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EXISTENCE OF MULTIPLE SOLUTIONS FOR THREE-POINT BOUNDARY-VALUE PROBLEMS ON INFINITE INTERVALS IN BANACH SPACES

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Abstract. We prove the existence of at least three solutions for a second-order three-point boundary-value problem on infinite intervals in Banach spaces. We use the unbounded upper and lower solution method, and the topological degree theory of strict-set-contractions. To illustrate our results, we present an example.

1. Introduction

The purpose of this article is to investigate the existence of multiple solutions for the following nonlinear second-order three-point boundary-value problem (BVP) on the unbounded domain $[0,+\infty)$,

$$\begin{gathered} u''(t)+q(t)f(t,u(t),u'(t))=\theta, \quad t\in J_0, \\ u(0)-au'(0)-bu(\eta)=x_0, \quad u'(\infty)=y_\infty \end{gathered} \tag{1.1}$$

in a Banach space E, where θ is the zero element of E, $a\geq 0$, $b\geq 0,\eta>0$, $x_0,y_\infty\in E$, $q:J\to J_0, f:J\times E^2\to E$ are continuous, in which $J_0=(0,+\infty)$, $J=[0,+\infty)$.

Boundary-value problems on the half-line, arising naturally in the study of radially symmetric solutions of nonlinear elliptic equations and various physical phenomena [1], have been studied extensively in the literature and there are many excellent results about the existence of solutions for some boundary value problems of differential equations on infinite intervals (see, for instance, [2, 3, 5, 9, 10, 11, 12, 13, 14, 15, 16] and references therein).

In scalar space, the existence of solutions as well as the positive ones for second order boundary value problems on infinite intervals has been studied in a number of papers, see [2, 3, 5, 9, 12, 13, 14] and references therein. Bosiud [2] applied a diagonalization procedure to obtained the existence of bounded solutions for the

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following BVP on the half-line

$$\begin{gathered} u''(t)+q(t)f(t,u(t))=0, \quad t\in(\eta,+\infty),\\ u(\eta)=0, \quad \lim_{t\to+\infty}u'(t)=0. \end{gathered} \tag{1.2}$$

Chen and Zhang [3] established sufficient and necessary conditions for the existence of positive solutions for (1.2). Eloe, Kaufmann and Tisdell [5] studied the following BVP for the ordinary differential equation

$$\begin{gathered} u''(t)-q(t)u(t)+f(t,u(t))=0, \quad t\in(0,+\infty),\\ u(0)=x_0\geq 0, \quad x(t) \text{ bounded on } [0,+\infty). \end{gathered}$$

By employing the technique of lower and upper solutions and the theory of fixed point index, the authors obtained the existence of at least three solutions on sequential arguments.

When the nonlinear term f involves u', Yan, Agarwal and O'Regan [14] established a upper and lower solution theory for the boundary value problem

$$\begin{gathered} u''(t)+q(t)f(t,u(t), \quad u'(t))=0,\ t\in J_0,\\ au(0)-bu'(0)=x_0\geq 0, \quad \lim_{t\to+\infty}u'(t)=y_\infty\geq 0, \end{gathered} \tag{1.3}$$

where $a>0$, $b>0$. By using the upper and lower solutions method, the author presented sufficient conditions for the existence of at least one unbounded positive solution and at least two positive solutions for (1.3). In [9], with $x_0, y_\infty\in(-\infty,+\infty)$, Lian, Wang and Ge established a unbounded upper and lower solution theory for (1.3). By using the Schäuder fixed point theorem, they obtained sufficient conditions for the existence of solutions and of positive solutions.

For abstract spaces, Liu [10] investigated the existence of solutions of the following second-order two-point boundary-value problems on infinite intervals in a Banach space E,

$$\begin{gathered} u''(t)+f(t,u(t),u'(t))=\theta, \quad t\in J,\\ u(0)=x_0, \quad u'(\infty)=y_\infty. \end{gathered} \tag{1.4}$$

By employing the Sadovskii fixed point theorem, the author established sufficient conditions for the existence of at least one solution. Recently, Zhang [15] investigated the positive solutions of (1.4) with boundary conditions

$$u(0)=\sum_{i=1}^{m-2}a_iu(\eta_i), \quad u'(\infty)=y_\infty,$$

on an infinite interval in Banach spaces, where $a_i\in[0,+\infty)$, $\eta_i\in(0,+\infty)$. The main tool used is Mönch fixed point and monotone iterative technique. Zhao and Chen [16] studied the multiplicity of positive solutions for a class of nonlinear multi-point boundary value problem of second-order differentials equations in Banach spaces.

Inspired by [14, 15, 16], in the present article, we will show the existence of at least three solutions to (1.1) in a Banach space E. Note that the nonlinear term f depends on u and its derivative u'. We will use the topological degree theory of strict-set-contractions and the unbounded upper and lower solution method rather than the fixed-point theorem of strict-set-contraction used in [7, 10, 11, 16, 17, 18] to establish multiplicity results for (1.1).

2. Preliminaries

Basic facts about an ordered Banach space E can be found in [4, 6, 8]. Here, we just recall a few of them. Let the real Banach space E with norm $\|\cdot\|$ be partially ordered by a cone P of E; i.e., $u \le v$ if and only if $v - u \in P$. Let E^* be the dual space of E, and P^* denote the dual cone of P; i.e., $P^* = \{\varphi | \varphi \in E^*, \varphi(u) \ge 0, u \in P\}$. For $\varphi \in P^*$, let $C_\varphi = \{u \in E : \varphi(u) \ge 0\}$. If $S \subset P^*$ satisfying $P = \cap\{C_\varphi : \varphi \in S\}$, then P may be generated by S. the closure of S in the weak*-topology of E^* is denoted by $\bar{S}^*$.

Definition 2.1. $f(t,u,u')$ is quasi-monotone nondecreasing, if $u \le v$ and for any $\varphi \in S$, such that $\varphi(u) = \varphi(v), \varphi(u') = \varphi(v')$ imply $\varphi(f(t,u,u')) \le \varphi(f(t,v,v'))$.

Definition 2.2. A function $\alpha(t) \in C^1[J,E] \cap C^2[J_0,E]$ is called a lower solution of (1.1) if

$$\begin{gathered}\alpha''(t) + q(t)f(t,\alpha(t), \quad \alpha'(t)) \ge \theta,\ t \in J_0,\\ \alpha(0) - a\alpha'(0) - b\alpha(\eta) \le x_0, \quad \alpha'(\infty) \le y_\infty.\end{gathered}$$

Similarly, a function $\beta(t) \in C^1[J,E] \cap C^2[J_0,E]$ is called an upper solution of (1.1) if

$$\begin{gathered}\beta''(t) + h(t)f(t,\beta(t), \quad \beta'(t)) \le \theta,\ t \in J_0,\\ \beta(0) - a\beta'(0) - b\beta(\eta) \ge x_0, \quad \beta'(\infty) \ge y_\infty.\end{gathered}$$

Consider the space

$$X = \{u \in C^1[J,E] : \sup_{t\in J} \frac{\|u(t)\|}{1+t} < +\infty \text{ and } \sup_{t\in J} \|u'(t)\| < +\infty\}. \tag{2.1}$$

with the norm $\|u\|_X = \max\{\|u\|_1, \|u'\|_\infty\}$, where $\|u\|_1 = \sup_{t\in J} \frac{\|u(t)\|}{1+t}, \|u'\|_\infty = \sup_{t\in J} \|u'(t)\|$. By the standard arguments, it is easy to prove that $(X, \|\cdot\|_X)$ is a Banach space. A function $u \in X$ is called a solution of the boundary value problem (1.1) if it satisfies (1.1).

Definition 2.3 (Kuratovski Noncompactness measure)**.** Let V be a bounded set in a real Banach space E, and $\alpha(V) = \inf\{\delta > 0 : V = \cup_{i=1}^m V_i$, all the diameters of $V_i \le \delta\}$. Clearly, $0 \le \alpha(V) < \infty$. $\alpha(V)$ is called the Kuratovski measure of noncompactness; see [4, 6, 8].

Let $T : D \to E(D \subset E)$ is a continuous and bounded operator. If there exists a constant $k \ge 0$, such that $\alpha(T(D)) \le k\alpha(D)$, then T is called a k-set contraction operator, when $k < 1$, T is called a strict-set contraction operator. The Kuratowski measure of noncompactness of bounded set in $E, C[J,E]$ and X are denoted by $\alpha_E(\cdot), \alpha_C(\cdot)$ and $\alpha_X(\cdot)$, respectively.

Lemma 2.4 ([4, 6])**.** *Let $D \subset E$ be an open, bounded and convex set, T is a strict-set-contraction from $\bar{D}$ into E, and $T(\bar{D}) \subset D$. Then $\deg(T,D,E) = 1$.*

Lemma 2.5 ([4, 6])**.** *Let $S \subset \{\varphi \in E^* : \|\varphi\| \le 1\}, u \in E$, and $d = \inf\{\varphi(u) : \varphi \in S\}$. Then there exists $\varphi_0 \in \bar{S}^*$ such that $\varphi_0(u) = d$.*

Lemma 2.6 ([13])**.** *Let E be a Banach space and $H \subset C[J,E]$. If H is countable and there exist a $\rho \in L[J,J_0]$ such that $\|u(t)\| \le \rho(t) a.e. t \in J$ for all $u \in H$. Then*

$\alpha_E(u(t))$ is integrable on J, and

$$\alpha_E(\{\int_0^{+\infty} u(t)dt : u \in H\}) \leq 2\int_0^{+\infty} \alpha_E(H(t))dt,$$

where $H(t) = \{u(t) : u \in H, t \in J\}$.

3. Main results

To abbreviate our presentation, we define the following assumptions:

(C0) $f \in C[J \times E \times E, E]$, and f is quasi-monotone nondecreasing with respect to P;

(C1) $\alpha, \beta \in X$ is a pair lower and upper solutions for (1.1) satisfying $\alpha(t) \leq \beta(t)$ on $t \in J$;

(C2) There exist a continuous function $h : J_0 \to E$ and $A, B \in E$, for any $\varphi \in S$, $t \in J_0, \alpha(t) \leq u \leq \beta(t)$ and $v \in E$, such that

$$|\varphi(f(t,u,v))| \leq |\varphi(A)\varphi(v)| \cdot \varphi(h|\varphi(v)|) + |\varphi(B)|; \tag{3.1}$$

(C3) $q \in L^1(0,+\infty)$ satisfying $\int_0^{+\infty} q(s)ds < +\infty$, and there exists $k > 1$ such that

$$M = \sup_{0 \leq t < +\infty} (1+t)^k q(s) < +\infty; \tag{3.2}$$

(C4) For any $\varphi \in S$, there exists $N, \xi \in P$ ($\varphi(N) > \varphi(\xi)$) and $\tau > \varphi(y_\infty)$, such that

$$\int_{\varphi(\xi)}^{\varphi(N)} \frac{ds}{\varphi(h(s))} > \Gamma \tag{3.3}$$

where

$$\varphi(\xi) = \max\big\{\sup_{\tau \leq t < +\infty} |\frac{\varphi(\beta(t)) - \varphi(\alpha(0))}{t}|, \sup_{\tau \leq t < +\infty} |\frac{\varphi(\beta(0)) - \varphi(\alpha(t))}{t}|\big\},$$

$$\begin{aligned} \Gamma := {}& M|\varphi(A)|\Big(\sup_{t\in J} \frac{\varphi(\beta(t))}{(1+t)^k} - \inf_{t \in J} \frac{\varphi(\alpha(t))}{(1+t)^k} + \frac{k}{k-1} \sup_{t\in J} \frac{\varphi(\beta(t))}{1+t}\Big) \\ &+ \frac{M|\varphi(B)|}{k-1} \sup_{s \geq \varphi(\xi)} \frac{2}{h(s)}, \end{aligned}$$

where k is given in (C3).

(C5) There exists $l_0, l_1 \in L^1[0,+\infty)$ such that

$$\alpha(f(t,D_0,D_1)) \leq l_0(t)\alpha_E(D_0) + l_1(t)\alpha_E(D_2), \quad \forall t \in J \tag{3.4}$$

for all bounded subsets $D_0, D_1 \subset E$, and

$$\delta \sup_{s\in J} q(s) \cdot \int_0^{+\infty} [l_0(s)(1+s) + l_1(s)]ds < \frac{1}{2},$$

where δ will be given in (3.6).

Lemma 3.1. *If $1 - b \neq 0$, then for any $z \in L^1[J,E]$, the problem*

$$\begin{gathered} u''(t) + q(t)z(t) = \theta, \quad t \in J_0, \\ u(0) - au'(0) - bu(\eta) = x_0, \quad u'(\infty) = y_\infty \end{gathered} \tag{3.5}$$

has a unique solution in X. Moreover this solution can be expressed as

$$u(t) = \frac{x_0 + (a + b\eta)y_\infty}{1-b} + ty_\infty + \int_0^{+\infty} G(t,s)q(s)z(s)ds,$$

where

$$G(t,s)=\frac{1}{1-b}\begin{cases} a+s, & 0\le s\le \min\{\eta,t\},\\ a+t+b(s-t), & t\le s\le \eta,\\ a+s+b(\eta-s), & \eta\le s\le t,\\ a+t+b(\eta-t), & \max\{\eta,t\}\le s<+\infty. \end{cases}$$

The proof of the above lemma is standard; so we omit it.

If $1-b>0$, by directly computation, we have

$$0\le \frac{G(t,s)}{1+t}\le \delta,$$

where

$$\delta:=\frac{1}{1-b}\max\{1+a,a+\eta,1+a+b\eta-b\}. \tag{3.6}$$

Obviously, $\delta\ge 1$.

Lemma 3.2. *Assume that $1-b>0$, and* (C1)–(C4) *hold. If $u\in C^2[J,E]$ is a solution of* (1.1) *and $\alpha(t)\le u(t)\le\beta(t), t\in J$. Then for any $\varphi\in S, t\in J$,*

$$|\varphi(u'(t))|\le\varphi(N),\quad t\in J. \tag{3.7}$$

Proof. Choose $N_0\in P$ such that

$$\varphi(N_0)\ge\max\{\varphi(N),2\varphi(\xi),\sup_{t\in J}|\varphi(\alpha'(t))|,\sup_{t\in J}|\varphi(\beta'(t))|\} \tag{3.8}$$

Assume that $|\varphi(u')|>\varphi(N)$ for some $t\in J$. Without loss of generality, we may assume that $\varphi(u')>\varphi(N)$. Then there exists $t^*\in(0,+\infty)$ such that for any $t^*\ge\tau>0$,

$$\varphi(u'(t^*))=\frac{\varphi(u(t^*))-\varphi(u(0))}{t^*-0}\le\frac{\varphi(\beta(t^*))-\varphi(\alpha(0))}{t^*}\le\varphi(\xi)<\varphi(N_0).$$

Since $u\in C^1[J,E]\cap C^2[J_0,E]$ and for any $\varphi\in S$, there exists $[t_1,t_2]\subset(0,+\infty)$ (or $[t_2,t_1]\subset(0,+\infty)$) such that

$$\varphi(u'(t_1))=\varphi(\xi),\quad \varphi(u'(t_2))=\varphi(N_0),\quad \varphi(\xi)<\varphi(u'(t))<\varphi(N_0),\quad t\in(t_1,t_2).$$

It follows from assumption (C1), (C2) that

$$\varphi(u''(t))\le q(t)|\varphi(A)||\varphi(u')|\cdot\varphi(h(|\varphi(u')|))+|\varphi(B)|q(t),\qquad \text{for } t\in(t_1,t_2).$$

This implies

$$\begin{aligned}
&\Big|\int_{t_1}^{t_2}\frac{\varphi(u''(t))dt}{\varphi(h(|\varphi(u')|))}\Big|\\
&\le|\varphi(A)|\Big|\int_{t_1}^{t_2}q(t)\varphi(u'(t))dt\Big|+|\varphi(B)|\Big|\int_{t_1}^{t_2}\frac{q(t)dt}{\varphi(h(|\varphi(u')|))}\Big|\\
&\le M|\varphi(A)|\Big|\int_{t_1}^{t_2}\frac{\varphi(u'(t))dt}{(1+t)^k}\Big|+M|\varphi(B)|\Big|\int_{t_1}^{t_2}\frac{dt}{(1+t)^k\varphi(h(|\varphi(u')|))}\Big|\\
&\le M|\varphi(A)|\Big|\int_{t_1}^{t_2}\varphi\big(\big(\frac{u(t)}{(1+t)^k}\big)'\big)dt\Big|+Mk|\varphi(A)|\Big|\int_{t_1}^{t_2}\varphi\big(\frac{u(t)}{(1+t)^{1+k}}\big)dt\Big|\\
&\quad+M|\varphi(B)|\sup_{s\ge\varphi(\xi)}\frac{1}{\varphi(h(s))}\Big|\int_{t_1}^{t_2}\frac{1}{(1+t)^k}dt\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le M|\varphi(A)|\Big(\sup_{t\in J}\frac{\varphi(\beta(t))}{(1+t)^k}-\inf_{t\in J}\frac{\varphi(\alpha(t))}{(1+t)^k}+\frac{k}{k-1}\sup_{t\in J}\frac{\varphi(\beta(t))}{1+t}\Big)\\
&\quad+\frac{M|\varphi(B)|}{k-1}\sup_{s\ge\varphi(\xi)}\frac{2}{\varphi(h(s))}.
\end{aligned}
$$

On the other hand, from (C4), we have

$$
\begin{aligned}
\Big|\int_{t_1}^{t_2}\frac{\varphi(u''(t))dt}{\varphi(h(|\varphi(u')|))}\Big| &= \Big|\int_{\varphi(\xi)}^{\varphi(N_0)}\frac{ds}{\varphi(h(s))}\Big|\\
&> M|\varphi(A)|\Big(\sup_{t\in J}\frac{\varphi(\beta(t))}{(1+t)^k}-\inf_{t\in J}\frac{\varphi(\alpha(t))}{(1+t)^k}+\frac{k}{k-1}\sup_{t\in J}\frac{\varphi(\beta(t))}{1+t}\Big)\\
&\quad+\frac{M|\varphi(B)|}{k-1}\sup_{s\ge\varphi(\xi)}\frac{2}{\varphi(h(s))}.
\end{aligned}
$$

which is a contradiction. The proof is complete. □

Theorem 3.3. *Suppose that $1-b>0$, and* (C0)–(C5) *hold. Then then* (1.1) *has at least one solution $u\in C^1[J,E]\cap C^2[J_0,E]$ such that*

$$
\alpha(t)\le u(t)\le\beta(t),\quad t\in J. \tag{3.9}
$$

Moreover, there exists a $L\in P$ such that for any $\varphi\in S$,

$$
|\varphi(u'(t))|\le\varphi(L),\quad t\in J. \tag{3.10}
$$

Proof. Take L such that $L\ge N_0$, where N_0 is given by (3.8). By Lemma 3.2, for any $\varphi\in S$, $t\in J$, (3.10) is true. Define a modified function $F^*: J\times E\times E\to E$ as follows. For any $(t,u,u')\in J\times E\times E$, set

$$
F^*(t,u,u')=f(t,u,\bar u'), \tag{3.11}
$$

where $\bar u'$ is given by

$$
\varphi(\bar u')=\begin{cases}\varphi(L) & \text{if } \varphi(u')>\varphi(L),\\ \varphi(u'), & \text{if } -\varphi(L)\le\varphi(u')\le\varphi(L),\\ \varphi(-L), & \text{if } \varphi(u')<\varphi(-L),\end{cases} \tag{3.12}
$$

for any $\varphi\in S$. Since P may be generated by S, it is easy to see that an element $\bar u'$ of E can be given uniquely by (3.12).

According to $F^*(t,u,\bar u')$, we next define a map $F(t,u,u')$ as follows: for any $(t,u,u')\in J\times E\times E$ and for any $\varphi\in S$, $F(t,u(t),u'(t))$ satisfying

$$
\varphi(F(t,u,u'))=\begin{cases}\varphi(F^*(t,\bar u,u'))+\frac{\lambda\varphi(\beta(t)-u)}{1+(\varphi(u))^2}, & \text{if } \varphi(u)>\varphi(\beta(t)),\\ \varphi(F^*(t,u,u')), & \text{if } \varphi(\alpha(t))\le\varphi(u)\le\varphi(\beta(t)),\\ \varphi(F^*(t,\bar u,u'))-\frac{\lambda\varphi(u-\alpha(t))}{1+(\varphi(u))^2}, & \text{if } \varphi(u)<\varphi(\alpha(t)),\end{cases}
$$

where $\lambda>0$ satisfies

$$
\delta\sup_{s\in J}q(s)\Big\{\int_0^{+\infty}[l_0(s)(1+s)+l_1(s)]ds+\lambda\Big\}<\frac{1}{2}. \tag{3.13}
$$

and $\bar u$ is defined by

$$
\varphi(\bar u)=\begin{cases}\varphi(\beta(t)) & \text{if } \varphi(u)>\varphi(\beta(t)),\\ \varphi(u), & \text{if } \varphi(\alpha(t))\le\varphi(u)\le\varphi(\beta(t)),\\ \varphi(\alpha(t)), & \text{if } \varphi(u)<\varphi(\alpha(t)).\end{cases}
$$

Obviously, for each fixed $(t,u,u') \in J \times E \times E$, $F(t,u,u')$ is unique. It is clear from the definition that F is continuous and bounded on $J \times E \times E$. Consider the modified boundary value problem

$$\begin{gathered} u''(t) + q(t)F(t,u(t),u'(t)) = \theta, \quad t \in J_0, \\ u(0) - au'(0) - bu(\eta) = x_0, \quad u'(\infty) = y_\infty. \end{gathered} \tag{3.14}$$

From the definitions of F, it suffices to show that (3.14) has at least one solution u such that

$$\alpha(t) \le u(t) \le \beta(t), \quad |\varphi(u'(t))| \le \varphi(L), \quad t \in J. \tag{3.15}$$

Since $F = f$ in the region, we divide the proof into two steps.

Step 1. (3.14) has at least one solution u. For $u \in X$(X be defined in (2.1)), we define an operator $T : X \to X$ by

$$(Tu)(t) := \frac{x_0 + (a+b\eta)y_\infty}{1-b} + ty_\infty + \int_0^{+\infty} G(t,s)q(s)F(s,u(s),u'(s))ds. \tag{3.16}$$

From Lemma 3.1 and the definitions of F, it is easy to see that the fixed point of T coincide with the solutions of (3.14). So it is suffices to show that T has at least one fixed point. We claim that $T : X \to X$ is a strict-set-constraction operator.

First, we will show that $T : X \to X$ is well defined. For any $u \in X, \varphi \in S$, we have

$$\begin{aligned} \int_0^{+\infty} q(s)|\varphi(F(s,u(s),u'(s)))|ds &\le \int_0^{+\infty} q(s)[|\varphi(A)||\varphi(v)|\varphi(h|\varphi(v)|) + |\varphi(B)|]ds \\ &\le A^* \int_0^{+\infty} q(s)ds < +\infty, \end{aligned} \tag{3.17}$$

where

$$A^* := |\varphi(A)| \sup_{0 \le s < \infty}\{s\varphi(h(s))\} + 1 + |\varphi(B)|. \tag{3.18}$$

For any $t \in J_0, u \in X$, by Lebesgue dominated convergent theorem and Hahn-Banach theorem, there exists $\phi \in S$ with $\|\phi\| = 1$, such that

$$\begin{aligned} &\|\frac{(Tu)(t)}{1+t}\| \\ &= \|\phi(\frac{(Tu)(t)}{1+t})\| \le \frac{\|\phi(x_0 + (a+b\eta)y_\infty)\|}{(1-b)(1+t)} + \frac{\|\phi(ty_\infty)\|}{1+t} \\ &\quad + \int_0^{+\infty} \frac{G(t,s)}{1+t} q(s)|\phi(F(s,u(s),u'(s)))|ds \\ &\le \|\frac{x_0 + (a+b\eta)y_\infty}{1-b}\| + \|y_\infty\| + \delta \int_0^{+\infty} q(s)|\phi(F(s,u(s),u'(s)))|ds < +\infty. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|(Tu)'(t)\| = \|\phi((Tu)'(t))\| &= \|\phi(y_\infty - \int_t^{+\infty} q(s)F(s,u(s),u'(s))ds)\| \\ &\le \|y_\infty\| + \int_0^{+\infty} q(s)|\phi(F(s,u(s),u'(s)))|ds < +\infty. \end{aligned}$$

Hence, $T : X \to X$ is well defined. Moreover, $Tu \in X$. Let

$$R = \|\frac{x_0 + (a + b\eta)y_\infty}{1-b}\| + \|y_\infty\| + \delta A^* \int_0^{+\infty} q(s)ds,$$
$$\Omega = \{u \in X : \|u\|_X \le R\},$$

where A^* be given in (3.18).

It is easy to see that Ω is a bounded closed convex set in space X. Obviously, Ω is not empty since $(1+t)y_\infty \in \Omega$, and $\Omega \subset X$. It follows from (3.16) and Lemma 3.1 that $u \in \Omega$ implies $Tu \in \Omega$; i.e., $T(\Omega) \subset \Omega$.

Next we show that T is continuous. Let $u_m, \bar{u} \in \Omega$, $\|u_m - \bar{u}\|_X \to 0 (m \to \infty)$. Then $\{u_m\}$ is a bounded subset of Ω, thus there exists $r > 0$ such that $\|u_m\|_X < r$ for $m \ge 1$. Taking limit, we have $\|\bar{u}\|_X \le r$. Similarly, It follows from (3.16) that

$$\begin{aligned}
\|Tu_m - T\bar{u}\|_X &= \max\{\|Tu_m - T\bar{u}\|_1, \|Tu_m - T\bar{u}\|_\infty\} \\
&\le \int_0^{+\infty} \delta q(s)|\varphi(F(s, u_m(s), u_m'(s)) - F(s, \bar{u}(s), \bar{u}'(s)))|ds \\
&\to 0, \quad \text{as } m \to +\infty.
\end{aligned}$$

Hence, the continuity of T is proved. By (3.16),(3.17) and the definite of the set Ω, similar to the proofs of [10, Lemma 2.4], we have

$$\alpha_X(T\Omega) = \max\big\{\sup_{t\in J} \alpha_E\big(\frac{(T\Omega)(t)}{1+t}\big), \sup_{t\in J} \alpha_E((T\Omega)'(t))\big\}. \tag{3.19}$$

Let $V = \{u_m : m = 1, 2, \dots\} \subset \Omega$. Then $\|u_m\|_X \le R$. It follows from (3.16) that

$$(Tu_m)(t) = \frac{x_0 + (a + b\eta)y_\infty}{1-b} + ty_\infty + \int_0^{+\infty} G(t, s)q(s)F(s, u_m(s), u_m'(s))ds, \tag{3.20}$$

and

$$(Tu_m)'(t) = y_\infty - \int_t^{+\infty} q(s)F(s, u_m(s), u_m'(s))ds. \tag{3.21}$$

By (3.19), we obtain

$$\alpha_X(TV) = \max\big\{\sup_{t\in J} \alpha_E\big(\frac{(TV)(t)}{1+t}\big), \sup_{t\in J} \alpha_E((TV)'(t))\big\}, \tag{3.22}$$

where $TV = \{Tu_m : m = 1, 2, \dots\}$ and $(TV)'(t) = \{(Tu_m)' : m = 1, 2, \dots\}$. From (3.17), we can see that the infinite integral $\int_0^{+\infty} h(s)|\varphi(F(s, u_m(s), u_m'(s)))|ds$ is convergent uniformly for $m = 1, 2, \dots$. Hence, for all $\varepsilon > 0$ and $u \in V$, there exists a sufficiently large $T_0 > 0$ such that

$$\int_{T_0}^{+\infty} q(s)|\varphi(F(s, u_m(s), u_m'(s)))|ds < \varepsilon. \tag{3.23}$$

On the other hand, It is easy to prove that $(TV)(t)/(1 + t)$ and $(TV)'(t)$ are equicontinuous on any finite subinterval of J.

By Lemma 2.6, (3.20), (3.21), (3.22) and (C5), for any $t \in J$ and $u \in V$, we obtain

$$\begin{aligned}
\alpha_E\Big(\frac{(TV)(t)}{1+t}\Big) &\le \alpha\Big(\Big\{\int_0^{+\infty} \frac{G(t, s)}{1+t} q(s)F(s, u_m(s), u_m'(s))ds : u_m \in V\Big\}\Big) \\
&\le 2\delta \sup_{s\in J} q(s) \int_0^{+\infty} \alpha(F(s, V(s), V'(s)))ds
\end{aligned}$$

$$\begin{aligned}&\le 2\delta \sup_{s\in J} q(s)\Big\{\int_0^{+\infty}[l_0(s)\alpha(V)+l_1(s)\alpha(V')]ds+\lambda\alpha(V)\Big\}\\&\le 2\delta \sup_{s\in J} q(s)\cdot\alpha_X(V)\cdot\Big\{\int_0^{+\infty}[l_0(s)(1+s)+l_1(s)]ds+\lambda\Big\},\end{aligned}$$

and

$$\begin{aligned}\alpha_E(TV)'(t)&\le 2\sup_{s\in J} q(s)\cdot\int_0^{+\infty}\alpha(F(s,V(s),V'(s)))ds\\&\le 2\sup_{s\in J} q(s)\cdot\alpha_X(V)\cdot\Big\{\int_0^{+\infty}[l_0(s)(1+s)+l_1(s)]ds+\lambda\Big\}.\end{aligned}$$

It follows from(3.14),(3.21) and (C3) that

$$\alpha_X(TV)\le 2\delta\sup_{s\in J} q(s)\cdot\alpha_X(V)\cdot\Big\{\int_0^{+\infty}[l_0(s)(1+s)+l_1(s)]ds+\lambda\Big\}<\alpha_X(V),$$

which implies T is a strict-set-contraction operator. Hence, Lemma 2.4 implies that T has a fixed point $u\in V\subset\Omega$.

Step 2. Suppose that (3.14) has a solution u, then u satisfies (3.15). Moreover, u is a solution of (1.1). For this we need to prove that $\alpha(t)\le u(t)\le\beta(t), t\in J$. Obviously, we only show that $\alpha(t)\le u(t), t\in J$. A similar argument may be used to prove $u(t)\le\beta(t), t\in J$.

If $\alpha(t)\le u(t), t\in J$ does not hold, by Lemma 2.5, there exists $\phi\in\bar{S}^*$ and $t_0\in J$ such that

$$p(t_0)=\inf_{t\in J}\{p(t)=\phi(u(t)-\alpha(t))\}<0. \tag{3.24}$$

Then, there are three cases.

Case 1. If $t_0\in(0,+\infty)$, then we have $p'(t_0)=0, p''(t_0)\ge 0$. Hence,

$$\phi(u(t_0)-\alpha(t_0))<0,\quad \phi(u'(t_0)-\alpha'(t_0))=0,\quad \phi(u''(t_0)-\alpha''(t_0))\ge 0,$$

and consequently,

$$\begin{aligned}&\phi(u''(t_0)-\alpha''(t_0))\\&\le q(t)\phi(f(t_0,\alpha(t_0),\alpha'(t_0))-F(t_0,u(t_0),u'(t_0)))\\&=q(t)\phi\Big(f(t_0,\alpha(t_0),\alpha'(t_0))-f(t_0,\bar u(t_0),\bar u'(t_0))+\frac{\lambda(u(t_0)-\alpha(t_0))}{1+(\phi(u(t_0)))^2}\Big).\end{aligned}$$

Note that

$$\bar u(t_0)\ge\alpha(t_0)),\phi(\bar u(t_0))=\phi(\alpha(t_0)),\text{and}\ \ \phi(\bar u'(t_0))=\phi(\alpha'(t_0)).$$

By (C0) and Definition 2.1, $\phi(f(t_0,\alpha(t_0),\alpha'(t_0))\le\phi(f(t_0,\bar u(t_0),\bar u'(t_0)))$, which implies

$$p''(t_0)=\phi(u''(t_0)-\alpha''(t_0))\le q(t)\cdot\phi\Big(\frac{\lambda(u(t_0)-\alpha(t_0))}{1+(\phi(u(t_0)))^2}\Big)<0,$$

which is a contradiction.

Case 2. If $t_0=0$. Obviously, it holds $p'(0^+)\ge 0$, while by the boundary conditions, we have

$$p(0)=\phi(u(0)-\alpha(0))\ge\phi(u'(0)-\alpha'(0))+b\phi(u(\eta)-\alpha(\eta))\ge bp(\eta). \tag{3.25}$$

If $b = 0$, then we obtain $p(0) \geq 0$, which is a contradiction with $p(0) < 0$. If $0 < b < 1$, since $p(0) < 0, p'(0^+) \geq 0$ holds, then by (3.25), we have

$$p(\eta) < 0,$$

together with (3.25), we obtain

$$p(\eta) \leq \frac{1}{b}p(0) - \frac{a}{b}p'(0) \leq \frac{1}{b}p(0) < p(0). \tag{3.26}$$

Let

$$t^* = \sup\{t : t > 0, \phi(u(s) - \alpha(s)) < 0, \text{for} \quad s \in [0, t]\}.$$

Then $p(s) < 0$ for all $t \in [0, t^*)$. If $t^* < +\infty$, then $p(t^*) = 0$. By the definition of lower solution and the proof of Case 1, we have

$$p''(t) = \phi(u''(t) - \alpha''(t)) < 0, \quad \text{for } t \in (0, +\infty),$$

which implies that $p(t)$ is a concave function. Since $p(0) < 0$ and $p(t^*) = 0$ for $t^* < +\infty$, according to the concavity of $p(t)$ for $t^* < +\infty$, we obtain

$$\frac{p(\eta) - p(0)}{\eta} \geq \frac{p(t^*) - p(0)}{t^*} = \frac{-p(0)}{t^*} > 0.$$

That is,

$$p(\eta) > p(0),$$

which is a contradiction with (3.26).

Case 3. If $t_0 = +\infty$. Obviously, from the boundary conditions, we obtain $p'(+\infty) = \phi(u'(+\infty) - \alpha'(+\infty)) \geq 0$. By (3.24), we have that $p(+\infty) < 0$, so $p'(+\infty) = 0, p''(+\infty) \geq 0$. Similar to the proof of Case1, we have

$$p''(+\infty) = \phi(u''(+\infty) - \alpha''(+\infty)) < 0,$$

which is a contradiction. Thus, we establish $\alpha(t) \leq u(t) \leq \beta(t)$ on J. Therefore it follows that

$$u''(t) + q(t)F(t, u(t), u'(t)) = u''(t) + q(t)f(t, u(t), \bar{u}'(t)) = \theta.$$

By (C1), for any $\varphi \in S$, we obtain that

$$|\varphi(f(t, u, \bar{u}'))| \leq |\varphi(A)||\varphi(\bar{u}')| \cdot \varphi(h|\varphi(\bar{u}')|) + |\varphi(B)|,$$

whenever $\alpha(t) \leq u(t) \leq \beta(t)$ on J. From Lemma 3.2, we have $|\varphi(u'(t))| \leq \varphi(L)$, which implies $F(t, u, u') = f(t, u, u')$. So, u is also a solution of (1.1). □

Theorem 3.4. *Assume that* $1 - b > 0$, *and* (C0)–(C5) *hold. Suppose further that*

- (H1) $\alpha_1 \in C^1[J, E] \cap C^2[J_0, E]$ *is a lower solution and* $\beta_1 \in C^1[J, E] \cap C^2[J_0, E]$ *is an upper solution of* (1.1) *satisfying*

$$\alpha \leq \alpha_1 \leq \beta, \quad \alpha \leq \beta_1 \leq \beta, \quad \alpha_1 \not\leq \beta_1 \quad \text{on } J.$$

- (H2) α_1 *and* β_1 *are not solutions of* (1.1).

Then (1.1) *has at least three solutions* $u_i \in C^1[J, E] \cap C^2[J_0, E] (i = 1, 2, 3)$ *such that*

$$\alpha(t) \leq u_1(t) \leq \beta_1(t), \quad \alpha_1(t) \leq u_2(t) \leq \beta(t), \quad \alpha_1(t) \not\leq u_3(t) \not\leq \beta_1(t), \quad t \in J.$$

Moreover, there exists a $L \in P$ *and for any* $\varphi \in S$ *such that*

$$|\varphi(u_i'(t))| \leq \varphi(L), (i = 1, 2, 3), \quad t \in J.$$

Proof. According to the definition of lower and upper solutions, we can obtain that $\alpha_1, \beta_1 \in X$. Let

$$\Omega_{\alpha_1} = \{u \in \Omega : \|u\|_X > \|\alpha_1\|_X\}, \quad \Omega^{\beta_1} = \{u \in \Omega : \|u\|_X < \|\beta_1\|_X\}.$$

By (H1), $\alpha_1 \not\leq \beta_1$ on J, therefore, $\bar{\Omega}_{\alpha_1} \cap \bar{\Omega}^{\beta_1} = \emptyset$ and the set $\Omega \backslash \{\bar{\Omega}_{\alpha_1} \cup \bar{\Omega}^{\beta_1}\} \neq \emptyset$. From (H2), it can be seen that T has no solution on $\partial\Omega_{\alpha_1} \cap \partial\Omega^{\beta_1}$. The additivity of degree implies that

$$\begin{aligned} \deg(I - T, \Omega, \theta) &= \deg(I - T, \Omega_{\alpha_1}, \theta) + \deg(I - T, \Omega^{\beta_1}, \theta) \\ &\quad + \deg(I - T, \Omega \backslash \{\bar{\Omega}_{\alpha_1} \cup \bar{\Omega}^{\beta_1}\}, \theta). \end{aligned} \tag{3.27}$$

First, we show that $\deg(I - T, \Omega_{\alpha_1}, \theta) = 1$. Define the function $F^{**}(t, u, u')$ as follows. Let

$$f_1^*(t, u, u') = f(t, u, \bar{u}'), \quad \text{for } (t, u, u') \in J \times E \times E,$$

where $\bar{u}'$ is given by

$$\varphi(\bar{u}') = \max\{\varphi(-L), \min\{\varphi(u'), \varphi(L)\}\},$$

and

$$\varphi(F^{**}(t, u, u')) = \begin{cases} \varphi(f_1^*(t, \bar{u}, u')) + \frac{\lambda\varphi(\beta(t) - u)}{1 + (\varphi(u))^2}, & \text{if } \varphi(u) > \varphi(\beta(t)), \\ \varphi(f_1^*(t, u, u')), & \text{if } \varphi(\alpha_1(t)) \leq \varphi(u) \leq \varphi(\beta(t)), \\ \varphi(f_1^*(t, \bar{u}, u')) - \frac{\lambda\varphi(u - \alpha_1(t))}{1 + (\varphi(u))^2}, & \text{if } \varphi(u) < \varphi(\alpha_1(t)), \end{cases}$$

where λ satisfies (3.13) and $\bar{u}$ is given by

$$\varphi(\bar{u}) = \max\{\varphi(\alpha_1(t)), \min\{\varphi(u), \varphi(\beta(t))\}\}.$$

Consider the auxiliary problem

$$\begin{gathered} u''(t) + q(t)F^{**}(t, u(t), u'(t)) = \theta, \quad t \in J_0, \\ u(0) - au'(0) - bu(\eta) = x_0, \quad u'(\infty) = y_\infty, \end{gathered} \tag{3.28}$$

and an operator $T^* : X \to X$ defined by

$$(Tu)(t) := \frac{x_0 + (a + b\eta)y_\infty}{1 - b} + ty_\infty + \int_0^{+\infty} G(t, s)q(s)F^{**}(s, u(s), u'(s))ds. \tag{3.29}$$

As for proof of T, the operator T^* is also well defined and maps X into X and is a strict-set-contract operator.

In a way similar to that for the proof of Theorem 3.3, it is easy to show that any solution u of (3.28) satisfies $u \geq \alpha_1$ on J. It follows from the condition (H2) that $u \neq \alpha_1$ on J. Hence, $u \in \Omega_{\alpha_1}$. Moreover, we can prove $T^*(\bar{\Omega}) \subset \Omega$. Then we have

$$\deg(I - T^*, \Omega, \theta) = 1. \tag{3.30}$$

Since $F^{**} = f$ in the region Ω_{α_1}, thus

$$\begin{aligned} \deg(I - T, \Omega, \theta) &= \deg(I - T^*, \Omega_{\alpha_1}, \theta) \\ &= \deg(I - T^*, \Omega \backslash \bar{\Omega}_{\alpha_1}, \theta) + \deg(I - T^*, \Omega_{\alpha_1}, \theta) = 1. \end{aligned} \tag{3.31}$$

Similar to the proof of (3.31), we have

$$\deg(I - T, \Omega^{\beta_1}, \theta) = 1. \tag{3.32}$$

By (3.27), (3.31) and (3.32), we obtain

$$\deg(I - T, \Omega \backslash \{\bar{\Omega}_{\alpha_1} \cup \bar{\Omega}^{\beta_1}\}, \theta) = -1.$$

So (1.1) has at least three solutions $u_1 \in \Omega_{\alpha_1}, u_2 \in \Omega^{\beta_1}$ and $u_3 \in \Omega\backslash\{\bar{\Omega}_{\alpha_1} \cup \bar{\Omega}^{\beta_1}\}$. Then the proof is complete. $\square$

4. An example

To illustrate our main results, consider the following boundary-value problem, in $[0, +\infty)$,

$$\begin{gathered}
u_n''(t) - \frac{2(1+u_n'(t))(u_n'(t)-\frac{1}{2n})}{36(1+t)^4}\Big(\frac{\arctan u_n(t)}{1+t^3} + \sqrt[3]{u_{2n}'(t)}\Big) \\
- \frac{\pi(u_n'(t)-\frac{1}{2n})}{144(1+t)^4(1+t^2)} = 0, \quad ht \in J, \\
u_n(0) - 3u_n'(0) - \frac{2}{3}u_n(1) = 0, \quad u_n'(\infty) = \frac{1}{2n}, \quad (n = 1, 2, 3, \dots).
\end{gathered} \tag{4.1}$$

We claim tat the above equation has at least three solutions.

Proof. Let

$$E = l^\infty = \{u = (u_1, \dots, u_n, \dots) : \sup_n |u_n| < +\infty\}$$

with norm $\|u\| = \sup_n |u_n|$. Then (4.1) can be regarded as a BVP of form (1.1) in E. In this situation, $u = (u_1, \dots, u_n, \dots)$, $x_0 = (0, \dots, 0, \dots)$, $y_\infty = (\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2n}, \dots)$, $f = (f_1, \dots, f_n, \dots)$, $q = (q_1, \dots, q_n, \dots)$, in which $q_n(t) = \frac{1}{36(1+t)^4}$, and

$$f_n(t, u, v) = 2(1+v_n)(v_n - \frac{1}{2n})\big(\frac{\arctan u_n}{1+t^3} + \sqrt[3]{v_{2n}}\big) + \frac{\pi(v_n - \frac{1}{2n})}{4(1+t^2)}. \tag{4.2}$$

Choose $k = 3$, $\tau = \frac{3}{4}$, and $S \subset \{\varphi \in E^* : \|\varphi\| = 1\}$. Obviously, we have that $f \in C[J \times E \times E, E]$, and $f_n(t, u, v)$ is quasi-monotone nondecreasing in u and v. $q \in C([0, +\infty), (0, +\infty))$, and

$$\int_0^{+\infty} q_n(s)ds = \int_0^{+\infty} \frac{1}{36(1+s)^4} ds = \frac{1}{144} < +\infty,$$

$$M_n = \sup_{0 \le t < +\infty} (1+t)^3 q_n(t) = \sup_{0 \le t < +\infty} \frac{1}{36(1+t)} = \frac{1}{36} < +\infty.$$

Let $\alpha(t) = (\alpha_1^0(t), \alpha_2^0(t), \dots, \alpha_n^0(t), \dots)$, $\alpha_1(t) = (\alpha_1^1(t), \alpha_2^1(t), \dots, \alpha_n^1(t), \dots)$, $\beta_1(t) = (\beta_1^1(t), \beta_2^1(t), \dots, \beta_n^1(t), \dots)$, $\beta(t) = (\beta_1^0(t), \beta_2^0(t), \dots, \beta_n^0(t), \dots)$, where

$$\alpha_n^0(t) = -11 - t, \quad \alpha_n^1(t) = \frac{t}{2n}, \quad \forall t \in J, \ (n = 1, 2, 3, \dots).$$

and

$$\beta_n^1(t) = \begin{cases} -\frac{t}{2n} - 1, & 0 \le t \le 1, \\ \frac{t}{2n} - \frac{1}{n} - 1, & t \ge 1, \end{cases} \quad \beta_n^0(t) = t + 60, \quad t \ge 0, \ (n = 1, 2, 3, \dots).$$

It is easy to prove that $\alpha(t), \alpha_1(t)$ and $\beta_1(t), \beta(t)$ are two pairs of lower and upper solutions of (4.1). Moreover, $\alpha, \alpha_1, \beta_1, \beta \in X$,

$$\alpha < \alpha_1 < \beta, \quad \alpha < \beta_1 < \beta, \quad \alpha_1 \not\le \beta_1 \quad \text{on} J,$$

and α_1 and β_1 are not solutions of (4.1).

Meanwhile, when $0 \le t < +\infty$, $\alpha \le u \le \beta$, $-1 \le v \le 1$, it follows from $\sqrt[n]{y} < 1 + \frac{y}{n}$ that

$$\begin{aligned}|f_n(t,u,v))| &= |2(1+v_n)(v_n - \frac{1}{2n})\big(\frac{\arctan u_n}{1+t^3} + \sqrt[3]{v_{2n}}\big) + \frac{\pi(v_n - \frac{1}{2n})}{4(1+t^2)}| \\ &\le 3 + \frac{13\pi}{8} + 3|v_n|(1 + \frac{3\pi}{4} + \sqrt[3]{|v_n|}), \quad (n = 1,2,3,\dots).\end{aligned}$$

If we choose $A = (A_1, \dots, A_n, \dots)$, $B = (B_1, \dots, B_n, \dots)$ and $h = (h_1, \dots, h_n, \dots)$, in which $A_n = 3, B_n = 3 + \frac{13\pi}{8}$, $h_n(s) = 1 + \frac{3\pi}{4} + \sqrt[3]{s}$. Then $|f_n(t,u,v)| \le A_n|v_n|h_n(|v_n|) + B_n$, $(n = 1,2,3,\dots)$, for all $u \in [\alpha,\beta]$, which implies for any $\varphi \in S$ with $\|\varphi\| = 1$ that

$$\begin{aligned}|\varphi(f(t,u,v))| &= \|f(t,u,v)\| = \sup_n |f_n(t,u,v)| \\ &\le |\varphi(A)\varphi(v)| \cdot \varphi(h|\varphi(v)|) + |\varphi(B)|.\end{aligned}$$

Noting that

$$\begin{aligned}\varphi(\xi) &= \max\Big\{\sup_{\tau\le t<+\infty} |\frac{\varphi(\beta_n^0(t)) - \varphi(\alpha_n^0(0))}{t}|, \sup_{\tau\le t<+\infty} |\frac{\varphi(\beta_n^0(0)) - \varphi(\alpha_n^0(t))}{t}|\Big\} \\ &= 1 + \frac{71}{\tau} = \frac{287}{3},\end{aligned}$$

we have

$$\int_\xi^{+\infty} \frac{ds}{h_n(s)} = +\infty, \quad \sup_{s\ge\xi} \frac{1}{h_n(s)} \le 1, \quad (n = 1,2,3,\dots).$$

Since

$$\sup_{t\in J} \frac{\varphi(\beta_n^0(t))}{(1+t)^3} = \sup_{t\in J} \frac{\varphi(\beta_n^0(t))}{1+t} = 60, \quad \inf_{t\in J} \frac{\varphi(\alpha_n^0(t))}{(1+t)^3} = -11,$$

it follows that

$$\begin{aligned}\Gamma &= M_nA_n\Big(\sup_{t\in J} \frac{\varphi(\beta_n^0(t))}{(1+t)^3} - \inf_{t\in J} \frac{\varphi(\alpha_n^0(t))}{(1+t)^3} + \frac{3}{2}\sup_{t\in J} \frac{\varphi(\beta_n^0(t))}{1+t}\Big) \\ &\quad + \frac{M_n \cdot B_n}{2} \sup_{s\ge\xi^*} \frac{2}{\varphi(h_n(s))} \\ &\le \frac{167}{12} + \frac{1}{36}(3 + \frac{13\pi}{8}) < +\infty;\end{aligned}$$

that is, there exist $N > \xi$, such that the condition (C4) holds with respect to $\alpha(t)$ and $\beta(t)$.

Finally, we check condition (C5). Let $f = f^{(1)} + f^{(2)}$, where

$$f^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)}, \dots), \quad f^{(2)} = (f_1^{(2)}, \dots, f_n^{(2)}, \dots),$$

in which

$$f_n^{(1)}(t,u,v) = 2(1+v_n)(v_n - \frac{1}{2n})(\frac{\arctan u_n}{1+t^3} + \sqrt[3]{v_{2n}}), \quad (n = 1,2,3,\dots), \tag{4.3}$$

and

$$f_n^{(2)}(t,u,v) = \frac{\pi(v_n - \frac{1}{2n})}{4(1+t^2)}, \quad (n = 1,2,3,\dots). \tag{4.4}$$

For any $t \in J$ and bounded subsets $D_0, D_1 \subset E$, by (4.3),(4.4), we know

$$\alpha(f^{(2)}(J,D_0,D_1)) \le \frac{\pi}{4(1+t^2)}\alpha(D_1), \quad \forall t \in J, D_0, D_1 \subset E, \tag{4.5}$$

and

$$\begin{aligned}0 \le \|f^{(1)}(t,u,v)\| &= \sup_n |f^{(1)}(t,u_n,v_n)| \\ &\le 2(1+\|v\|)(\|v\|+\frac{1}{2n})(\frac{\pi}{2(1+t^2)}+\|v\|^{\frac{1}{3}}), \quad \forall t\in J, u,v\in E.\end{aligned}$$

Similar to the proof of [6, Example 2.12], we have

$$\alpha(f^{(1)}(t,D_0,D_1)) = 0, \quad \forall t\in J,\ \text{bounded sets } D_0,D_1\subset E. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\alpha(f(J,D_0,D_1)) \le \frac{\pi}{4(1+t^2)}\alpha(D_1), \quad \forall t\in J, D_0,D_1\subset E,$$

$$\delta \sup_{s\in J} q(s)\cdot \int_0^{+\infty}[l_0(s)(1+s)+l_1(s)] = \frac{\pi^2}{24} < \frac{1}{2};$$

i.e., condition (C5) holds for $l_0(t)=0, l_1(t)=\frac{\pi}{4(1+t^2)}$. So by Theorem 3.4, (4.1) has at least three solutions. $\square$

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