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MULTIPLE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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Dedicated to the memory of Professor Riichi Iino

Abstract. We consider the elliptic problem with nonlinear boundary conditions:

$$\begin{aligned} -\Delta u + bu &= f(x,u) \quad \text{in } \Omega, \\ -\partial_\nu u &= |u|^{q-1}u - g(u) \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$. Proving the existence of solutions of this problem relies essentially on a variational argument. However, since $L^{q+1}(\partial\Omega) \subset H^1(\Omega)$ does not hold for large q, the standard variational method can not be applied directly. To overcome this difficulty, we use approximation methods and uniform a priori estimates for solutions of approximate equations.

1. Introduction

We consider the heat equations with nonlinear boundary conditions of the form:

$$\begin{gathered} u_t = \Delta u + bu, \quad (x,t) \in \Omega \times (0,T), \\ -\partial_\nu u = \beta(u), \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where ∂_ν denotes the outward normal derivative on the boundary, which appears in models describing diffusion systems governed by some radiation law on the boundary. The standard boundary conditions for heat equations are usually assumed to be Dirichlet-type, Neumann-type or mixed-type boundary conditions. This convention could be meaningful when the total system on the boundary is controlled so as to keep the prescribed boundary conditions. However when the whole system is very large, it would be no more possible to control the flux of heat through the boundary. For such a case, the boundary condition is expected to be posed by considering the heat radiation law on the boundary. The typical example of this kind of radiation law on the boundary is derived from the so-called Stefan-Boltzmann's radiation law, which says that the heat energy radiation from the surface of the

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body J is given by $J = \sigma(T^4 - T_s^4)$, where $\sigma > 0$ is a physical constant, T is the surface temperature and T_s is the outside temperature. Thus Stefan-Boltzmann's law gives an example where $\beta(u)$ is a monotone increasing function. For this case, the unique solvability for parabolic equations(1.1) is completely covered by the abstract (subdifferential operator) theory by Brézis [1].

However, Stefan-Boltzmann's radiation law is valid only for an idealized situation, in other words, the radiation law rulling real situations might be perturbed from Stefan-Boltzmann's law. In particular, if we consider the case where the heat flux radiated from the surface is reflected by its surrounding materials, then we must consider also the absorption effect. For such a case, $\beta(u)$ could not be a monotone increasing function anymore, but monotone increasing with small perturbation; i.e., the boundary condition should be altered by

$$-\partial_\nu u = \beta(u) - g(u) \quad x \in \partial\Omega,$$

where $\beta(u)$ is a monotone increasing function and $g(u)$ is its perturbation. In fact, such a kind of non-monotone radiation-absorption models are already proposed from the view point of engineering (see e.g. [2]).

In this article, we are concerned with such non-monotone radiation-absorption models and study the stationary problem of a general form:

$$\begin{gathered} -\Delta u + bu = f(x,u) \quad \text{in } \Omega, \\ -\partial_\nu u = \beta(u) - g(u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $b > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. In [8], the existence and the H^2-regularity of solutions of (1.2) is studied for the special case $f(x,u) = f(x)$ under the following conditions on $\beta(u)$ and $g(u)$.

- (A1) $\beta(u)$ is a continuous and monotone increasing function,
- (A2) $\lim_{|u|\to\infty} \beta(u)/u = \infty$,
- (A3) $g(u)$ is a locally Lipschitz continuous function and there exist $\theta \in (0,1)$ and $c_1 > 0$ such that $|g'(u)| \le \theta\beta'(u) + c_1 \quad \forall u \in \mathbb{R}^1$,
- (A4) there exists $c_2 > 0$ such that $|u\beta(u)| \le c_2(j(u) + u^2 + 1)$ for all $u \in \mathbb{R}^1$, where $j(u) = \int_0^u \beta(s)ds$.

The following results were presented in [8].

Theorem 1.1. *Let* (A1)–(A3) *be satisfied and let* $f(x,u) = f(x) \in L^2(\Omega)$. *Then there exists a solution* $u \in H^2(\Omega)$ *of* (1.2). *Moreover there exists* $c > 0$ *such that every solution* u *of* (1.2) *belonging to* $H^2(\Omega)$ *satisfies*

$$\|u\|_{H^2(\Omega)} \le c\big(1 + \|f\|_{L^2(\Omega)}\big). \tag{1.3}$$

Furthermore the elliptic estimates for weak solutions of (1.2) is also shown in [8].

A function $u \in \{u \in H^1(\Omega); \beta(u),\ g(u) \in L^1(\partial\Omega)\}$ is said to be a weak solution of (1.2) if u satisfies

$$\int_\Omega \big(\nabla u \cdot \nabla\phi + bu\phi\big)\, dx + \int_{\partial\Omega} \big(\beta(u) - g(u)\big)\phi\, d\sigma = \int_\Omega f\phi\, dx \tag{1.4}$$

for any $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$. We set $j(u) = \int_0^u \beta(s)\, ds$ and

$$D(j) = \{u \in H^1(\Omega); j(u) \in L^1(\partial\Omega)\}.$$

Theorem 1.2. *Let* (A1)–(A4) *be satisfied and let* $f(x,u) = f(x) \in L^2(\Omega)$. *Then every weak solution* u *of* (1.2) *with* $u \in D(j)$ *satisfies* (1.3).

In this article, we consider the case where $f(x,u)$ satisfies the following conditions.

- (B1) $f(x,t) \in C(\bar{\Omega} \times \mathbb{R}^1; \mathbb{R}^1)$,
- (B2) thee exist $p \in (1, 2^*)$ and $c > 0$ such that $|f(x,u)| \le c(1 + |u|^{p-1})$, where

$$2^* = \begin{cases} \infty & \text{if } n = 1 \text{ or } 2, \\ \frac{2n}{n-2} & \text{if } n \ge 3. \end{cases}$$

- (B3) $\lim_{u \to 0} \frac{f(x,u)}{u} = 0$ uniformly on $x \in \Omega$,
- (B4) there exist $\mu > 2$ and $r > 0$ such that $0 < \mu F(x,u) \le u f(x,u)$ for $|u| \ge r$, where $F(x,u) = \int_0^u f(x,s) ds$.

A typical example of a function satisfying(B1)–(B4) is $f(x,u) = a(x)|u|^{p-2}u$ with $a(\cdot) \in L^\infty(\Omega)$ and $1 < p < 2^*$.

The function $\beta(u)$ is assumed to have the power nonlinearity $\beta(u) = |u|^{q-2}u$; i.e., we are concerned with the equation

$$\begin{gathered} -\Delta u + bu = f(x,u) \quad \text{in } \Omega, \\ -\partial_\nu u = |u|^{q-2}u - g(u) \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

We further impose the following conditions on $g(u)$.

- (A5) $\lim_{u \to 0} \frac{g(u)}{u} = 0$.
- (A6) $g(u)$ is a continuous function and for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|g(u)| \le \varepsilon |u|^{q-1} + c_\varepsilon \quad \forall u \in \mathbb{R}^1.$$

Then our existence results are stated a follows.

Theorem 1.3. *Let* (B1)–(B4), (A5) *and* (A6) *be satisfied and let* $2 < q < \mu$. *Then there exists a nontrivial weak solution* u *of* (1.5) *belonging to* $H^1(\Omega) \cap L^\infty(\Omega)$.

Theorem 1.4. *Let the assumptions in Theorem 1.3 be satisfied and let* $f(x,u)$ *and* $g(u)$ *be odd in* u. *Then there exist infinitely many weak solutions* $\{u_k\}_{k=1}^\infty$ *of* (1.5) *in* $H^1(\Omega) \cap L^\infty(\Omega)$ *satisfying*

$$\lim_{k \to \infty} I(u_k) = \infty.$$

Here $I(u)$ *is a functional associated with* (1.5) *defined by*

$$I(u) = \int_\Omega \Big(\frac{1}{2} \left(|\nabla u|^2 + bu^2 \right) - F(x,u) \Big) dx + \int_{\partial\Omega} \Big(\frac{1}{q} |u|^q - G(u) \Big) d\sigma, \tag{1.6}$$

where $G(u) = \int_0^u g(s)\, ds$.

2. Proofs of main theorems

Proof of Theorem 1.3. We rely on the variational approach (mountain pass lemma) to prove the existence of nontrivial solutions of (1.5). However the functional $I(u)$ given by (1.6) may not be well defined on $H^1(\Omega)$ in general, since the functions appearing in the boundary integral might not be integrable for any $u \in H^1(\Omega)$. To cope with this difficulty, we first introduce the following approximation problems:

$$\begin{gathered} -\Delta u + bu = f(x,u) \quad \text{in } \Omega, \\ -\partial_\nu u = \beta_k(u) - g_k(u) \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where the approximation functions β_k and g_k for β and g are given by

$$\beta_k(u) = \begin{cases} k^{q-1} & u > k, \\ |u|^{q-2}u & |u| \le k, \\ -k^{q-1} & u < -k, \end{cases} \qquad g_k(u) = \begin{cases} g(k) & u > k, \\ g(u) & |u| \le k, \\ g(-k) & u < -k. \end{cases}$$

Then the functional I_k associated with (2.1) is

$$I_k(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + bu^2\right)\,dx + \int_{\partial\Omega} \left(j_k(u) - G_k(u)\right)\,d\sigma - \int_\Omega F(x,u)\,dx,$$

where $j_k(u) = \int_0^u \beta_k(s)\,ds$, $G_k(u) = \int_0^u g_k(s)\,ds$. Since $\beta_k,\ g_k \in L^\infty(\mathbb{R})$ and $p \in (1, 2^*)$, it is clear that I_k is well defined on $H^1(\Omega)$. From (A6), there exists $r_0 > 0$ independent of $k \in \mathbb{N}$ such that $j_k(u) - G_k(u) > 0$ for $|u| > r_0$. Hence by (A5) and the trace theorem, for any $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ independent of $k \in \mathbb{N}$ such that

$$\int_{\partial\Omega} (j_k(u) - G_k(u))\,d\sigma \ge -\eta\|u\|^2_{H^1(\Omega)} \quad \forall u \in \{u \in H^1(\Omega); \|u\|_{H^1(\Omega)} < \delta\}.$$

Therefore, by (B3), there exists $\mu, \rho > 0$ independent of $k \in \mathbb{N}$ such that

$$I_k(u) \ge \mu\|u\|^2_{H^1(\Omega)} \quad \forall u \in \{u \in H^1(\Omega); \|u\|_{H^1(\Omega)} = \rho\}. \tag{2.2}$$

Next we are going to check the (PS) condition. We note that for $u \in H^1(\Omega)$,

$$\begin{aligned} I_k(u) - \frac{(\nabla I_k(u), u)}{\mu} &= \big(\frac{1}{2} - \frac{1}{\mu}\big)\int_\Omega \left(|\nabla u|^2 + bu^2\right)\,dx - \int_\Omega \big(F(x,u) - \frac{uf(x,u)}{\mu}\big)\,dx \\ &\quad + \int_{\partial\Omega} \big(j_k(u) - \frac{\beta_k(u)u}{\mu}\big)d\sigma - \int_{\partial\Omega}\big(G_k(u) - \frac{g_k(u)u}{\mu}\big)d\sigma, \end{aligned}$$

where $\nabla I_k(u)$ denotes the Fréchet derivative of $I_k(u)$. From (A6), for any $\eta > 0$ there exists $c_\eta > 0$ such that

$$\begin{aligned} & j_k(u) - \frac{\beta_k(u)u}{\mu} - \big(G_k(u) - \frac{g_k(u)u}{\mu}\big) \\ & \ge \begin{cases} \big(\frac{1}{q+1} - \frac{1}{\mu}\big)|u|^{q+1} - \big(\eta|u|^{q+1} + c_\eta\big) & \text{if } \ |u| \le k, \\ \big(\frac{1}{q+1} - \frac{1}{\mu}\big)k^q|u| - (\eta k^q|u| + c_\eta) & \text{if } \ |u| > k. \end{cases} \end{aligned}$$

Since $1 < q < \mu - 1$, by choosing $\eta > 0$ small enough, we deduce that

$$\int_{\partial\Omega}\Big(j_k(u) - \frac{\beta_k(u)u}{\mu}\Big)d\sigma - \int_{\partial\Omega}\Big(G_k(u) - \frac{g_k(u)u}{\mu}\Big)d\sigma \ge -c_\eta|\partial\Omega|.$$

Therefore by (B4), there exists a constant C_Ω depending on $|\Omega|$ and $|\partial\Omega|$ such that

$$\big(\frac{1}{2} - \frac{1}{\mu}\big)\int_\Omega \left(|\nabla u|^2 + bu^2\right)dx \le I_k(u) - \frac{(\nabla I_k(u), u)}{\mu} + C_\Omega. \tag{2.3}$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence such that $I_k(u_j) \to c$ and $\nabla I_k(u_j) \to 0$ in $(H^1(\Omega))^*$ as $j \to \infty$. From (2.3), the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Hence, there exists a subsequence of $\{u_j\}_{j\in\mathbb{N}}$ denoted again by $\{u_j\}_{j\in\mathbb{N}}$ which converges to u weakly in $H^1(\Omega)$.

Here we note the identity

$$\begin{aligned} &(\nabla I_k(u_j) - \nabla I_k(u), u_j - u) \\ &= |\nabla(u_j - u)|_{L^2}^2 + b|u_j - u|_{L^2}^2 + \big(f(x,u_j) - f(x,u), u_j - u\big) \\ &\quad \times \int_{\partial\Omega} (\beta_k(u_j) - \beta_k(u), u_j - u)\, d\sigma - \int_{\partial\Omega} (g_k(u_j) - g_k(u), u_j - u)\, d\sigma. \end{aligned} \tag{2.4}$$

Furthermore, by Rellich's compactnes theorem with (B2) and the trace theorem, we obtain

$$\begin{gathered} f(x,u_j) \to f(x,u) \quad \text{strongly in} \quad L^{(p+1)/p}(\Omega), \\ \beta_k(u_j) \to \beta_k(u),\ g_k(u_j) \to g_k(u) \quad \text{strongly in} \quad L^2(\partial\Omega). \end{gathered}$$

Hence, by letting $j \to \infty$ in (2.4), we find that u_j converges to u strongly in $H^1(\Omega)$. Thus it is verified that I_k satisfies (PS)-condition for any $k \in \mathbb{N}$. Now we define

$$I_0(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 + bu^2\right) dx - \int_\Omega F(x,u) dx. \tag{2.5}$$

Since $I_k(u) = I_0(u)$ for $u \in C_0^\infty(\Omega)$ and (B4) implies that there exist constants $\gamma_1,\ \gamma_2 > 0$ such that (see [7])

$$F(x,\xi) \ge \gamma_1 |\xi|^\mu - \gamma_2 \quad \forall x \in \Omega,\ \forall \xi \in \mathbb{R}^1, \tag{2.6}$$

it is easy to see that there exists $\phi_0 \in C_0^\infty(\Omega)$ independent of $k \in \mathbb{N}$ such that

$$I_k(\phi_0) \le 0 = I_k(0).$$

Note that due to mountain pass lemma [7], there exists a critical value of I_k characterized by

$$c_k = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_k(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1]; H^1(\Omega)); \gamma(0) = 0,\ \gamma(1) = \phi_0\}$. Take $\gamma(t) = t\phi_0$ as a test path in Γ for all $k \in \mathbb{N}$, then we obtain

$$c_k \le \max_{t \in [0,1]} I_k(t\phi_0) = \max_{t \in [0,1]} I_0(t\phi_0) =: c^*, \tag{2.7}$$

which implies the boundedness of $\{c_k\}_{k=1}^\infty$. Moreover, from (2.2) it is clear that for $k \in \mathbb{N}$,

$$c_k \ge \mu\rho^2.$$

Therefore, the critical point with the critical value c_k gives a nontrivial solution of (2.1). Let u_k be a critical point of I_k with the critical value c_k, then by using (2.7), we can derive the H^1-boundedness of $\{u_k\}_{k=1}^\infty$. In fact, taking $u = u_k$ in (2.3), then from $\nabla I_k(u_k) = 0$, we have

$$\Big(\frac{1}{2} - \frac{1}{\mu}\Big) \int_\Omega (|\nabla u_k|^2 + bu_k^2) dx \le I_k(u_k) + C_\Omega \le c^* + C_\Omega. \tag{2.8}$$

Furthermore, we can derive the following L^∞-estimates for u_k.

Lemma 2.1. *Let $n \ge 2$. Then there exist $c = c(n,p,g) > 0$ and $\gamma = \gamma(n,p) \ge 1$ such that any weak solution $u_k \in H^1(\Omega)$ of (2.1) with $\|u_k\|_{H^1(\Omega)} \le K$ satisfies*

$$\|u_k\|_{L^\infty(\Omega)} \le cK^\gamma.$$

Proof. Our proof is based on Moser's iteration argument; see [4, Lemma 3.1] or [3, Theorem 8.17]. Here we use the notation $\|u\|_r = \|u\|_{L^r(\Omega)}$ for $r \in [1,\infty]$. From (A6), we can choose $R_0 > 0$ such that $(\beta(u) - g(u))u \geq 0$ for $|u| \geq R_0$. We set $w_k = \max\{u_k, 0\}$ and $m_0 = \sup_{|u|\leq R_0} |g(u)|/|u|$. Then we see that

$$(\beta_k(u_k) - g_k(u_k))w_k \geq -m_0 w_k^2.$$

Hence multiplying (2.1) by w^α ($\alpha \geq 1$), we obtain

$$\min(\frac{4\alpha}{(\alpha+1)^2}, b)\|w_k^{(\alpha+1)/2}\|_{H^1(\Omega)}^2 \leq \int_\Omega w_k^\alpha F(x,u_k)dx + m_0 \int_{\partial\Omega} w_k^{\alpha+1} d\sigma.$$

From (B2) and (B3), there exists $c > 0$ such that $|F(x,u)| \leq c(|u|+|u|^p)$. Therefore,

$$\min(\frac{4\alpha}{(\alpha+1)^2}, b)\|w_k^{(\alpha+1)/2}\|_{H^1(\Omega)}^2 \leq c\int_\Omega \left(w_k^{\alpha+1} + w_k^{\alpha+p}\right)dx + m_0 \int_{\partial\Omega} w_k^{\alpha+1} d\sigma. \tag{2.9}$$

Furthermore, from the trace inequality, we obtain

$$\begin{aligned}\int_{\partial\Omega} w_k^{\alpha+1} d\sigma &= \|w^{(\alpha+1)/2}\|_{L^2(\partial\Omega)}^2 \\ &\leq \big(\frac{\varepsilon}{\alpha+1}\big)\|\nabla w_k^{(\alpha+1)/2}\|_2^2 + c\big(\frac{\alpha+1}{\varepsilon}\big)\|w_k^{(\alpha+1)/2}\|_2^2.\end{aligned} \tag{2.10}$$

Let $p^* = 2n/(n-2)$ if $n \geq 3$ and $p^* = 2p$ if $n = 2$. We choose $\theta \in (0,1)$ such that $1/2p = \theta/2 + (1-\theta)/p^*$ and set $\kappa = p^*/2 > 1$. By the Hölder inequality and the interpolation inequality, we have

$$\int_\Omega w_k^{\alpha+p} dx \leq \|w_k^{p-1}\|_{p/(p-1)}\|w_k^{\alpha+1}\|_p \leq \|w_k\|_p^{p-1}\|w_k^{\alpha+1}\|_\kappa^{1-\theta}\|w_k^{\alpha+1}\|_1^\theta.$$

By the Sobolev inequality and the assumption $\|u_k\|_{H^1(\Omega)} \leq K$, we see that

$$\begin{aligned}&\int_\Omega w_k^{\alpha+p} dx \\ &\leq cK^{p-1}\|w_k^{\alpha+1}\|_\kappa^{1-\theta}\|w_k^{\alpha+1}\|_1^\theta \\ &\leq c\Big(\big(\frac{\varepsilon}{\alpha+1}\big)\|w_k^{\alpha+1}\|_\kappa + K^{(p-1)/\theta}\big(\frac{\varepsilon}{\alpha+1}\big)^{-(1-\theta)/\theta}\|w_k^{\alpha+1}\|_1\Big) \\ &\leq c\Big(\big(\frac{\varepsilon}{\alpha+1}\big)\|w_k^{(\alpha+1)/2}\|_{2^*}^2 + K^{(p-1)/\theta}\big(\frac{\varepsilon}{\alpha+1}\big)^{-(1-\theta)/\theta}\|w_k\|_{\alpha+1}^{\alpha+1}\Big) \\ &\leq c\Big(\big(\frac{\varepsilon}{\alpha+1}\big)\|w_k^{(\alpha+1)/2}\|_{H^1(\Omega)}^2 + K^{(p-1)/\theta}\big(\frac{\varepsilon}{\alpha+1}\big)^{-(1-\theta)/\theta}\|w_k\|_{\alpha+1}^{\alpha+1}\Big).\end{aligned} \tag{2.11}$$

Hence, in view of (2.9), (2.10) and (2.11), taking $\varepsilon > 0$ small enough, we obtain

$$\|w_k^{(\alpha+1)/2}\|_{H^1(\Omega)}^2 \leq cK^{(p-1)/\theta}(\alpha+1)^\nu\|w_k\|_{\alpha+1}^{\alpha+1}$$

for some $\nu \geq 1$. Therefore, from the Sobolev inequality, it follows that

$$\|w_k\|_{\kappa(\alpha+1)}^{\alpha+1} \leq cK^{(p-1)/\theta}(\alpha+1)^\nu\|w_k\|_{\alpha+1}^{\alpha+1}.$$

By the same iteration argument as in the proof in [3, Theorem 8.17], we can show that here exists $\gamma > 0$ such that

$$\|w_k\|_\infty \leq cK^\gamma\|w_k\|_2,$$

which assures the L^∞-estimates for $w_k = \max\{u_k, 0\}$. By the arguments similar to those above, we can also derive the L^∞-estimates for $\min\{u_k, 0\}$. Thus the proof is completed. $\square$

For the case $n \geq 2$, Lemma 2.1 and (2.8) assure that $\{u_k\}_{k=1}^{\infty}$ is bounded in $L^{\infty}(\Omega)$. As for the case $n = 1$, (2.8) with the embedding $H^1(\Omega) \subset L^{\infty}(\Omega)$ assures the boundedness of $\{u_k\}_{k=1}^{\infty}$ in $L^{\infty}(\Omega)$. Hence there exists $A > 0$ such that $\|u_k\|_{L^{\infty}(\Omega)} \leq A$. Therefore if solutions u_k are classical ones, then it is clear that by the construction of β_k and g_k, we find that

$$\beta_k(u_k) = \beta(u_k), \quad g_k(u_k) = g(u_k) \quad \text{for } k > A.$$

Thus it is easy to see that u_k with $k > A$ satisfies (1.4) and gives the desired solution. To verify this rigorously for $u_k \in H^1(\Omega) \cap L^{\infty}(\Omega)$, we prepare the following lemma, which completes the proof of Theorem 1.3. □

Lemma 2.2. *Let Ω be a domain in $\mathbb{R}^n$ such that the trace theorem in $L^p(\Omega)$ holds true for some $p \in [1, \infty)$ and let u belong to $W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$. Then u belongs to $L^{\infty}(\partial\Omega)$ and satisfies*

$$\|u\|_{L^{\infty}(\partial\Omega)} \leq \|u\|_{L^{\infty}(\Omega)}.$$

Proof. We are going to apply the "L^{∞}-energy method" developed in [5, 6]. Since $u \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ assures that $|u|^r \in W^{1,p}(\Omega)$ for all $r \in [1, \infty)$ and the trace theorem works in $L^p(\Omega)$, we have

$$\begin{aligned} \|u\|_{L^{ps}(\partial\Omega)} &= \||u|^s\|_{L^p(\partial\Omega)}^{1/s} \\ &\leq \Big(C_p\Big\{\Big(\int_{\Omega} s^p |u|^{(s-1)p} |\nabla u|^p dx\Big)^{1/p} + \Big(\int_{\Omega} (|u|^s)^p dx\Big)^{1/p}\Big\}\Big)^{1/s} \\ &\leq C_p^{1/s} s^{1/s} \|u\|_{L^{\infty}(\Omega)}^{\frac{s-1}{s}} \big(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\big)^{1/s} \to \|u\|_{L^{\infty}(\Omega)}, \end{aligned}$$

as $s \to \infty$. Then the conclusion follows from [6, Lemma 2.2]. □

Proof of Theorem 1.4. We again consider approximation problems (2.1). Since $g(u)$ and $f(\cdot, u)$ are assumed to be odd functions, we can apply the symmetric mountain pass lemma to obtain infinitely many solutions for approximation problems. In fact, let $< \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition and let e_i be the corresponding i-th eigenfunctions. We claim that for a sufficiently large $k_0 \in \mathbb{N}$, there exist $\rho > 0$, $\alpha > 0$ such that $I_k(u) \geq \alpha$ for all $u \in V^+ :=$ span $\{e_k; k \geq k_0\}$ with $\|u\|_{H^1(\Omega)} = \rho$. Indeed, by (B2) and the interpolation inequality, we obtain

$$\begin{aligned} I_k(u) &\geq \int_{\Omega} \big(\frac{1}{2}|\nabla u|^2 + b|u|^2\big) dx - C\int_{\Omega} |u|^p dx - C \\ &\geq \tilde{b}\|u\|_{H^1}^2 - C\|u\|_{L^2}^r \|u\|_{L^{2^*}}^{p-r} - C \\ &\geq \Big(\tilde{b} - C_1 \lambda_{k_0}^{-r/2} \|u\|_{H^1}^{p-2}\Big) \|u\|_{H^1}^2 - C_2, \end{aligned}$$

where $\tilde{b} = \min(1/2, b)$, $\frac{r}{2} + \frac{p-r}{2^*} = 1$. Then since $r = n(1 - p/2^*) > 0$, taking $\rho = \sqrt{2(C_2 + 1)/\tilde{b}}$ and choose $k_0 \in \mathbb{N}$ such that $C_1 \lambda_{k_0}^{-r/2} \rho^{p-2} \leq \tilde{b}/2$, we find that $I_k(u) \geq 1$ for all $u \in V^+$ with$\|u\|_{H^1} = \rho$.

Now we put $V^- :=$ span $\{e_k; k < k_0\}$, the orthogonal complement of V^+ in $H_0^1(\Omega)$. Since $I_k|_{V^-} = I_0|_{V^-}$ and V^- is finite dimensional, by virtue of (2.6), there exists $R > 0$ independent of $k \in \mathbb{N}$ such that

$$I_k(u) \leq 0 \quad \text{for } u \in V^- \setminus B_{V^-}(R),$$

where $B_{V^-}(R) = \{u \in V^-; \|u\|_{H^1(\Omega)} < R\}$. Then by the symmetric mountain pass lemma [7, Theorem 9.12], there exist infinitely many critical points $\{u_k^j\}_{j=1}^\infty$ of I_k whose critical values $\{c_k^j\}_{j=1}^\infty$ are unbounded and characterized by

$$c_k^j = I_k(u_k^j) = \inf_{h\in\Gamma} \max_{u\in V^-} I_k(h(u)),$$

where $\Gamma = \{h \in C(V^-; H^1(\Omega)); h \text{ is odd}, \ h(u) = u \text{ if } u \in V^- \setminus B_{V^-}(R)\}$. Take $\mathrm{id} \in \Gamma$ as a test path, then we obtain

$$c_k^j \le \max_{u\in V^-} I_k(u) = \max_{u\in V^-} I_0(u) =: c_j^*,$$

whence follows the boundedness of $\{c_k^j\}_{k=1}^\infty$. Hence the H^1-boundedness of $\{u_k^j\}_{k\in\mathbb{N}}$ follows from (2.8); i.e., there exists C such that

$$\|u_k^j\|_{H^1(\Omega)}^2 \le C\left(I_k(u_k^j) + 1\right) \le C(c_j^* + 1).$$

Moreover from Lemma 2.1, we obtain the L^∞-estimates for $\{u_k^j\}_{k\in\mathbb{N}}$.

$$\|u_k^j\|_{L^\infty(\Omega)} \le c(1 + c_j^*)^{\gamma/2} =: A_j.$$

Hence by the construction of β_k and g_k and Lemma 2.2, we note that for $k > A_j$,

$$\beta_k(u_k^j) = \beta(u_k^j), \quad g_k(u_k^j) = g(u_k^j).$$

Therefore, the critical point u_k^j of the functional I_k with $k \ge A_j$ turn out to be the critical points of I. Thus, we can find infinitely many critical points $\{u_k\}_{k=1}^\infty$ of I whose critical values are unbounded, which completes the proof. □

Remark 2.3. (1) If we assume that g is a locally Lipschitz continuous function on $\mathbb{R}$, then every solution in $H^1(\Omega) \cap L^\infty(\Omega)$ of (1.5) given in Theorems 1.3 and 1.4 belongs to $H^2(\Omega)$. In fact, Since (A3) and (A4) are satisfied with β, g replaced by β_k, g_k respectively, we can apply Theorem 1.2.

(2) In Theorems 1.3 and 1.4, if $n = 1$, then (B2) can be dropped while if $n = 2$, then it suffices that $|f(x,\xi)| \le c \exp \varphi(\xi)$, with $\varphi(\xi)\xi^{-2} \to 0$ as $|\xi| \to \infty$; see [7].

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