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WEAK SOLUTIONS FOR DEGENERATE SEMILINEAR ELLIPTIC BVPS IN UNBOUNDED DOMAINS

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Abstract. In this article, we prove the existence of a weak solution for the degenerate semilinear elliptic Dirichlet boundary-value problem

$$\begin{gathered} Lu(x) + \sum_{i=1}^{n} g(x)h(u(x))D_i u(x) = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

in a suitable weighted Sobolev space. Here $\Omega \subset \mathbb{R}^n$, $1 \le n \le 3$, is not necessarily bounded.

1. Introduction

For $1 \le n \le 3$, let $\Omega \subset \mathbb{R}^n$, be a domain (not necessarily bounded) with boundary $\partial\Omega$. We assume $\Omega = \cup_{i=1}^{\infty}\Omega_i$, $\overline{\Omega}_i \subseteq \Omega_{i+1} \subseteq \overline{\Omega}_{i+1} \subset \Omega$, each $\Omega_i \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega_i$. Let L be an elliptic operator in divergence form

$$Lu(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(x)), \quad D_j = \frac{\partial}{\partial x_j},$$

where the coefficients a_{ij} are measurable, real valued functions, the matrix $\mathcal{A} = (a_{ij})$ is symmetric and satisfy the degenerate ellipticity condition

$$\lambda|\xi|^2\omega(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2\omega(x), \quad \text{a.e. } x \in \Omega, \tag{1.1}$$

for all $\xi \in \mathbb{R}^n$ and ω is an weight function ($\lambda > 0, \Lambda > 0$). When $\omega = 1$ in (1.1), the condition (1.1) reduces to the usual ellipticity condition. However, such an ellipticity condition may not hold if a_{ij} are functions vanishing at some point $x \in \overline{\Omega}$ leading to the degeneracy of the ellipticity condition. Let $f \in L^2(\Omega)$. In this paper, we study the existence of weak solutions to the degenerate semilinear elliptic BVP

$$\begin{gathered} Lu(x) + \sum_{i=1}^{n} g(x)h(u(x))D_i u(x) = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

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where $g/\sqrt{\omega} \in L^\infty(\Omega)$ and h is bounded and Lipschitz continuous. The tools used are pseudomonotone operators as introduced by of Brézis [6], the compact embedding theorem in weighted Sobolev spaces in a domain of $\mathbb{R}^n, n \leq 3$ and a well-known technique used for unbounded domain as in Noussair and Swanson[23]. Where as the restriction on dimension of the domain has yields us a required compactness condition. The study is inspired by a non-degenerate problem in bounded domain given in the book by Zeidler [27].

In general, the Sobolev spaces $W^{k,p}(\Omega)$(without weights) occurs as spaces of solutions for elliptic and parabolic PDEs. For degenerate problems with various types of singularities in the coefficients it is natural to look for solutions in weighted Sobolev spaces; see [9, 10, 11, 15, 16, 17]. Elliptic BVPs in unbounded domains present specific difficulties, primarily due to lack of compactness. Another difficulty in the study of the elliptic BVPs is due to the non-availability of the Poincare-inequality in the Sobolev spaces $W_0^{1,p}(\Omega)$ for a general unbounded domain say Ω. One of the classical technique employed is extracting a solution on unbounded domain Ω by solutions on bounded subdomains of Ω under the assumption the suitable upper and lower solutions exist. The related literature are found in Noussair and Swanson[23] and Cac [8]. Secondly, the use of Sobolev spaces of highly symmetric functions, which admit compact embeddings, as in Berestycki and Lions [2, 3]. Thirdly, the use of weighted-norm Sobolev spaces which admit compact embeddings, as in Benci [1], Bongers, Heinz and Kiipper [5].

In [4], Berger and Schechter have shown that a substitute for such embedding results can be obtained when Ω is unbounded, by introducing appropriate weighted L^p norms. These results are then applied by them to establish an existence theorem for the Dirichlet problem for quasilinear elliptic equations in an unbounded domain. A few references for nonlinear boundary value problems in unbounded domains with aid of pseudomonotone operators are found in [7, 12, 14, 22]. The equation (1.2) considered in the present study is not a subclass of the equations studied in [7, 12, 14, 22]. The compactness condition for weighted Sobolev spaces has been assumed in [12], and it is shown how the assumption of monotonicity can be weakened still guaranteeing the pseudo-monotonicity of certain nonlinear degenerated or singular elliptic differential operators.

Section 2 deals with preliminaries. Section 3 deals with the existence of a solution (1.2) in an arbitrary bounded domain say G. In section 4, we obtain a uniform bound for the solutions $\{u_i\}$ of (1.2) in each bounded subdomains Ω_i and finally, extraction of a solution for (1.2) from the sequence $\{u_i\}$ has been shown.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$ be an open connected set. Let $\omega : \mathbb{R}^n \to \mathbb{R}^+$ be a weight function(*i.e.* locally integrable non negative function with $0 < \omega < \infty$ a.e) in Ω satisfying the conditions

$$\omega \in L^1_{\rm loc}(\Omega), \quad \omega^{-1/(p-1)} \in L^1_{\rm loc}(\Omega), \quad 1 < p < \infty. \tag{2.1}$$

We denote by $L^p(\Omega)$ $(1 \leq p < \infty)$ the usual Banach space of measurable real valued functions, u, defined in Ω for which

$$\|u\|_{p,\Omega} = \Big(\int_\Omega |u|^p dx \Big)^{1/p} < \infty. \tag{2.2}$$

For $p \geq 1$, the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined by

$$W^{1,p}(\Omega, \omega) := \{u \in L^p(\Omega) : D_j u \in L^p(\Omega, \omega), j = 1, 2 \dots, n\}$$

with the associated norm

$$\|u\|_{1,p,\Omega} = \Big(\int_\Omega |u|^p dx + \int_\Omega |Du|^p \omega \, dx \Big)^{1/p}, \tag{2.3}$$

where $Du = (D_1 u, \dots, D_n u)$. The space $W_0^{1,p}(\Omega, \omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). We also note that $W^{1,2}(\Omega, \omega)$ and $W_0^{1,2}(\Omega, \omega)$, are Hilbert spaces.

Proposition 2.1. *For abounded domain $\Omega \subset \mathbb{R}^n$, we have the compact embedding*

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^{p+\eta}(\Omega) \quad \textit{for } 0 \leq \eta < p_s^* - p \tag{2.4}$$

provided

$$\omega^{-s} \in L^1(\Omega) \quad \textit{and} \quad s \in \big(\frac{n}{p}, \infty\big) \cap \big[\frac{1}{p-1}, \infty\big), \tag{2.5}$$

where

$$p_s = \frac{ps}{s+1} \quad \textit{and} \quad p_s^* = \frac{np_s}{n - p_s}. \tag{2.6}$$

For more details, we refer [13]. It follows from the weighted Friedrichs inequality [13, p.27] the norm

$$\|u\|_{0,1,p,\Omega} = \Big(\int_\Omega |Du|^p \omega dx \Big)^{1/p}. \tag{2.7}$$

on the space $W_0^{1,p}(\Omega, \omega)$($\Omega$ bounded) is equivalent to the norm $\|u\|_{1,p,\Omega}$ defined by (2.3) provided (2.5) holds. Hereafter, we assume the weight function ω satisfies conditions (2.1) and (2.5). We note in the following remark that the Proposition 2.1 restricts the dimension n given the weight ω and the exponent p.

Remark 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. From (2.6), we note that

$$2_s^* = \frac{2ns}{n(s+1) - 2s}.$$

Let

$$\omega^{-s} \in L^1(\Omega) \quad \text{and} \quad s \in \big(\frac{n}{p}, \infty\big) \cap \big[\frac{1}{p-1}, \infty\big).$$

For $\eta = 2$, from (2.4), we have

$$W_0^{1,2}(\Omega, \omega) \hookrightarrow\hookrightarrow L^4(\Omega) \quad \text{for } 0 \leq 2 < 2_s^* - 2.$$

Then

$$2_s^* - 2 > 2 \Rightarrow \frac{2ns}{n(s+1) - 2s} > 4. \tag{2.8}$$

Now, the inequality (2.8) holds, when $n \leq 3$.

Example 2.3. Let $\Omega = \{x \in \mathbb{R}^n, n \leq 3 : |x| < 1\}$ and $p = 2$. Then $\omega(x) = |x|^\eta$, $0 < \eta < 1$ is an admissible weight function.

For more details on weighted Sobolev spaces, we refer [13, 18, 20, 25]. At each step, a generic constant is denoted by c or β_0 in order to avoid too many suffices.

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be an open connected set. We say that $u \in W_0^{1,2}(\Omega,\omega)$ is a weak solution of (1.2) if

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u(x) D_j \phi(x) dx + \int_\Omega \sum_{i=1}^n g(x)h(u(x))D_i u\phi(x)dx = \int_\Omega f(x)\phi(x)dx$$

for every $\phi \in W_0^{1,2}(\Omega,\omega)$.

Definition 2.5 (Pseudomonotone operators)**.** Let $A : X \to X^*$ be an operator on the real reflexive Banach space X. The operator A is called pseudomonotone if $u_n \rightharpoonup u$ as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \le 0$$

implies

$$\langle Au, u - w\rangle \le \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle \quad \text{for all } w \in X.$$

We consider the operator equation

$$Au = b, \quad u \in X. \tag{2.9}$$

In section 3, we use the following result.

Proposition 2.6 (Brézis(1968))**.** *Assume that the operator $A : X \to X^*$ is pseudomonotone, bounded and coercive on the real,separable reflexive Banach space X. Then, for each $b \in X^*$, the equation* (2.9) *has a solution.*

For a proof of the above Theorem, we refer the reader to [26, Theorem 27.A].

3. Bounded domain

Let G be a bounded domain in $\mathbb{R}^n$ with $1 \le n \le 3$. We consider the degenerate semilinear elliptic BVP

$$\begin{gathered} Lu(x) + \sum_{i=1}^n g(x)h(u(x))D_i u(x) = f(x) \quad \text{in } G, \\ u(x) = 0 \quad \text{on } \partial G. \end{gathered} \tag{3.1}$$

We need the following hypotheses for further study.

- (H1) Assume $g/\sqrt{\omega} \in L^\infty(G)$ and $f \in L^2(G)$.
- (H2) Let $h : \mathbb{R} \to \mathbb{R}$ is a bounded ($|h(t)| \le \mu, \forall t \in \mathbb{R}$, $\mu > 0$), and Lipschitz continuous with Lipschitz constant $A > 0$ (e.g., $h(t) = \sin(t)$, $\forall t \in \mathbb{R}$).

We define the functionals $B_1, B_2 : W_0^{1,2}(G,\omega) \times W_0^{1,2}(G,\omega) \to \mathbb{R}$ by

$$B_1(u,\phi) = \int_G \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j \phi(x) dx$$

$$B_2(u,\phi) = r(u,u,\phi), \quad r(u,v,\phi) := \int_G \sum_{i=1}^n g(x)h(u(x))D_i v(x)\phi(x)dx.$$

Also, define the functional $T : W_0^{1,2}(G,\omega) \to \mathbb{R}$ by

$$T(\phi) = \int_G f(x)\phi(x)dx.$$

A function $u \in W_0^{1,2}(G,\omega)$ is a weak solution of (3.1) if
$$B_1(u,\phi) + B_2(u,\phi) = T(\phi), \quad \text{for all } \phi \in W_0^{1,2}(G,\omega). \tag{3.2}$$

Theorem 3.1. *Assume* (H1) *and* (H2). *In addition, let the condition*
$$\mu C_G \|g/\sqrt{\omega}\|_{\infty,G} < \lambda,$$
where C_G is a constant (depending on G) arising out of weighted Fredrichs inequality. Then the BVP (3.1) *has a weak solution.*

Proof. First we write the BVP (3.1) as operator equation
$$u \in W_0^{1,2}(\Omega,\omega) : Bu + Nu = T \quad \text{in } [W_0^{1,2}(\Omega,\omega)]^*, \tag{3.3}$$
where $T \in [W_0^{1,2}(\Omega,\omega)]^*$, $B : W_0^{1,2}(\Omega,\omega) \to [W_0^{1,2}(\Omega,\omega)]^*$ is linear, uniformly monotone and continuous, $N : W_0^{1,2}(\Omega,\omega) \to [W_0^{1,2}(\Omega,\omega)]^*$ is strongly continuous and $B+N$ is coercive. Further we put Propositions 2.6 to this operator equation. The realization of this idea is split into 5 steps for convenience.

Step 1: Since $|a_{ij}(x)| \le c\omega(x)$, we have by Hölder's inequality
$$\begin{aligned}
B_1(u,v) &= \int_G \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j v(x) dx \\
&\le c \int_G \sum_{i,j=1}^n |D_i u(x)||D_j v(x)| \omega dx \\
&\le c\|u\|_{0,1,2,G} \|v\|_{0,1,2,G}, \quad \text{for all } u,v \in W_0^{1,2}(G,\omega).
\end{aligned}$$
We define the operator $B : W_0^{1,2}(G,\omega) \to [W_0^{1,2}(G,\omega)]^*$ as
$$(Bu|\phi) = B_1(u,\phi), \quad \text{for } u,\phi \in W_0^{1,2}(G,\omega).$$
Hence, the operator B is well defined, linear, and continuous. It follows from (1.1) that
$$\begin{aligned}
\big(Bu - Bv | u - v\big) &= B_1(u-v, u-v) \\
&= \int_G \sum_{i,j=1}^n a_{ij} D_i(u-v) D_j(u-v) dx \\
&\ge \lambda \int_G |D(u-v)|^2 w dx \\
&= \lambda \|u-v\|_{0,1,2,G}^2 \text{ for all } u,v \in X.
\end{aligned}$$
Consequently, B is uniformly monotone(and hence coercive). For more details on monotone operators, we refer[27].

Step 2: By (H1) and (H2), it follows from Hölder's inequality,
$$\begin{aligned}
&\Big|\int_G g(x)\ h(u(x)) D_i u(x) v(x) dx\Big| \\
&\le \int_G |g/\sqrt{\omega}||h(u(x))||D_i u(x)\sqrt{\omega}||v(x)| dx \\
&\le \mu\, \|g/\sqrt{\omega}\|_{\infty,G} \int_G |D_i u\, \omega^{1/2}||v| dx \\
&\le \mu\, \|g/\sqrt{\omega}\|_{\infty,G} \Big(\int_G |D_i u|^2 w dx\Big)^{1/2} \Big(\int_G |v|^2 dx\Big)^{1/2}
\end{aligned}$$

and hence, by the weighted Friedrichs inequality [13, p.27],

$$|B_2(u,v)| \le C_G\|u\|_{0,1,2,G}\|v\|_{0,1,2,G} \quad \text{for all } u,v \in W_0^{1,2}(G,\omega).$$

where $C_G > 0$ is a constant(depending on domain G). Since $B_2(u,.)$ is linear and bounded, there exists an operator $N : W_0^{1,2}(G,\omega) \to [W_0^{1,2}(G,\omega)]^*$ such that

$$(Nu|v) = B_2(u,v) \quad \text{for all } u,v \in W_0^{1,2}(G,\omega).$$

Then, problem (3.1) is equivalent to operator equation

$$Bu + Nu = T,\ u \in W_0^{1,2}(G,\omega).$$

Step 3: (I) By (2.4), the embedding $W_0^{1,2}(G,\omega) \hookrightarrow\hookrightarrow L^4(G)$ is compact. (II) Let $u_n \rightharpoonup u$ in $W_0^{1,2}(G,\omega)$ as $n \to \infty$. Then, the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(G,\omega)$. By (I), $u_n \to u$ in $L^4(G)$ as $n \to \infty$. We claim that

$$Nu_n \to Nu \quad \text{in } [W_0^{1,2}(G,\omega)]^* \quad \text{as } n \to \infty.$$

or

$$\|Nu_n - Nu\|_{[W_0^{1,2}(G,\omega)]^*} = \sup_{\|v\|_{0,1,2,G}\le 1} |(Nu_n - Nu|v)| \to 0 \quad \text{as } n \to \infty.$$

Otherwise, there exists an $\epsilon_0 > 0$ and a sequence $\{v_n'\}$, which we denote briefly by $\{v_n\}$, such that $\|v_n\|_{0,1,2,G} \le 1$ for all n, with

$$(Nu_n - Nu|v_n) \ge \epsilon_0 \quad \text{for all } n.$$

Passing to a subsequence, if necessary, we assume that $v_n \rightharpoonup v$ in $W_0^{1,2}(G,\omega)$ and it follows that $v_n \to v$ in $L^4(G)$ as $n \to \infty$. We note that

$$\begin{aligned} &h(u_n)(D_iu_n)v_n - h(u)(D_iu)v_n \\ &= (h(u_n) - h(u))(D_iu_n)v_n + h(u)(D_iu_n)(v_n - v) \\ &\quad + h(u)(D_iu_n - D_iu)v + h(u)(D_iu)(v - v_n). \end{aligned} \tag{3.4}$$

Since h is Lipschitz, we have

$$|h(u_n) - h(u)| \le A|u_n - u|,$$

by (I) and by the generalized Hölder's inequality, we obtain

$$\begin{aligned} &\Big|\int_G g(x)(h(u_n) - h(u))(D_iu_n)v_n dx\Big| \\ &\le \|g/\sqrt{\omega}\|_{\infty,G} \int_G |h(u_n) - h(u)||D_iu_n\omega^{1/2}||v_n|dx \\ &\le A\|g/\sqrt{\omega}\|_{\infty,G}\Big(\int_G |u_n - u|^4 dx\Big)^{1/4}\Big(\int_G |D_iu_n|^2\omega dx\Big)^{1/2}\Big(\int_G |v_n|^4 dx\Big)^{1/4} \\ &\le C_G\|u_n - u\|_{4,G}\|u_n\|_{0,1,2,G}\|v_n\|_{0,1,2,G}, \end{aligned} \tag{3.5}$$

where C_G is a constant (depending on G) arising out of weighted Fredrichs inequality. We have $u_n \to u$ and $v_n \to v$ in $L^4(G)$ as $n \to \infty$; i.e., $\|u_n - u\|_{4,G} \to 0$ and $\|v_n - v\|_{4,G} \to 0$. Moreover, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $W_0^{1,2}(G,\omega)$. Again, we have $u_n \rightharpoonup u$ in $W_0^{1,2}(G,\omega)$ and

$$\begin{aligned} |r(u,w,v)| &= \Big|\int_G \sum_{i=1}^n g(x)h(u(x))D_iw(x)v(x)dx\Big| \\ &\le C_G\|w\|_{0,1,2,G}\|v\|_{0,1,2,G} \quad \text{for all } u,v,w \in W_0^{1,2}(G,\omega), \end{aligned}$$

due to the Hölder's inequality, and hence, the linear functional $w \mapsto r(u, w, v)$ is continuous on $W_0^{1,2}(G, \omega)$. Finally, we have

$$r(u, u_n - u, v) \to 0 \quad \text{as } n \to \infty. \tag{3.6}$$

By (3.4),(3.6) and by similar arguments as in(3.5), we have

$$\begin{aligned}
|(Nu_n - Nu|v_n)| &= \sum_{i=1}^{n} \Big| \int_G g(x)\{h(u_n)(D_i u_n)v_n - h(u)(D_i u)v_n\} dx \Big| \\
&\leq \sum_{i=1}^{n} \int_G |g/\sqrt{\omega}||h(u_n)(D_i u_n)v_n - h(u)(D_i u)v_n|\omega^{\frac{1}{2}} dx \\
&\leq \|g/\sqrt{\omega}\|_{\infty,G} C_G\{A\|u_n - u\|_4 \|u_n\|_{0,1,2,G} \|v_n\|_{0,1,2,G} \\
&\quad + \mu\|v_n - v\|_{4,G}\|u_n\|_{0,1,2,G} + \mu\|v_n - v\|_{4,G}\|u\|_{0,1,2,G}\} \\
&\quad + |r(u, u_n - u, v)| \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.7}$$

Relation (3.7) contradicts (3) and hence, N is strongly continuous.

Step 4: For all $u \in W_0^{1,2}(G, \omega)$,

$$\begin{aligned}
|B_2(u, u)| &\leq \Big| \int_G \sum_{i=1}^{n} g \; h(u)(D_i u) u dx \Big| \\
&\leq \mu\|g/\sqrt{\omega}\|_{\infty,G} \int_G \sum_{i=1}^{n} |D_i u \, \omega^{1/2}||u| dx \\
&\leq \mu\|g/\sqrt{\omega}\|_{\infty,G} \sum_{i=1}^{n} \Big(\int_G |D_i u|^2 \omega dx \Big)^{1/2} \Big(\int_G |u|^2 dx \Big)^{1/2} \\
&\leq \mu C_G \|g/\sqrt{\omega}\|_{\infty,G} \|u\|_{0,1,2,G}^2,
\end{aligned}$$

where C_G is a constant(depending on G) arising out of weighted Fredrichs inequality. By (1.1), there exists a constant $\lambda > 0$ such that

$$B_1(u, u) \geq \lambda\|u\|_{0,1,2,G}^2 \quad \text{for all } u \in W_0^{1,2}(G, \omega).$$

This implies

$$\begin{aligned}
\big(Bu + Nu|u\big) &= B_1(u, u) + B_2(u, u) \\
&\geq (\lambda - \mu C_G \|g/\sqrt{\omega}\|_\infty)\|u\|_{0,1,2,G}^2 \quad \text{for all } u \in W_0^{1,2}(G, \omega);
\end{aligned}$$

i.e., $B + N$ is coercive if $\mu C_G \|g/\sqrt{\omega}\|_\infty < \lambda$.

Step 5: Since B is uniformly monotone and continuous, N is strongly continuous and $B + N$ is coercive, by [27, Proposition 26.16, p.576], we note that the operator $B + N$ is pseudomonotone. Also, we have $B + N$ is continuous, and bounded. Now, for $\mu C_G \|g/\sqrt{\omega}\|_\infty < \lambda$, by Proposition 2.6, problem (3.1) has a weak solution in $W_0^{1,2}(G, \omega)$. □

4. Unbounded domain

Let Ω be a domain (not necessarily bounded) in $\mathbb{R}^n$ with $1 \le n \le 3$. We consider the degenerate semilinear elliptic BVP

$$\begin{gathered} Lu(x) + \sum_{i=1}^{n} g(x)h(u(x))D_i u(x) = f(x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

(H1') Assume $g/\sqrt{\omega} \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$.

Lemma 4.1. *Assume* (H1') *and* (H2). *If* $\mu C_{\Omega_l} \|g/\sqrt{\omega}\|_{\infty,\Omega} < \lambda$, *then the BVP*

$$\begin{gathered} Lu + \sum_{i=1}^{n} g\ h(u)D_i u = f \quad \text{in } \Omega_l, \\ u = 0 \quad \text{on } \partial\Omega_l \end{gathered} \tag{4.2}$$

has a weak solution $u = u_l \in W_0^{1,2}(\Omega_l, \omega)$ *for* $l = 1, 2, 3, \dots$. *In addition, for* $k \ge l$, $\|u_k\|_{0,1,2,\Omega_l} \le \beta_0$, *where* β_0 *is independent of* k.

Proof. We use arguments similar to those in Theorem 3.1, Let $u_k \in W_0^{1,2}(\Omega_k, \omega)$ be the solutions of (4.2) in each bounded subdomains Ω_k. Also B_1, B_2 and T are defined in a similar way as in section-3. Then, from the hypotheses and relation (2.4), we note that for $k \ge l$,

$$\begin{gathered} |B_1(u_k, u_k)| \le c\|u_k\|^2_{0,1,2,\Omega_l} \\ |B_2(u_k, u_k)| \le \mu C_{\Omega_l} \|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega_l} \|u_k\|^2_{0,1,2,\Omega_l} \\ |T(u_k)| \le C_{\Omega_l} \|f\|_{2,\Omega_l} \|u_k\|_{0,1,2,\Omega_l}, \end{gathered}$$

where C_{Ω_l} (is the constant depending on the domain Ω_l) independent of k. Also, we have for $k \ge l$

$$B_1(u_k, u_k) \ge \lambda \int_{\Omega_l} |Du_k|^2 \omega dx = \lambda \|u_k\|^2_{0,1,2,\Omega_l}.$$

We obtain

$$\|u_k\|^2_{0,1,2,\Omega_l} \le \frac{1}{\lambda} B_1(u_k, u_k) \tag{4.3}$$

Also, we note that

$$\begin{aligned} (Bu_k + Nu_k | u_k) &= B_1(u_k, u_k) + B_2(u_k, u_k) \\ &\ge (\lambda - \mu C_{\Omega_l} \|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega_l}) \|u_k\|^2_{0,1,2,\Omega_l} \end{aligned}$$

As, $T(u_k) = B_1(u_k, u_k) + B_2(u_k, u_k)$, we have

$$(\lambda - \mu C_{\Omega_l} \|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega_l}) \|u_k\|^2_{0,1,2,\Omega_l} \le C_{\Omega_l} \|f\|_{2,\Omega_l} \|u_k\|_{0,1,2,\Omega_l}. \tag{4.4}$$

Since $\lambda > \mu C_{\Omega_l} \|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega_l}$, By (4.3) and (4.4), we have

$$\begin{aligned} \|u_k\|_{0,1,2,\Omega_l} &\le \frac{C_{\Omega_l} \|f\|_{2,\Omega_l}}{(\lambda - \mu C_{\Omega_l} \|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega_l})} \\ &\le \frac{C_{\Omega_l} \|f\|_{2,\Omega}}{(\lambda - \mu C_{\Omega_l} \|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega})} = \beta_0, \end{aligned}$$

where β_0 is independent of k. Hence,

$$\|u_k\|_{0,1,2,\Omega_l} \leq \beta_0, \quad \text{for all } k \geq l \tag{4.5}$$

$\square$

Theorem 4.2. *Let $\Omega = \cup_{l=1}^{\infty}\Omega_l$, $\Omega_l \subseteq \overline{\Omega}_l \subseteq \Omega_{l+1} \subseteq \overline{\Omega}_{l+1}$ be bounded domains in Ω, for $l \geq 1$ and let the condition $\mu C_{\Omega_l}\|\frac{g}{\sqrt{\omega}}\|_{\infty,\Omega} < \lambda$ be fulfilled. Under the hypotheses* (H1') *and* (H2), (4.1) has a weak solution $u \in W_0^{1,2}(\Omega,\omega)$.

Proof. A part of this proof follows from [19, 23, 24]. Let $\{u_k\}$ be the sequence of solutions of (4.2) in $W_0^{1,2}(\Omega_k,\omega), (k \geq 1)$. Let $\tilde{u}_k$, for $k \geq 1$, denote the extension of u_k by zero outside Ω_k, which we continue to denote it by u_k. From (4.5), we have

$$\|u_k\|_{0,1,2,\Omega_l} \leq \beta_0, \quad \text{for } k \geq l.$$

Then, $\{u_k\}$ has a subsequence $\{u_{k_m^1}\}$ which converges weakly to u^1, as $m \to \infty$, in $W_0^{1,2}(\Omega_1,\omega)$. Since $\{u_{k_m^1}\}$ is bounded in $W_0^{1,2}(\Omega_2,\omega)$, it has a convergent subsequence $\{u_{k_m^2}\}$ converging weakly to u^2 in $W_0^{1,2}(\Omega_2,\omega)$. By induction, we have $\{u_{k_m^{l-1}}\}$ has a subsequence $\{u_{k_m^l}\}$ which weakly converges to u^l in $W_0^{1,2}(\Omega_l,\omega)$; i.e., in short, we have $u_{k_m^l} \rightharpoonup u^l$ in $W_0^{1,2}(\Omega_l,\omega),\ l \geq 1$. Define $u : \Omega \to \mathbb{R}$ by

$$u(x) := u^l(x), \quad \text{for } x \in \Omega_l.$$

(Here, there is no confusion since $u^l(x) = u^m(x)$, $x \in \Omega$, for any $m \geq l$).

Let M be any fixed (but arbitrary) bounded domain such that $M \subseteq \Omega$. Then, there exists an integer l such that $M \subseteq \Omega_l$. We note that, the diagonal sequence $\{u_{k_m^m}; m \geq l\}$ converges weakly to $u = u^l$ in $W_0^{1,2}(M,\omega)$, as $m \to \infty$.

We still need to show that u is the required weak solution. It is sufficient to show that u is a weak solution of (4.1) for an arbitrary bounded domain M in Ω. Since $u_{k_m^m} \rightharpoonup u^l$ in $W_0^{1,2}(M,\omega)$, we have

$$\int_M D(u_{k_m^m} - u).D\phi\,\omega dx \to 0, \quad \text{as } m \to \infty,$$

implies

$$\int_M D_i(u_{k_m^m} - u)D_j\phi\omega dx \to 0, \quad \text{as } m \to \infty.$$

From (1.1), for a constant c, we have $|a_{ij}| \leq c\omega$. We observe that

$$\int_M \sum_{i,j=1}^n a_{ij}D_i(u_{k_m^m} - u)D_j\phi\,dx \leq c\sum_{i,j=1}^n \int_M D_i(u_{k_m^m} - u)D_j\phi w\,dx \to 0, \tag{4.6}$$

as $m \to \infty$. Also, by (2.4), $u_{k_m^m} \to u$ in $L^4(M)$. We have, by the generalized Hölder's inequality

$$
\begin{aligned}
&\Big|\int_M g(h(u_{k_m^m}) - h(u)) D_i(u_{k_m^m} - u)\phi dx\Big| \\
&\le A\int_M |g/\sqrt{\omega}||(u_{k_m^m} - u)||D_i(u_{k_m^m} - u)\sqrt{\omega}||\phi| dx \\
&\le A\|\frac{g}{\sqrt{\omega}}\|_{\infty,M}\int_M |(u_{k_m^m} - u)||D_i(u_{k_m^m} - u)\sqrt{\omega}||\phi| dx \\
&\le A\|\frac{g}{\sqrt{\omega}}\|_{\infty,M}\Big(\int_M |(u_{k_m^m} - u)|^4 dx\Big)^{1/4}\Big(\int_M |D_i(u_{k_m^m} - u)|^2\omega dx\Big)^{1/2} \\
&\quad\times\Big(\int_M |\phi|^4 dx\Big)^{1/4} \\
&\le AC_M\|\frac{g}{\sqrt{\omega}}\|_{\infty,M}\|u_{k_m^m} - u\|_{4,M}\|u_{k_m^m} - u\|_{0,1,2,M}\|\phi\|_{2,M} \to 0,
\end{aligned}
\tag{4.7}
$$

as $m \to \infty$. Since M is an arbitrary bounded domain in Ω, it follows from (4.6) and (4.7),

$$
\begin{aligned}
&\int_\Omega \sum_{i,j=1}^n a_{ij}(x) D_i u(x) D_j\phi(x) dx + \int_\Omega \sum_{i=1}^n g(x)h(u(x)) D_i u(x)\phi(x) dx \\
&= \int_\Omega f(x)\phi(x) dx
\end{aligned}
$$

for every $\phi \in W_0^{1,2}(\Omega,\omega)$, which completes the proof. $\square$

Remark 4.3. The above results still hold if h is a bounded and continuous (not necessarily Lipschitz). We have to slightly modify the argument used in the inequalities (3.5) and (4.7) and the rest of the proof remains same. For a bounded domain G and bounded function h, if $u \in L^2(G)$, we have $h(u) \in L^4(G)$. Define the Nemytskii operator $h_u : L^2(G) \to L^4(G)$ by $h_u(x) = h(u(x))$; we have h_u is continuous [21, Theorem 2.1]. Let $u_n \rightharpoonup u$ in $W_0^{1,2}(G,\omega)$, then

$$
\begin{aligned}
&\Big|\int_G g(x)(h(u_n) - h(u))(D_i u_n) v_n dx\Big| \\
&\le \|g/\sqrt{\omega}\|_{\infty,G}\int_G |h(u_n) - h(u)||D_i u_n \omega^{1/2}||v_n| dx \\
&\le C_G\|g/\sqrt{\omega}\|_{\infty,G}\|h(u_n) - h(u)\|_{4,G}\|u_n\|_{0,1,2,G}\|v_n\|_{0,1,2,G} \to 0, \quad \text{as } m \to \infty.
\end{aligned}
$$

Similar argument can be use to prove the inequality (4.7) in section 4.

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