = -w(t)f(t,0,Au(t),Bu(t)) \le 0$ and $\varphi_p((Su)'(t))$ is monotone decreasing on $[t_3,t_4]$, we can obtain $(Su)'(t)$ is monotone decreasing on $[t_3,t_4]$. It is easy to show that

$$(Su)'(t) \le (Su)'(t_3) \le 0,\ t \in [t_3,t_4].$$

Hence, $t_4 = 1$ and $Su(1) < 0$. Since $\xi \le \eta$, we have $(Su)'(t) \le 0,\ t \in [\eta,1]$ and

$$0 > Su(1) = -\int_{\eta}^{1} h(\eta,s)(Su)'(s)\,\mathrm{d}s \ge 0,$$

which is a contradiction.

Therefore, $t_2 < 1$ is not true. We have $t_2 = 1$.

Case 2: If $t_1 > 0$, we have $Su(t) = 0$ for $t \in [t_1,1]$, $Su(t) < 0$ for $t \in (t_1,1)$ and $Su(t_1) = 0$. Thus $(Su)'(t_1) \le 0$. We have $[\varphi_p((Su)'(t))]' = -f(t,0,Au(t),Bu(t)) \le 0$ by (H2). This implies $(Su)'(t) \le 0$ and $Su(t) < 0$ for $t \in (t_1,1]$ and $Su(1) = \min_{t\in[t_1,1]} Su(t)$.

We can prove that

$$Su(t) \ge 0 \text{ for } t \in [0,t_1]. \tag{2.8}$$

If there exists a $t_5 \in [0,t_1]$ such that $Su(t_5) < 0$ and there is a maximal interval $[t_6,t_7]$ which contains t_5 such that $Su(t) < 0$ for $t \in (t_6,t_7)$. Obviously $[t_6,t_7) \cap [t_1,1] = \emptyset$, so $1 \notin (t_6,t_7)$; i.e., $t_7 < 1$, this is a contradiction with the above discussion. Thus we can show $Su(t) \ge 0$ for $t \in [0,t_1]$.

For $Su(1) < 0$, we have

$$Su(1) = -\int_{\eta}^{1} h(\eta,s)(Su)'(s)\,\mathrm{d}s.$$

Then, if $\eta \ge t_1$, we have

$$0 > Su(1) = -\int_{\eta}^{1} h(\eta,s)(Su)'(s)\,\mathrm{d}s \ge 0,$$

which is a contradiction.

If $\eta < t_1$, by using mean value theorem of integral, there exist $\eta_1 \in [\eta, t_1] \subset [0, t_1]$ and $\eta_2 \in [t_1, 1]$ such that

$$\begin{aligned}
Su(1) &= -\int_{\eta}^{t_1} h(\eta,s)(Su)'(s)\,\mathrm{d}s - \int_{t_1}^{1} h(\eta,s)(Su)'(s)\,\mathrm{d}s \\
&= -h(\eta,t_1)\int_{\eta_1}^{t_1}(Su)'(s)\,\mathrm{d}s - h(\eta,\eta_2)\int_{t_1}^{1}(Su)'(s)\,\mathrm{d}s \\
&= -h(\eta,t_1)(Su(t_1) - Su(\eta_1)) - h(\eta,\eta_2)(Su(1) - Su(t_1)) \\
&= h(\eta,t_1)Su(\eta_1)) - h(\eta,\eta_2)Su(1),
\end{aligned}$$

and

$$0 > (1 + h(\eta,\eta_2))Su(1) = h(\eta,t_1)Su(\eta_1). \tag{2.9}$$

By (2.8), we have $Su(\eta_1) \geq 0$. Hence, (2.9) is a contradiction. Therefore $t_1 = 0$.

The above also contradicts $[t_1, t_2] \neq [0, 1]$. Thus $Su(t) \geq 0$ for $t \in [0, 1]$. That is $u(t)$ is a fixed point of operator S. □

Next we state the Krasnosel'skii Fixed Point Theorem [10].

Lemma 2.10. *Let E be a Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ and $A : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator. In addition, suppose either*

$$\|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1, \text{ and } \|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2;$$

or

$$\|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1, \text{ and } \|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2.$$

hold. Then A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main result

Denote

$$\begin{gathered}
M = \min\Big\{\int_{\delta}^{1/2}\varphi_q\Big(\int_{s}^{1/2} w(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s, \int_{1/2}^{1-\delta}\varphi_q\Big(\int_{1/2}^{s} w(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big\}, \\
N = \max\Big\{(1 + \int_0^1 g(\xi,s)\,\mathrm{d}s)\varphi_q\Big(\int_0^1 w(\tau)\,\mathrm{d}\tau\Big), \\
(1 + \int_0^1 h(\eta,s)\,\mathrm{d}s)\varphi_q\Big(\int_0^1 w(\tau)\,\mathrm{d}\tau\Big)\Big\}.
\end{gathered}$$

For the next theorem, we assume that f satisfies the following growth conditions:

- (H4) $f(t,u,x,y) \geq 0$ for $(t,u,x,y) \in [0,1] \times [c_1,c_3] \times \mathbb{R}^2$;
- (H5) $f(t,u,x,y) < \varphi_p(\frac{c_2}{N})$ for $(t,u,x,y) \in [0,1] \times [0,c_2] \times \mathbb{R}^2$;
- (H6) $f(t,u,x,y) \geq \varphi_p(\frac{c_3}{M})$ for $(t,u,x,y) \in [\delta, 1-\delta] \times [\delta c_3, c_3] \times \mathbb{R}^2$.

Theorem 3.1. *Suppose* (H1)–(H6) *hold. There exist constants c_1, c_2, c_3 such that*

$$0 < c_1 \leq \min\Big\{\frac{g(\xi,\xi)}{1 + g(\xi,0)}, \frac{h(\eta,\eta)}{1 + h(\eta,1)}\Big\}\delta c_2, \quad c_2 < \delta c_3 < c_3\,.$$

Then (1.1) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < c_2 \leq \|u_2\| \leq c_3.$$

Proof. Let $\Omega_1 = \{u \in K : \|u\| < c_2\}$. For any $u \in \overline{\Omega}_1$, we have $u \in K$ and $\|u\| \le c_2$. Denote

$$\|Tu\| = \max_{0 \le t \le 1} |Tu(t)| = Tu(\bar{t}).$$

If $\bar{t} < \sigma_u$, it follows from (H_5) that

$$\begin{aligned}
Tu(\bar{t}) &= \Big[\int_0^{\xi} g(\xi,s)\varphi_q\Big(\int_s^{\sigma_u} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\quad + \int_0^{\bar{t}} \varphi_q\Big(\int_s^{\sigma_u} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big]^+ \\
&< \int_0^1 g(\xi,s)\varphi_q\Big(\int_0^1 w(\tau)\varphi_p(\frac{c_2}{N})\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_0^1 \varphi_q\Big(\int_0^1 w(\tau)\varphi_p(\frac{c_2}{N})\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&= \frac{c_2}{N}\varphi_q\Big(\int_0^1 w(\tau)\,\mathrm{d}\tau\Big)\Big)\Big[1 + \int_0^1 g(\xi,s)\,\mathrm{d}s\Big] \\
&\le c_2,
\end{aligned}$$

and if $\bar{t} > \sigma_u$, we have

$$\begin{aligned}
Tu(\bar{t}) &= \Big[\int_{\eta}^1 h(\eta,s)\varphi_q\Big(\int_{\sigma_u}^{s} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&\quad + \int_{\bar{t}}^1 \varphi_q\Big(\int_{\sigma_u}^{s} w(\tau)f(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\Big]^+ \\
&< \int_0^1 h(\eta,s)\varphi_q\Big(\int_0^1 w(\tau)\varphi_p(\frac{c_2}{N})\,\mathrm{d}\tau\Big)\,\mathrm{d}s + \int_0^1 \varphi_q\Big(\int_0^1 w(\tau)\varphi_p(\frac{c_2}{N})\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&= \frac{c_2}{N}\varphi_q\Big(\int_0^1 w(\tau)\,\mathrm{d}\tau\Big)\Big[1 + \int_0^1 h(\eta,s)\,\mathrm{d}s\Big] \\
&\le c_2.
\end{aligned}$$

We have

$$\|Tu\| < c_2 = \|u\| \text{ for } u \in \overline{\Omega}_1. \tag{3.1}$$

By using Schauder fixed point theorem, we can get the T has at least one fixed point u_1 in Ω_1. That is, $Tu_1 = u_1$ and $\|u_1\| < c_2$. If $\|u_1\| = 0$, we have $u_1 \equiv 0$, $t \in [0,1]$, which is a contradiction with (H2). Hence, $0 < \|u_1\| < c_2$.

It follows that (1.1) has at least one positive solutions u_1 such that $0 < \|u_1\| < c_2$ from Lemma 2.9. Let

$$\Omega_2 = \{u \in K : \|u\| < c_3\}.$$

For any $u \in \partial\Omega_2$, we have $u \in K$ and $\|u\| = c_3$ and $\delta c_3 \le u(t) \le c_3$ for $\delta \le t \le 1-\delta$. By Lemma 2.4, we have

$$\|T^*u\| = T^*u(\sigma_u) \ge T^*u(\frac{1}{2}).$$

If $\sigma_u \ge 1/2$, it follows from (H6) that

$$\begin{aligned}
\|T^*u\| \ge &\int_0^{\xi} g(\xi,s)\varphi_q\Big(\int_s^{\sigma_u} w(\tau)f^+(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s \\
&+ \int_0^{1/2} \varphi_q\Big(\int_s^{\sigma_u} w(\tau)f^+(\tau,u(\tau),Au(\tau),Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_\delta^{1/2} \varphi_q\Big(\int_s^{1/2} w(\tau) f^+(\tau, u(\tau), Au(\tau), Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geq \int_\delta^{1/2} \varphi_q\Big(\int_s^{1/2} w(\tau)\varphi_p(\frac{c_3}{M})\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{c_3}{M}\int_\delta^{1/2} \varphi_q\Big(\int_s^{1/2} w(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geq c_3.
\end{aligned}
$$

If $\sigma_u < 1/2$, it follows from (H6) that

$$
\begin{aligned}
\|T^*u\| &\geq \int_\eta^1 h(\eta, s)\varphi_q\Big(\int_{\sigma_u}^s w(\tau) f^+(\tau, u(\tau), Au(\tau), Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\quad + \int_{1/2}^1 \varphi_q\Big(\int_{\sigma_u}^s w(\tau) f^+(\tau, u(\tau), Au(\tau), Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geq \int_{1/2}^{1-\delta} \varphi_q\Big(\int_{1/2}^s w(\tau) f^+(\tau, u(\tau), Au(\tau), Bu(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geq \int_{1/2}^{1-\delta} \varphi_q\Big(\int_{1/2}^s w(\tau)\varphi_p(\frac{c_3}{M})\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&= \frac{c_3}{M}\int_{1/2}^{1-\delta} \varphi_q\Big(\int_{1/2}^s w(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s\\
&\geq c_3.
\end{aligned}
$$

Hence, we can show that

$$
\|T^*u\| \geq c_3 = \|u\| \quad \text{for} u \in \partial\Omega_2. \tag{3.2}
$$

As in the proof of (3.1), we obtain

$$
\|T^*u\| < c_2 = \|u\| \quad \text{for } u \in \partial\Omega_1.
$$

Therefore, by using Krasnosel'skii fixed point theorem, T^* has at least one fixed point $u_2 \in \overline{\Omega}_2\backslash\Omega_1$ with $c_2 \leq \|u_2\| \leq c_3$. Subsequently, we prove u_2 is also a fixed point of S.

If $u \in \overline{\Omega}_2\backslash\Omega_1$ and $T^*u = u$, then $u \in K$, $c_2 \leq \|u\| \leq c_3$ and $\min_{t\in[\delta,1-\delta]} u(t) \geq \delta\|u\| \geq \delta c_2$. By Lemma 2.4, Lemma 2.7 and the definition of T^*, we obtain

$$
\begin{gathered}
(T^*u)'(t) = \varphi_q\Big(\int_t^{\sigma_u} w(\tau) f^+(\tau, u(\tau), Au(\tau), Bu(\tau))\,\mathrm{d}\tau)\,\mathrm{d}s\Big), \quad t \in [0,1]\\
(T^*u)'(\sigma_u) = 0,\ (T^*u)'(t) \geq 0 \text{ if } t \in [0, \sigma_u],\ (T^*u)'(t) \leq 0 \text{ if } t \in [\sigma_u, 1]
\end{gathered}
$$

and

$$
\min_{0\leq t\leq 1} u(t) = \min\{u(0), u(1)\} \geq 0.
$$

If $\min_{0\leq t\leq 1} u(t) = u(0)$, when $\xi \leq \sigma_u$, by (H3) and mean value theorem of integral, there exists a $\xi_1 \in [0, \xi]$ such that

$$
u(0) = Au(\xi) = \int_0^\xi g(\xi, s)u'(s)\,\mathrm{d}s = g(\xi, \xi_1)\int_0^\xi u'(s)\,\mathrm{d}s \geq g(\xi, \xi)(u(\xi) - u(0))
$$

and

$$
u(0) \geq \frac{g(\xi,\xi)u(\xi)}{1 + g(\xi,\xi)} \geq \frac{\delta g(\xi,\xi)}{1 + g(\xi,\xi)}\|u\| \geq \frac{\delta g(\xi,\xi)c_2}{1 + g(\xi,\xi)} \geq \frac{\delta g(\xi,\xi)c_2}{1 + g(\xi,0)}. \tag{3.3}
$$

When $\xi > \sigma_u$, by (H3) and mean value theorem of integral, there exist $\xi_2 \in [0, \sigma_u]$ and $\xi_3 \in [\sigma_u, \xi]$ such that

$$\begin{aligned} u(0) =& Au(\xi) = \int_0^{\sigma_u} g(\xi, s)u'(s)\,\mathrm{d}s + \int_{\sigma_u}^{\xi} g(\xi, s)u'(s)\,\mathrm{d}s \\ &= g(\xi, \xi_2)\int_0^{\sigma_u} u'(s)\,\mathrm{d}s + g(\xi, \xi_3)\int_{\sigma_u}^{\xi} u'(s)\,\mathrm{d}s \\ &= g(\xi, \xi_2)(u(\sigma_u) - u(0)) + g(\xi, \xi_3)(u(\xi) - u(\sigma_u)) \\ \geq& g(\xi, \xi)u(\xi) - g(\xi, 0)u(0) \end{aligned}$$

and

$$u(0) \geq \frac{g(\xi, \xi)u(\xi)}{1 + g(\xi, 0)} \geq \frac{\delta g(\xi, \xi)}{1 + g(\xi, 0)}\|u\| \geq \frac{\delta g(\xi, \xi)c_2}{1 + g(\xi, 0)}. \tag{3.4}$$

If $\min_{0\leq t\leq 1} u(t) = u(1)$, when $\eta \leq \sigma_u$, by (H3) and mean value theorem of integral, there exist $\eta_1 \in [\eta, \sigma_u]$ and $\eta_2 \in [\sigma_u, 1]$ such that

$$\begin{aligned} u(1) &= -Bu(\eta) = -\int_{\eta}^{\sigma_u} h(\eta, s)u'(s)\,\mathrm{d}s - \int_{\sigma_u}^{1} h(\eta, s)u'(s)\,\mathrm{d}s \\ &= -h(\eta, \eta_1)\int_{\eta}^{\sigma_u} u'(s)\,\mathrm{d}s - h(\eta, \eta_2)\int_{\sigma_u}^{1} u'(s)\,\mathrm{d}s \\ &= -h(\eta, \eta_1)(u(\sigma_u) - u(\eta)) - h(\eta, \eta_2)(u(1) - u(\sigma_u)) \\ &= h(\eta, \eta_2)u(\eta) - h(\eta, \eta_1)u(1) + (h(\eta, \eta_2) - h(\eta, \eta_1))u(\sigma_u) \\ &\geq h(\eta, \eta)u(\eta) - h(\eta, 1)u(1) \end{aligned}$$

and

$$u(1) \geq \frac{h(\eta, \eta)u(\eta)}{1 + h(\eta, 1)} \geq \frac{\delta h(\eta, \eta)}{1 + h(\eta, 1)}\|u\| \geq \frac{\delta h(\eta, \eta)c_2}{1 + h(\eta, 1)}. \tag{3.5}$$

When $\eta > \sigma_u$, by (H3) and mean value theorem for integrals, there exists $\eta_3 \in [\eta, 1]$ such that

$$\begin{aligned} u(1) &= -Bu(\eta) = -\int_{\eta}^{1} h(\eta, s)u'(s)\,\mathrm{d}s \\ &= h(\eta, \eta_3)(u(\eta) - u(1)) \geq h(\eta, \eta)(u(\eta) - u(1)) \end{aligned}$$

and

$$u(1) \geq \frac{h(\eta, \eta)u(\eta)}{1 + h(\eta, \eta)} \geq \frac{\delta h(\eta, \eta)}{1 + h(\eta, \eta)}\|u\| \geq \frac{\delta h(\eta, \eta)c_2}{1 + h(\eta, \eta))} \geq \frac{\delta h(\eta, \eta)c_2}{1 + h(\eta, 1)}. \tag{3.6}$$

Hence, it follows

$$\min_{0\leq t\leq 1} u(t) \geq \min\Big\{\frac{g(\xi, \xi)}{1 + g(\xi, 0)}, \frac{h(\eta, \eta)}{1 + h(\eta, 1)}\Big\}\delta c_2 \geq c_1$$

from (3.3)–(3.6).

Therefore, if $u \in \overline{\Omega}_2 \setminus \Omega_1$ and $T^*u = u$, we have

$$c_1 \leq u(t) \leq \|u\| \leq c_3.$$

It follows $f(t, u(t), Au(t), Bu(t)) \geq 0$, $t \in [0, 1]$ from (H4). Thus, $T^*u = Su$. That is, the fixed point of T^* on $\overline{\Omega}_2 \setminus \Omega_1$ is also a fixed point of S. We can get the boundary-value problem (1.1) has at least one positive solutions u_2 such that $c_2 \leq \|u_2\| \leq c_3$. The proof is complete. □

For the next theorem, we have a new assumption:

(H7) For each $t \in [0,1]$, $g(t,s)$ and $h(t,s)$ are monotone with respect to s.

We denote

$$M_g = \max\{\max_{t\in[0,1]} g(t,0), \max_{t\in[0,1]} g(t,t)\}, \quad M_h = \max\{\max_{t\in[0,1]} h(t,1), \max_{t\in[0,1]} h(t,t)\}.$$

If (H7) holds, for each $t \in [0,1]$, when $g(t,s)$ is decreasing with respect to s, there exists a $\bar{\xi}_1 \in [0,t]$ such that

$$|(Au)(t)| = |\int_0^t g(t,s)u'(s)\,\mathrm{d}s| = g(t,0)|\int_0^{\bar{\xi}_1} u'(s)\,\mathrm{d}s| \le 2g(t,0)\|u\| \le 2M_g\|u\|,$$

and when $g(t,s)$ is monotone creasing with respect to s, there exists a $\bar{\xi}_2 \in [0,t]$ such that

$$|(Au)(t)| = |\int_0^t g(t,s)u'(s)\,\mathrm{d}s| = g(t,t)|\int_{\bar{\xi}_2}^t u'(s)\,\mathrm{d}s| \le 2g(t,t)\|u\| \le 2M_g\|u\|.$$

As above, if (H7) holds, we can show that

$$|(Bu)(t)| \le 2M_h\|u\|, \quad \text{for each } t \in [0,1].$$

As in the proof of Theorem 3.1, we obtain the following theorem under the assumption taht f satisfies the following growth conditions:

(H8) $f(t,u,x,y) \ge 0$ for $(t,u,x,y) \in [0,1] \times [c_1,c_3] \times [-2M_gc_3, 2M_gc_3] \times [-2M_hc_3, 2M_hc_3]$;

(H9) $f(t,u,x,y) < \varphi_p(\frac{c_2}{N})$ for $(t,u,x,y) \in [0,1] \times [0,c_2] \times [-2M_gc_2, 2M_gc_2] \times [-2M_hc_2, 2M_hc_2]$;

(H10) $f(t,u,x,y) \ge \varphi_p(\frac{c_3}{M})$ for $(t,u,x,y) \in [\delta,1-\delta] \times [\delta c_3, c_3] \times [-2M_gc_3, 2M_gc_3] \times [-2M_hc_3, 2M_hc_3]$.

Theorem 3.2. *Suppose* (H1)–(H3) *and* (H7)–(H10) *hold. Then there exist constants* c_1, c_2, c_3 *such that such that*

$$0 < c_1 \le \min\Big\{\frac{g(\xi,\xi)}{1+g(\xi,0)}, \frac{h(\eta,\eta)}{1+h(\eta,1)}\Big\}\delta c_2, \text{ and } c_2 < \delta c_3 < c_3$$

Then (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < c_2 \le \|u_2\| \le c_3.$$

We have a new assumption:

(H11) For each $t,s \in [0,1]$, $g_s(t,s)$ and $h_s(t,s)$ are bounded.

We denote

$$N_g = \max\{\max_{t\in[0,1]} g(t,0), \max_{t\in[0,1]} g(t,t), \sup_{t,s\in[0,1]} g_s(t,s)\},$$
$$N_h = \max\{\max_{t\in[0,1]} h(t,1), \max_{t\in[0,1]} h(t,t), \sup_{t,s\in[0,1]} h_s(t,s)\}$$

If (H11) holds, for each $t \in [0,1]$, we have

$$\begin{aligned} |(Au)(t)| &= |\int_0^t g(t,s)u'(s)\,\mathrm{d}s| \\ &= |(g(t,t)u(t) - g(t,0)u(0)) - \int_0^t g_s(t,s)u(s)\,\mathrm{d}s| \le 3N_g\|u\|. \end{aligned}$$

As above, if (H11) holds, we can show that

$$|(Bu)(t)| \le 3N_h\|u\|, \quad \text{for each } t \in [0,1].$$

As in the proof of Theorem 3.1, we can get the following theorem, and f satisfies the following growth conditions:

- (H12) $f(t,u,x,y) \ge 0$ for $(t,u,x,y) \in [0,1] \times [c_1,c_3] \times [-3N_gc_3, 3N_gc_3] \times [-3N_hc_3, 3N_hc_3]$;
- (H13) $f(t,u,x,y) < \varphi_p(\frac{c_2}{N})$ for $(t,u,x,y) \in [0,1] \times [0,c_2] \times [-3N_gc_2, 3N_gc_2] \times [-3N_hc_2, 3N_hc_2]$;
- (H14) $f(t,u,x,y) \ge \varphi_p(\frac{c_3}{M})$ for $(t,u,x,y) \in [\delta, 1-\delta] \times [\delta c_3, c_3] \times [-3N_gc_3, 3N_gc_3] \times [-3N_hc_3, 3N_hc_3]$.

Theorem 3.3. *Suppose* (H1)–(H3) *and* (H11–(H14)) *hold. Then there exist nonnegative constants* c_1, c_2, c_3 *such that*

$$0 < c_1 \le \min\Big\{\frac{g(\xi,\xi)}{1+g(\xi,0)}, \frac{h(\eta,\eta)}{1+h(\eta,1)}\Big\}\delta c_2, \text{ and } c_2 < \delta c_3 < c_3$$

Then (1.1) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < c_2 \le \|u_2\| \le c_3.$$

References

[1] R.P. Agarwal, D. O'Regan; *Some new results for singular problems with sign changing nonlinearities*, J. Comput. Appl. Math. 113 (2000), 1-15.

[2] R .P. Agarwal, D. O'Regan, V. Lakshmikantham, S. Leela, *Nonresonant singular boundary-value problems with sign changing nonlinearities*, Appl. Math. Comput. 167 (2005), 1236-1248.

[3] B. Ahmad, S. Sivasundaram; *Existence and approximation of solutions of forced duffing type integro-differential equations with three-point nonlinear boundary conditions*, Nonlinear Anal.: Real World Applications 11 (2010), 2905-2912.

[4] Qasem M. Al-Mdallal; *Monotone iterative sequences for nonlinear integro-differential equations of second order*, Nonlinear Anal.: Real World Applications 12 (2011), 3665-3673

[5] Z. Bai, Z. Gui, W. Ge; *Multiple positive solutions for some p-Laplacian boundary-value problems*, J. Math. Anal. Appl. 300 (2004), 477-490.

[6] W. Cheung, J. Ren; *Twin positive solutions for quasi-linear multi-point boundary-value problems*, Nonlinear Anal. 62 (2005), 167-177.

[7] H. Feng, W. Ge; *Existence of three positive solutions for m-point boundary-value problems with one-dimensional p-Laplacian*, Nonlinear anal. 68 (2008), 2017-2026.

[8] J. M. Gallardo; *Second order differential operators with integral boundary conditions and generation of semigroups*, Rocky Mountain J. Math. 30 (2000), 1265-1292.

[9] W. Ge. *The boundary-value problems for nonlinear differential equations*. Science Press, Beijing. 2007, 147- 228. (In Chinese)

[10] D. Guo, V. Lakshmikantham; *Nonlinear problems in abstract cones*, Academic Press, San Diego, 1988.

[11] Y. Guo, W. Ge, S. Dong; *Two positive solutions for second order three point value problems with sign change nonlinearities*, Acta. Math. Appl. Sinica. 27 (2004), 522-529. (In Chinese)

[12] D. Ji, Y. Tian, W. Ge; *Positive solutions for one-dimensional p-Laplacian boundary-value problems with sign changing nonlinearity*, Nonlinear Anal., 71 (2009) 5406-5416.

[13] M. Jia, P. Wang; *Existence of positive solutions for boundary-value problems with one-dimensional p-Laplacian*, Proceedings of the 5th International Congress on Mathematical Biology (ICMB2011, Nanjing) Vol. 1, 125-130.

[14] G. L. Karakostas, P. Ch. Tsamatos; *Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems*, Electron. J. Differential Equations 30 (2002), 1-17.

[15] A. Lakmeche, A. Hammoudi; *Multiple positive solutions of the one-dimensional p-Laplacian*, J. Math. Anal. Appl. 317 (2006), 43-49.

[16] H. Lian, W. Ge; *Positive solutions for a four-point boundary-value problem with the p-Laplacian*, Nonlinear Anal., 68 (2008), 3493-3503
[17] S. Liu, M. Jia, Y. Tian; *Existence of positive solutions for boundary-value problems with integral boundary conditions and sign changing nonlinearities*, Electron. J. Differential Equations, Vol. 2010(2010), No. 163, pp. 1-12.
[18] A. Lomtatidze, L. Malaguti; *On a nonlocal boundary-value problems for second order nonlinear singular differential equations*, Georgian Math. J. 7 (2000), 133-154.
[19] D. Ma, Z. Du, W. Ge; *Existence and iteration of monotone positive solutions for multipoint boundary-value problem with p-Laplacian operator*, Comput. Math. Appl. 50 (2005), 729-739.
[20] R. Ma; *Positive solutions for second-order functional differential equations*, Dynam. Systems Appl. 10 (2001), 215–224.
[21] Y. Wang, C. Hou; *Existence of multiple positive solutions for one-dimensional p-Laplacian*, J. Math. Anal. Appl. 315 (2006), 144-153.
[22] A. Wazwaz; *A comparison study between the modified decomposition method and the traditional methods for solving nonlinear integral equations*, Appl. Math. Comput. 181 (2006), 1703-1712.
[23] J .R. L. Webb, G. Infante; *Positive solutions of nonlocal boundary-value problems involving integral conditions*, NoDEA Nonlinear Differential Equations Appl., 15(2008) 45–67.
[24] J. R .L. Webb, G. Infante; *Positive solutions of nonlocal boundary-value problems: a unified approach*, J. London Math. Soc. 74(2006) 673–693.
[25] S. Xi, M. Jia, H. Ji; *Positive solutions of boundary-value problems for systems of second-order differential equations with integral boundary condition on the half-line*, Electron. J. Qual. Theory Differ. Equ. 1 (2009), 1-12.

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