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EVOLUTION EQUATIONS IN GENERALIZED STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHIC SPACES

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Abstract. In this article, first we introduce and study the concept of $\mathbb{S}^p_\gamma$-pseudo almost automorphy (or generalized Stepanov-like pseudo almost automorphy), which is more general than the notion of Stepanov-like pseudo almost automorphy due to Diagana. We next study the existence of solutions to some classes of nonautonomous differential equations of Sobolev type in $\mathbb{S}^p_\gamma$-pseudo almost automorphic spaces. To illustrate our abstract result, we will study the existence and uniqueness of a pseudo almost automorphic solution to the heat equation with a negative time-dependent diffusion coefficient.

1. Introduction

Let $p \in [1, \infty)$ and let $\gamma : (0, \infty) \to (0, \infty)$ be a measurable function satisfying the following condition:

$$\gamma_0 := \lim_{\varepsilon \to 0} \int_\varepsilon^1 \gamma(\sigma) d\sigma = \int_0^1 \gamma(\sigma) d\sigma < \infty. \tag{1.1}$$

The impetus of this paper essentially comes from three papers. The first one is a paper by Liang *et al.* [23], in which, the powerful concept of pseudo almost automorphy was introduced and studied. Since its introduction in the literature, the concept of pseudo almost automorphy was utilized to investigate various types of differential, functional differential, and partial differential equations; see, e.g., [4, 14, 13, 17, 23, 27, 24] and the references therein.

The second source is a paper by Diagana [10], in which the concept of $\mathbb{S}^p$-pseudo almost automorphy (or Stepanov-like pseudo almost automorphy) was introduced and studied, which, in turn generalizes the notion of Stepanov-like almost automorphy which was introduced and studied by N'Guérékata and Pankov [30]. It should also be mentioned that some work on the notion of Stepanov-like almost automorphy has also been done notably in [3, 7, 17, 22].

The third and last source is a paper by Kostin and Pisareva [21], in which the concept of generalized Stepanov spaces was introduced and studied. In particular, in [21], the existence of generalized Stepanov almost periodic solutions to some

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differential differential equations with singularities was investigated. Other contributions on the concept of generalized Stepanov spaces include for instance the work of Kostin [20].

In this paper, we introduce and study the notion of $\mathbb{S}^p_\gamma$-pseudo almost automorphy (or generalized Stepanov-like pseudo almost automorphy), which, in turn generalizes all the above-mentioned concepts including the notion of $\mathbb{S}^p$-pseudo almost automorphy. As an illustration, we study and obtain the existence of pseudo almost automorphic solutions to the class of Sobolev type evolution equations given by

$$\frac{d}{dt}\big[u(t)+f(t,u(t))\big] = A(t)u(t) + g(t,u(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A(t) : D \subset \mathbb{X} \to \mathbb{X}$ for $t \in \mathbb{R}$ is a family of densely defined closed linear operator on a domain D, independent of t, and $f, g : \mathbb{R}\times\mathbb{X} \to \mathbb{X}$ belong to $PAA^p_\gamma(\mathbb{R}\times\mathbb{X},\mathbb{X}) \cap C(\mathbb{R}\times\mathbb{X},\mathbb{X})$ for $p > 1$. Such a result generalizes most of known results on the existence of pseudo almost automorphic (respectively, pseudo almost periodic) solutions to differential equations of type (1.2), in particular those in [6]. Let us also mention that Sobolev-type differential equations have various applications in particular in wave propagations or in dynamic of fluids [12]. Various formulations of these types of equations can be found in literature, in particular, we refer the reader to [2, 25].

This work will be heavily based upon the recent progress made by Xiao *et al.* [15, 16] notably on the composition of $\mathbb{S}^p$-pseudo almost automorphic spaces as well as the existence of pseudo almost automorphic solutions to differential equations with $\mathbb{S}^p$-pseudo almost automorphic coefficients. To illustrate our abstract results, the existence and uniqueness of a pseudo almost automorphic solution to the heat equation with a negative time-dependent diffusion coefficient will be investigated.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). The space $BC(\mathbb{R},\mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space. Furthermore, $C(\mathbb{R},\mathbb{Y})$ (respectively, $C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$).

Let $B(\mathbb{X},\mathbb{Y})$ stand for the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural operator topology; in particular, this is simply denoted $B(\mathbb{X})$ whenever $\mathbb{X} = \mathbb{Y}$.

2.1. $\mathbb{S}^p_\gamma$-pseudo almost automorphy.

Definition 2.1 ([31]). The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t,s) := f(t+s)$.

Remark 2.2. (i) A function $\varphi(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s) = f^b(t,s)$, if and only if $\varphi(t+\tau, s-\tau) = \varphi(s,t)$ for all $t \in \mathbb{R}$, $s \in [0,1]$ and $\tau \in [s-1, s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.3 ([6])**.** The Bochner transform $F^b(t,s,u)$, $t\in\mathbb{R}$, $s\in[0,1]$, $u\in\mathbb{X}$ of a function $F(t,u)$ on $\mathbb{R}\times\mathbb{X}$, with values in $\mathbb{X}$, is defined by $F^b(t,s,u):=F(t+s,u)$ for each $u\in\mathbb{X}$.

Definition 2.4 ([31])**.** Let $p\in[1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b\in L^\infty\big(\mathbb{R},L^p((0,1),d\tau)\big)$. This is a Banach space when it is equipped with the norm defined by

$$\|f\|_{S^p}=\|f^b\|_{L^\infty(\mathbb{R},L^p)}=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\|f(\tau)\|^p\,d\tau\Big)^{1/p}.$$

Let $\mathbb{U}$ denote the collection of all measurable (weights) functions $\gamma:(0,\infty)\to(0,\infty)$ satisfying (1.1). Let $\mathbb{U}_\infty$ be the collection of all functions $\gamma\in\mathbb{U}$, which are differentiable.

Define the set of weights

$$\mathbb{U}_\infty^+:=\Big\{\gamma\in\mathbb{U}_\infty:\frac{d\gamma}{dt}>0\text{ for all }t\in(0,\infty)\Big\},$$
$$\mathbb{U}_\infty^-:=\Big\{\gamma\in\mathbb{U}_\infty:\frac{d\gamma}{dt}<0\text{ for all }t\in(0,\infty)\Big\}$$

In addition to the above, we define the set of weights

$$\mathbb{U}_B:=\Big\{\gamma\in\mathbb{U}:\sup_{t\in(0,\infty)}\gamma(t)<\infty\Big\}.$$

Definition 2.5 ([5])**.** Let $\mu,\nu\in\mathbb{U}_\infty$. One says that μ is equivalent to ν and denote it $\mu\prec\nu$, if $\frac{\mu}{\nu}\in\mathbb{U}_B$.

Remark 2.6 ([5])**.** Let $\mu,\nu,\gamma\in\mathbb{U}_\infty$. Note that $\mu\prec\mu$ (reflexivity). If $\mu\prec\nu$, then $\nu\prec\mu$ (symmetry). If $\mu\prec\nu$ and $\nu\prec\gamma$, then $\mu\prec\gamma$ (transitivity). Therefore, $\prec$ is a binary equivalence relation on $\mathbb{U}_\infty$.

Theorem 2.7 ([21])**.** *If $\gamma\in\mathbb{U}_\infty^+$, then the norms $\|\cdot\|_{S^p}$ and $\|\cdot\|_{S^p_\gamma}$ are equivalent.*

Theorem 2.8 ([21])**.** *If $\gamma\in\mathbb{U}_\infty^-$ and if there exists $\varepsilon>0$ such that $\gamma^{1+\varepsilon}\notin L^1[0,1]$, then the norms $\|\cdot\|_{S^p}$ and $\|\cdot\|_{S^p_\gamma}$ are in general not equivalent.*

We now introduce the space $BS^p_\gamma(\mathbb{X})$ of all generalized Stepanov spaces as follows.

Definition 2.9. *Let $p\in[1,\infty)$ and let $\gamma\in\mathbb{U}$. The space $BS^p_\gamma(\mathbb{X})$ of all generalized Stepanov spaces, with the exponent p and weight γ, consists of all $\gamma d\tau$-measurable functions $f:\mathbb{R}\to\mathbb{X}$ such that $f^b\in L^\infty\big(\mathbb{R},L^p((0,1),\gamma d\tau)\big)$. This is a Banach space when it is equipped with the norm*

$$\|f\|_{S^p_\gamma}:=\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\gamma(\tau-t)\|f(\tau)\|^p\,d\tau\Big)^{1/p}=\sup_{t\in\mathbb{R}}\Big(\int_0^1\gamma(\tau)\|f(\tau+t)\|^p\,d\tau\Big)^{1/p}.$$

Remark 2.10. The assumption (1.1) on the weight γ does guarantee that identically constant functions belong to $BS^p_\gamma(\mathbb{X})$. Of course, if $\gamma(t)=1$ for all $t\in(0,\infty)$, then $BS^p_1(\mathbb{X})=BS^p(\mathbb{X})$.

Define the classes of functions:

$$PAP_0(\mathbb{X}):=\Big\{u\in BC(\mathbb{R},\mathbb{X}):\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|u(s)\|ds=0\Big\},$$

and $PAP_0(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is the collection of all functions $F\in BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ such that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\|F(s,u)\|ds=0$$

uniformly in $u\in\mathbb{K}$ where $\mathbb{K}\subset\mathbb{Y}$ is an arbitrary bounded subset.

Definition 2.11 (Bochner)**.** A function $f\in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t):=\lim_{n\to\infty}f(t+s_n)$$

is well defined for each $t\in\mathbb{R}$, and

$$\lim_{n\to\infty}g(t-s_n)=f(t)$$

for each $t\in\mathbb{R}$.

Remark 2.12. The function g in Definition 2.11 is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous. If the convergence above is uniform in $t\in\mathbb{R}$, then f is almost periodic. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R}\to\mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup norm, $\|\cdot\|_\infty$, turns out to be a Banach space.

We will denote by $AA_u(\mathbb{X})$ the closed subspace of all functions $f\in AA(\mathbb{X})$ with $g\in C(\mathbb{R},\mathbb{X})$. Equivalently, $f\in AA_u(\mathbb{X})$ if and only if f is almost automorphic and the convergence in Definition 2.11 are uniform on compact intervals, i.e. in the Fréchet space $C(\mathbb{R},\mathbb{X})$. Indeed, if f is almost automorphic, then, its range is relatively compact. Obviously, the following inclusions hold:

$$AP(\mathbb{X})\subset AA_u(\mathbb{X})\subset AA(\mathbb{X})\subset BC(\mathbb{X}).$$

Definition 2.13 (Xiao et al. [32])**.** A continuous function $L:\mathbb{R}\times\mathbb{R}\to\mathbb{X}$ is called bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$H(t,s):=\lim_{n\to\infty}L(t+s_n,s+s_n)$$

is well defined in $t,s\in\mathbb{R}$, and

$$\lim_{n\to\infty}H(t-s_n,s-s_n)=L(t,s)$$

for each $t,s\in\mathbb{R}$. The collection of such functions will be denoted $bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$.

We now introduce positively bi-almost automorphic functions. For that, let $\mathbb{T}$ be the set defined by:

$$\mathbb{T}:=\big\{(t,s)\in\mathbb{R}\times\mathbb{R}:t\geq s\big\}.$$

Definition 2.14. A continuous function $L:\mathbb{T}\to\mathbb{X}$ is called positively bi-almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, we can extract a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$H(t,s):=\lim_{n\to\infty}L(t+s_n,s+s_n)$$

is well defined in $t,s\in\mathbb{T}$, and

$$\lim_{n\to\infty}H(t-s_n,s-s_n)=L(t,s)$$

for each $(t,s)\in\mathbb{T}$. The collection of such functions will be denoted $bAA(\mathbb{T},\mathbb{X})$.

Obviously, every bi-almost automorphic function is positively bi-almost automorphic with the converse being false.

Definition 2.15 ([30])**.** The space $AS^p(\mathbb{X})$ of Stepanov-like almost automorphic functions (or $\mathbb{S}^p$-almost automorphic) consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA\big(L^p((0,1),ds)\big)$.

In other words, a function $f \in L^p_{\rm loc}(\mathbb{R},ds)$ is said to be $\mathbb{S}^p$-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p((0,1),ds)$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{\rm loc}(\mathbb{R};\mathbb{X})$ such that

$$\Big[\int_t^{t+1} \|f(s_n+s)-g(s)\|^p ds\Big]^{1/p} \to 0,$$
$$\Big[\int_t^{t+1} \|g(s-s_n)-f(s)\|^p ds\Big]^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Remark 2.16. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{\rm loc}(\mathbb{R},ds)$ is $\mathbb{S}^q$-almost automorphic, then f is $\mathbb{S}^p$-almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is $\mathbb{S}^p$-almost automorphic for any $1 \le p < \infty$.

It is also clear that $f \in AA_u(\mathbb{X})$ if and only if $f^b \in AA(L^\infty((0,1),ds))$. Thus, $AA_u(\mathbb{X})$ can be considered as $AS^\infty(\mathbb{X})$.

We now introduce the notion of $\mathbb{S}^p_\gamma$-almost automorphy, which generalizes that of $\mathbb{S}^p$-almost automorphy due to N'Guérékata and Pankov [30].

Definition 2.17. Let $p \ge 1$ and let $\gamma \in \mathbb{U}$. The space $AS^p_\gamma(\mathbb{X})$ of generalized Stepanov-like almost automorphic functions (or $\mathbb{S}^p_\gamma$-almost automorphic) consists of all $f \in BS^p_\gamma(\mathbb{X})$ such that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{\rm loc}(\mathbb{R},\gamma ds)$ such that

$$\begin{aligned}&\Big[\int_t^{t+1} \gamma(s-t)\|f(s_n+s)-g(s)\|^p ds\Big]^{1/p}\\ &= \Big[\int_0^1 \gamma(s)\|f(s_n+s+t)-g(s+t)\|^p ds\Big]^{1/p} \to 0,\end{aligned}$$

and

$$\begin{aligned}&\Big[\int_t^{t+1} \gamma(s-t)\|g(s-s_n)-f(s)\|^p ds\Big]^{1/p}\\ &= \Big[\int_0^1 \gamma(s)\|g(s+t-s_n)-f(s+t)\|^p ds\Big]^{1/p} \to 0\end{aligned}$$

as $n \to \infty$ for each $t \in \mathbb{R}$.

Remark 2.18. Let $\gamma \in \mathbb{U}$. As in the classical case (see Remark 2.16), if $1 \le p < q < \infty$ and $f \in L^q_{\rm loc}(\mathbb{R},\gamma ds)$ is $\mathbb{S}^q_\gamma$-almost automorphic, then f is $\mathbb{S}^p_\gamma$-almost automorphic. Also using (1.1), one can show that if $f \in AA(\mathbb{X})$, then f is $\mathbb{S}^p_\gamma$-almost automorphic for any $1 \le p < \infty$.

Definition 2.19. Let $\gamma \in \mathbb{U}$. A function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}, (t,u) \to F(t,u)$ with $F(\cdot,u) \in L^p_{\rm loc}(\mathbb{R},\gamma ds)$ for each $u \in \mathbb{Y}$, is said to be $\mathbb{S}^p_\gamma$-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \to F(t,u)$ is $\mathbb{S}^p_\gamma$-almost automorphic for each $u \in \mathbb{Y}$, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G(\cdot,u) \in L^p_{\rm loc}(\mathbb{R},\gamma ds)$ such that

$$\Big[\int_t^{t+1} \gamma(s-t)\|F(s_n+s,u)-G(s,u)\|^p ds\Big]^{1/p} \to 0,$$
$$\Big[\int_t^{t+1} \gamma(s-t)\|G(s-s_n,u)-F(s,u)\|^p ds\Big]^{1/p} \to 0$$

as $n\to\infty$ pointwise on $\mathbb{R}$ for each $u\in\mathbb{Y}$.

The collection of those $\mathbb{S}^p_\gamma$-almost automorphic functions $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ will be denoted by $AS^p_\gamma(\mathbb{R}\times\mathbb{Y},\mathbb{X})$.

Similarly, as in Ding *et al* [11], for each $K\subset\mathbb{Y}$ compact subset, we denote by $AS^p_{\gamma,K}(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ the collection of all functions $f\in AS^p_\gamma(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ satisfying that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ with $G(\cdot,u)\in L^p_{\rm loc}(\mathbb{R},\gamma ds)$ such that

$$\Big[\int_0^1 \gamma(s)\Big(\sup_{u\in K}\|F(s_n+s+t,u)-G(s+t,u)\|\Big)^p ds\Big]^{1/p} \to 0,$$
$$\Big[\int_0^1 \gamma(s)\Big(\sup_{u\in K}\|G(s+t-s_n,u)-F(s+t,u)\|\Big)^p ds\Big]^{1/p} \to 0$$

as $n\to\infty$ for each $t\in\mathbb{R}$.

Using similar arguments as in Ding *et al* [11], the following composition results can be established.

Lemma 2.20. *Let $f\in AS^p_\gamma(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and suppose f is Lipschitz; that is, there exists $L>0$ such that for all $u,v\in\mathbb{Y}$ and $t\in\mathbb{R}$*

$$\|f(t,u)-f(t,v)\| \le \|u-v\|_{\mathbb{Y}}. \tag{2.1}$$

Then for every $K\subset\mathbb{Y}$ a compact subset, $f\in AS^p_{\gamma,K}(\mathbb{R}\times\mathbb{Y},\mathbb{X})$.

Theorem 2.21 (). *Suppose $\varphi\in AS^p_\gamma(\mathbb{Y})$ such that $K=\overline{\{\varphi(t):t\in\mathbb{R}\}}\subset\mathbb{Y}$ is a compact subset. If $F\in AS^p_\gamma(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and satisfies the Lipschitz condition* (2.1), *then $t\to F(t,\varphi(t))$ belongs to $AS^p_\gamma(\mathbb{X})$.*

2.2. **Pseudo almost automorphy.** The concept of pseudo almost automorphy is a new notion due to Liang, Xiao and Zhang [23, 27, 24].

Definition 2.22 ([27]). A function $f\in C(\mathbb{R},\mathbb{X})$ is called pseudo almost automorphic if it can be expressed as $f=h+\varphi$, where $h\in AA(\mathbb{X})$ and $\varphi\in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{X})$.

Definition 2.23 ([27]). A function $F\in C(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ is said to pseudo almost automorphic if it can be expressed as $F=G+\Phi$, where $G\in AA(\mathbb{R}\times\mathbb{Y},\mathbb{X})$ and $\varphi\in PAP_0(\mathbb{R}\times\mathbb{Y},\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{R}\times Y,\mathbb{X})$.

A significant result is the next theorem, which is due to Liang et al. [27].

Theorem 2.24 ([27]). *The space $PAA(\mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.*

We also have the following composition result.

Theorem 2.25 ([27])**.** *If $f : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$ belongs to $PAA(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$ and if $x \to f(t, x)$ is uniformly continuous on any bounded subset K of $\mathbb{Y}$ for each $t \in \mathbb{R}$, then the function defined by $h(t) = f(t, \varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.*

3. $\mathbb{S}^p_\gamma$-pseudo almost automorphy

Let $\gamma \in \mathbb{U}$. This section is devoted to the concept of $\mathbb{S}^p_\gamma$-pseudo almost automorphy. Such a concept is new and generalizes the notion of $\mathbb{S}^p$-pseudo almost automorphy due to Diagana [10].

Definition 3.1. A function $f \in BS^p_\gamma(\mathbb{X})$ is called $\mathbb{S}^p_\gamma$-pseudo almost automorphic (or generalized Stepanov-like pseudo almost automorphic) if it can be expressed as

$$f = h + \varphi,$$

where $h^b \in AA\big(L^p((0, 1), \gamma ds)\big)$ and $\varphi^b \in PAP_0\big(L^p((0, 1), \gamma ds)\big)$. The collection of such functions will be denoted by $PAA^p_\gamma(\mathbb{X})$.

Clearly, a function $f \in L^p_{\rm loc}(\mathbb{R}, \gamma ds)$ is said to be $\mathbb{S}^p_\gamma$-pseudo almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p((0, 1), \gamma ds)$ is pseudo almost automorphic in the sense that there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h + \varphi$, where $h^b \in AA(L^p((0, 1), \gamma ds))$ and $\varphi^b \in PAP_0(L^p((0, 1), \gamma ds))$.

Remark 3.2. By definition, the decomposition of $\mathbb{S}^p_\gamma$-pseudo almost automorphic functions is unique. Furthermore, $\mathbb{S}^p_\gamma$-pseudo almost automorphic spaces are translation-invariant.

Theorem 3.3. *If $f \in PAA(\mathbb{X})$, then $f \in PAA^p_\gamma(\mathbb{X})$ for each $1 \le p < \infty$. In other words, $PAA(\mathbb{X}) \subset PAA^p_\gamma(\mathbb{X})$.*

Proof. Let $f \in PAA(\mathbb{X})$. Then, there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such $f = h + \varphi$ where $h \in AA(\mathbb{X})$ and $\varphi \in PAP_0(\mathbb{X})$. Clearly, $h^b \in AA(\mathbb{X})$. Using Remark 2.16 it follows that $h^b \in AA(\mathbb{X}) \subset AS^p_\gamma(\mathbb{X})$, that is, $h^b \in AA\big(L^p((0, 1), \gamma ds)\big)$. Let $q > 0$ such that $p^{-1} + q^{-1} = 1$. Then for $T > 0$,

$$\begin{aligned}
&\int_{-T}^{T}\Big(\int_0^1 \gamma(s)\|\varphi(t+s)\|^p ds\Big)^{1/p} dt \\
&\le (2T)^{1/q}\Big[\int_{-T}^{T}\Big(\int_0^1 \gamma(s)\|\varphi(s+t)\|^p ds\Big)dt\Big]^{1/p} \\
&\le (2T)^{1/q}\Big[\int_{-T}^{T}\Big(\int_0^1 \gamma(s)\|\varphi(s+t)\|\cdot\|\varphi\|_\infty^{p-1} ds\Big)dt\Big]^{1/p} \\
&= (2T)^{1/q}\|\varphi\|_\infty^{(p-1)/p}\Big[\int_{-T}^{T}\Big(\int_0^1 \gamma(s)\|\varphi(s+t)\| ds\Big)dt\Big]^{1/p} \\
&= (2T)^{1/q}\|\varphi\|_\infty^{(p-1)/p}\Big[\int_0^1 \gamma(s)\Big(\int_{-T}^{T}\|\varphi(s+t)\| dt\Big)ds\Big]^{1/p} \\
&= 2T\|\varphi\|_\infty^{(p-1)/p}\Big[\int_0^1 \gamma(s)\Big(\frac{1}{2T}\int_{-T}^{T}\|\varphi(s+t)\| dt\Big)ds\Big]^{1/p},
\end{aligned}$$

and hence

$$\frac{1}{2T}\int_{-T}^{T}\Big(\int_0^1 \gamma(s)\|\varphi(t+s)\|^p ds\Big)^{1/p} dt$$

$$\leq \|\varphi\|_\infty^{(p-1)/p}\Big[\int_0^1 \gamma(s)\Big(\frac{1}{2T}\int_{-T}^{T}\|\varphi(s+t)\|dt\Big)ds\Big]^{1/p}.$$

Since $PAP_0(\mathbb{X})$ is translation invariant, it follows that

$$\frac{1}{2T}\int_{-T}^{T}\|\varphi(t+s)\|dt \to 0 \quad \text{as } T\to\infty$$

for all $s\in[0,1]$. Using the Lebesgue Dominated Convergence Theorem it follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Big(\int_0^1 \gamma(s)\|\varphi(t+s)\|^p ds\Big)^{1/p}dt = 0.$$

□

Theorem 3.4. *Let $\gamma\in\mathbb{U}$. The space $PAA^p_\gamma(\mathbb{X})$ equipped with the norm $\|\cdot\|_{\mathbb{S}^p_\gamma}$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $PAA^p_\gamma(\mathbb{X})$. Let $(h_n)_{n\in\mathbb{N}},(\varphi_n)_{n\in\mathbb{N}}$ be sequences such that $f_n=h_n+\varphi_n$ where $(h_n^b)_{n\in\mathbb{N}}\subset AA\big(L^p((0,1),\gamma ds)\big)$ and $(\varphi_n^b)_{n\in\mathbb{N}}\subset PAP_0\big(L^p((0,1),\gamma ds)\big)$. Using similar ideas as in the proof of [27, Theorem 2.2] it can be shown that the following holds

$$\|h_n\|_{\mathbb{S}^p_\gamma}\leq\|f_n\|_{\mathbb{S}^p_\gamma}\quad\text{for all } n\in\mathbb{N}.$$

Thus there exists a function $h\in AS^p_\gamma(\mathbb{X})$ such that $\|h_n-h\|_{\mathbb{S}^p_\gamma}\to 0$ as $n\to\infty$. Using the previous fact, it easily follows that there exists a function $\varphi\in BS^p_\gamma(\mathbb{X})$ such that $\|\varphi_n-\varphi\|_{\mathbb{S}^p_\gamma}\to 0$ as $n\to\infty$. Now, for $T>0$, we have

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\gamma(s-t)\|\varphi(s)\|ds\Big)^{1/p}dt\\
&\leq \frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\gamma(s-t)\|\varphi_n(s)-\varphi(s)\|^p ds\Big)^{1/p}dt\\
&\quad+\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\gamma(s-t)\|\varphi_n(s)\|^p ds\Big)^{1/p}dt\\
&\leq \|\varphi_n-\varphi\|_{\mathbb{S}^p_\gamma}+\frac{1}{2T}\int_{-T}^{T}\Big(\int_t^{t+1}\gamma(s-t)\|\varphi_n(s)\|ds\Big)^{1/p}dt.
\end{aligned}$$

Letting $T\to\infty$ and then $n\to\infty$ in the previous inequality, it follows that $\varphi^b\in PAP_0\big(L^p((0,1),\gamma ds)\big)$; that is, $f=h+\varphi\in PAA^p_\gamma(\mathbb{X})$. □

Theorem 3.5. *Let $\gamma,\nu\in\mathbb{U}$. If $\gamma\prec\nu$, then $PAA^p_\gamma(\mathbb{X})=PAA^p_\nu(\mathbb{X})$.*

Corollary 3.6. *If $\gamma\in\mathbb{U}_B$, then $PAA_\gamma(\mathbb{X})=PAA(\mathbb{X})$.*

The proofs of the Theorem 3.5 and Corollary 3.6 are straightforward and hence omitted.

Definition 3.7. Let $\gamma\in\mathbb{U}_\infty$. A function $F:\mathbb{R}\times\mathbb{Y}\to\mathbb{X},(t,u)\to F(t,u)$ with $F(\cdot,u)\in L^p(\mathbb{R},\gamma ds)$ for each $u\in\mathbb{Y}$, is said to be $\mathbb{S}^p_\gamma$-pseudo almost automorphic if there exists two functions $H,\Phi:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ such that $F=H+\Phi$, where $H^b\in AA(\mathbb{R}\times\mathbb{Y},L^p((0,1),\gamma ds))$ and $\Phi^b\in PAP_0(\mathbb{R}\times\mathbb{Y},L^p((0,1),\gamma ds))$. The collection of those $\mathbb{S}^p_\gamma$-pseudo almost automorphic functions will be denoted by $PAA^p_\gamma(\mathbb{R}\times\mathbb{Y},\mathbb{X})$.

Using similar arguments as in Fan *et al.* [15] and in Theorem 2.21, the following composition result can be established.

Theorem 3.8. *Let $F = G + \Phi \in PAA^p_\gamma(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ such that $H^b \in AA(\mathbb{R} \times \mathbb{Y}, L^p((0,1), \gamma(s)ds))$ and $\Phi^b \in PAP_0(\mathbb{R} \times \mathbb{Y}, L^p((0,1), \gamma(s)ds))$. Moreover, we suppose that G satisfies (2.1) and that Φ satisfies: there exists $L > 0$ such that for all $u, v \in L^p_{\rm loc}(\mathbb{R}, \gamma ds)$ and $t \in \mathbb{R}$,*

$$\begin{aligned} &\Big(\int_0^1 \gamma(s)\|\Phi(t+s, u(s)) - \Phi(t+s, v(s))\|^p ds\Big)^{1/p} \\ &\leq L\Big(\int_0^1 \gamma(s)\|u(s) - v(s)\|^p ds\Big)^{1/p}. \end{aligned} \tag{3.1}$$

Furthermore, if $h = g + \varphi \in PAA^p_\gamma(\mathbb{Y})$ with $h^b \in AA(L^p((0,1), \gamma(s)ds))$ and $\varphi^b \in PAP_0(L^p((0,1), \gamma(s)ds))$ and such that $K = \overline{\{g(t) : t \in \mathbb{R}\}}$ is compact, then $t \mapsto F(t, h(t))$ belongs to $PAA^p_\gamma(\mathbb{X})$.

4. Existence of pseudo almost automorphic solutions

Fix $\gamma \in \mathbb{U}$ and $p > 1$. Throughout the rest of the paper, we set $q = 1 - p^{-1}$. Note that $q \neq 0$, as $p \neq 1$. Moreover, we suppose that $\gamma \in \mathbb{U}$ satisfies

$$\inf_{t \in (0,\infty)} \gamma(t) = m_0 > 0.$$

This section is devoted to the search of a pseudo almost automorphic solution to Eq. (1.2) with $\mathbb{S}^p_\gamma$-pseudo almost automorphic coefficients. For that, we suppose among others that there exists a Banach space $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ such that the embedding

$$(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}}) \hookrightarrow (\mathbb{X}, \|\cdot\|)$$

is continuous. Let $C > 0$ be the bound of this embedding. In addition to the above we assume that the following assumptions hold:

- (H1) The system
$$u'(t) = A(t)u(t), \quad t \geq s, \quad u(s) = \varphi \in \mathbb{X} \tag{4.1}$$
has an associated evolution family of operators $\{U(t,s) : t \geq s \text{ with } t, s \in \mathbb{R}\}$. In addition, we assume that the domains of the operators $A(t)$ are constant in t, that is, $D(A(t)) = D = \mathbb{Y}$ for all $t \in \mathbb{R}$ and that the evolution family $U(t,s)$ is asymptotically stable in the sense that there exist some constants $M, \delta > 0$ such that
$$\|U(t,s)\|_{B(\mathbb{X})} \leq Me^{-\delta(t-s)}$$
for all $t, s \in \mathbb{R}$ with $t \geq s$.
- (H2) The function $s \to A(s)U(t,s)$ defined from $(-\infty, t)$ into $B(\mathbb{Y}, \mathbb{X})$ is strongly measurable and there exist a measurable function $H : (0,\infty) \to (0,\infty)$ with $H \in L^1(0,\infty)$ and a constant $\omega > 0$ such that
$$\|A(s)U(t,s)\|_{B(\mathbb{Y},\mathbb{X})} \leq e^{-\omega(t-s)}H(t-s), \quad t, s \in \mathbb{R},\ t > s.$$
- (H3) The function $\mathbb{R} \times \mathbb{R} \to \mathbb{X}$, $(t,s) \to U(t,s)y \in bAA(\mathbb{T}, \mathbb{Y})$ uniformly for $y \in \mathbb{X}$.
- (H4) The function $\mathbb{R} \times \mathbb{R} \to \mathbb{X}$, $(t,s) \to A(s)U(t,s)y \in bAA(\mathbb{T}, \mathbb{X})$ uniformly for $y \in \mathbb{Y}$.

(H5) The function $f \in PAA(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ and $g \in PAA^p_\gamma(\mathbb{R} \times \mathbb{X}, \mathbb{X}) \cap C(\mathbb{R} \times \mathbb{X}, \mathbb{X})$. Moreover, there exists $L > 0$ such that

$$\|f(t,u) - f(t,v)\|_{\mathbb{Y}} \le L\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$, and

$$\|g(t,u) - g(t,v)\| \le L\|u - v\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

Definition 4.1. A family of linear operators $\{U(t,s) : t \ge s \text{with } t, s \in \mathbb{R}\} \subset B(\mathbb{X})$ is called an evolution family of operators for (4.1) whenever the following conditions hold:

(a) $U(t,s)U(s,r) = U(t,r)$ for all $t \ge s \ge r$;
(b) for each $x \in \mathbb{X}$, the function $(t,s) \mapsto U(t,s)x$ is continuous and $U(t,s) \in B(\mathbb{X}, D)$ for every $t > s$; and
(c) the function $(s,t] \to B(\mathbb{X})$, $t \mapsto U(t,s)$ is differentiable with

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s).$$

To study the existence and uniqueness of pseudo almost automorphic solutions to (1.2) we first introduce the notion of mild solution, which has been adapted from Diagana *et al.* [8, Defintion 3.1].

Definition 4.2. A continuous function $u : \mathbb{R} \to \mathbb{X}$ is said to be a mild solution of (1.2) provided that the function $s \to A(s)U(t,s)f(s,u(s))$ is integrable on (s,t), and

$$\begin{aligned} u(t) &= -f(t,u(t)) + U(t,s)\Big(u(s) + f(s,u(s))\Big) \\ &\quad - \int_s^t A(s)U(t,s)f(s,u(s))ds + \int_s^t U(t,s)g(s,u(s))ds \end{aligned}$$

for $t \ge s$ and for all $t, s \in \mathbb{R}$.

Under assumptions (H1)-(H2), it can be easily shown that the function

$$u(t) = -f(t,u(t)) + \int_{-\infty}^t U(t,s)g(s,u(s))ds - \int_{-\infty}^t A(s)U(t,s)f(s,u(s))ds$$

for each $t \in \mathbb{R}$, is a mild solution of (1.2).

Lemma 4.3. *Under assumptions* (H1), (H3), (H5)*, then the nonlinear integral operator* Γ *defined by*

$$(\Gamma u)(t) := \int_{-\infty}^t U(t,s)g(s,u(s))ds$$

maps $PAA(\mathbb{X})$ *into* $PAA(\mathbb{X})$.

Proof. Let $u \in PAA(\mathbb{X})$. Using Theorem 3.8 it follows that $G(t) := g(t,u(t))$ belongs to $PAA^p_\gamma(\mathbb{X})$. Now let $G = h + \varphi$, where $h^b \in AA\big(L^p((0,1), \gamma ds)\big)$ and $\varphi^b \in PAP_0\big(L^p((0,1), \gamma ds)\big)$. Consider for each $k = 1, 2, \dots$, the integral

$$V_k(t) = \int_{k-1}^k U(t, t-\xi)g(t-\xi)d\xi$$

$$= \int_{k-1}^{k} U(t,t-\xi)h(t-\xi)d\xi + \int_{k-1}^{k} U(t,t-\xi)\varphi(t-\xi)d\xi$$

and set $Y_k(t) = \int_{k-1}^{k} U(t,t-\xi)h(t-\xi)d\xi$ and $X_k(t) = \int_{k-1}^{k} U(t,t-\xi)\varphi(t-\xi)d\xi$.

Let us show that $Y_k \in AA(\mathbb{X})$. For that, letting $r = t-\xi$ one obtains

$$Y_k(t) = \int_{t-k}^{t-k+1} U(t,r)h(r)dr \quad \text{for each } t \in \mathbb{R}.$$

From (H1) it follows that the function $s \to U(t,r)h(r)$ is integrable over $(-\infty,t)$ for each $t \in \mathbb{R}$. Now using the Hölder's inequality, it follows that

$$\begin{aligned}
\|Y_k(t)\| &\le \int_{t-k}^{t-k+1} \|U(t,r)h(r)\|dr \\
&\le M \int_{t-k}^{t-k+1} e^{-\delta(t-r)}\|h(r)\|dr \\
&= M \int_{t-k}^{t-k+1} \gamma^{-1/p}(r-t+k)e^{-\delta(t-r)}\|h(r)\|\gamma^{1/p}(r-t+k)dr \\
&\le M\Big[\int_{t-k}^{t-k+1} \gamma^{-q/p}(r-t+k)e^{-q\delta(t-r)}dr\Big]^{1/q} \\
&\quad \times \Big[\int_{t-k}^{t-k+1} \gamma(r-t+k)\|h(r)\|^p dr\Big]^{1/p} \\
&\le M m_0^{-1/p}\Big[\int_{k-1}^{k} e^{-q\delta s}ds\Big]^{1/q}\|h\|_{\mathbb{S}^p_\gamma} \\
&\le \Big[e^{-\delta k} m_0^{-1/p}\ M\sqrt[q]{(1+e^{q\delta})/(q\delta)}\Big]\|h\|_{\mathbb{S}^p_\gamma}.
\end{aligned}$$

Using the fact that

$$m_0^{-1/p} M\sqrt[q]{(1+e^{q\delta})/(q\delta)} \sum_{k=1}^{\infty} e^{-\delta k} < \infty$$

we deduce from the well-known Weirstrass theorem that the series $\sum_{k=1}^{\infty} Y_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$Y(t) := \int_{-\infty}^{t} U(t,s)h(s)ds = \sum_{k=1}^{\infty} Y_k(t),$$

$Y \in C(\mathbb{R},\mathbb{X})$, and

$$\|Y(t)\| \le \sum_{k=1}^{\infty} \|Y_k(t)\| \le K_1\|h\|_{\mathbb{S}^p_\gamma},$$

where $K_1 > 0$ is a constant.

Fix $k \in \mathbb{N}$. Let $(s_m)_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},\mathbb{Y})$ and $h \in AS^p_\gamma(\mathbb{X})$, for every sequence $(s_m)_{m\in\mathbb{N}}$ there exists a subsequence $(s_{m_n})_{k\in\mathbb{N}}$ of $(s_m)_{m\in\mathbb{N}}$ and functions U_1 and $v \in AS^p_\gamma(\mathbb{X})$ such that

$$\lim_{n\to\infty} U(t+s_{m_n}, s+s_{m_n})x = U_1(t,s)x, \quad t,s \in \mathbb{R},\ x \in \mathbb{X}, \tag{4.2}$$

$$\lim_{n\to\infty} U_1(t-s_{m_n}, s-s_{m_n})x = U(t,s)x, \quad t,s \in \mathbb{R},\ x \in \mathbb{X}, \tag{4.3}$$

$$\lim_{n\to\infty} \|h(t+s_{m_n}+\cdot) - v(t+\cdot)\|_{\mathbb{S}^p_\gamma} = 0, \quad \text{for each } t \in \mathbb{R}, \tag{4.4}$$

$$\lim_{n\to\infty} \|v(t-s_{m_n}+\cdot)-h(t+\cdot)\|_{\mathbb{S}^p_\gamma}=0, \quad \text{for each } t\in\mathbb{R}. \tag{4.5}$$

Define

$$\begin{aligned} T_k(t) &= \int_{k-1}^{k} U_1(t,t-\xi)h(t-\xi)d\xi,\\ Z_k(t) &= \int_{k-1}^{k} U(t,t-\xi)v(t-\xi)d\xi. \end{aligned}$$

Now let

$$\begin{aligned} I_n^k(t) &:= \Big\| \int_{k-1}^{k} U(t+s_{m_n},t+s_{m_n}-\xi)\Big(h(t+s_{m_n}-\xi)-v(t-\xi)\Big)d\xi\Big\|,\\ J_n^k(t) &:= \Big\| \int_{k-1}^{k} \Big(U(t+s_{m_n},t+s_{m_n}-\xi)-U(t,t-\xi)\Big)v(t-\xi)d\xi\Big\|. \end{aligned}$$

Then

$$\|Y_k(t+s_{m_n})-Z_k(t)\| \le I_n^k(t)+J_n^k(t)\,.$$

Then using the Hölder's inequality we obtain

$$\begin{aligned} I_n^k(t) &\le M\int_{k-1}^{k} e^{-\delta\xi}\|h(t+s_{m_n}-\xi)-v(t-\xi)\|d\xi\\ &\le M\int_{k-1}^{k} e^{-\delta\xi}\|h(t+s_{m_n}-\xi)-v(t-\xi)\|d\xi\\ &\le M\int_{k-1}^{k} \gamma^{-1/p}(\xi-k+1)e^{-\delta\xi}\|h(t+s_{m_n}-\xi)-v(t-\xi)\|\gamma^{1/p}(\xi-k+1)d\xi\\ &\le K_2\Big[\int_{k-1}^{k}\gamma(\xi-k+1)\|h(t+s_{m_n}-\xi)-v(t-\xi)\|^p d\xi\Big]^{1/p} \end{aligned}$$

where $K_2>0$ is a constant.

Now using (4.4) it follows that $I_n^k(t)\to 0$ as $n\to\infty$ for each $t\in\mathbb{R}$. Similarly, using the Lebesgue Dominated Convergence theorem and (4.2) it follows that $J_n^k(t)\to 0$ as $n\to\infty$ for each $t\in\mathbb{R}$. Now,

$$\|Y_k(t+s_{m_n})-Z_n(t)\|\to 0 \quad \text{as } n\to\infty.$$

Similarly, using (4.3) and (4.5) it can be shown that

$$\|Z_k(t-s_{m_n})-Y_k(t)\|\to 0 \quad \text{as } n\to\infty.$$

Therefore each $Y_k\in AA(\mathbb{X})$ for each k and hence their uniform limit $Y(t)\in AA(\mathbb{X})$, by using [28, Theorem 2.1.10].

Let us show that each $X_n\in PAP_0(\mathbb{X})$. For that, note that

$$\begin{aligned} \|X_k(t)\| &\le M\int_{t-k}^{t-k+1} e^{-\delta(t-r)}\|\varphi(r)\|dr\\ &\le \Big[e^{-\delta k}m_0^{-1/p}\ M\sqrt[q]{\frac{1+e^{q\delta}}{q\delta}}\Big]\Big[\int_{t-k}^{t-k+1}\gamma(r-t+k)\|\varphi(r)\|^p dr\Big]^{1/p}\\ &\le K_3\Big[\int_{t-k}^{t-k+1}\gamma(r-t+k)\|\varphi(r)\|^p dr\Big]^{1/p} \end{aligned}$$

where $K_3 > 0$ is a constant. Now

$$\frac{1}{2T}\int_{-T}^{T}\|X_k(t)\|dt \le \frac{K_3}{2T}\int_{-T}^{T}\Big[\int_{t-k}^{t-k+1}\gamma(r-t+k)\|\varphi(r)\|^p dr\Big]^{1/p}dt.$$

Letting $T \to \infty$ in the previous inequality it follows that $X_k \in PAP_0(\mathbb{X})$, as $\varphi^b \in PAP_0(L^p((0,1),\gamma ds))$. Furthermore,

$$X(t) := \int_{-\infty}^{t} U(t,s)\varphi(s)ds = \sum_{k=1}^{\infty} X_k(t),$$

$X \in C(\mathbb{R},\mathbb{X})$, and

$$\|X(t)\| \le \sum_{k=1}^{\infty}\|X_k(t)\| \le K_4\ \|\varphi\|_{\mathbb{S}^p_\gamma},$$

where $K_4 > 0$ is a constant. Consequently the uniform limit $X(t) = \sum_{k=1}^{\infty} X_k(t) \in PAP_0(\mathbb{X})$, see [9, Lemma 2.5]. Therefore, $\Gamma u(t) = X(t) + Y(t) \in PAA(\mathbb{X})$. □

Lemma 4.4. *Under assumptions* (H1), (H2), (H4), (H5)*, then the nonlinear integral operator* Λ *defined by*

$$(\Lambda u)(t) := \int_{-\infty}^{t} A(s)U(t,s)f(s,u(s))ds$$

maps $PAA(\mathbb{X})$ *into itself whenever the series* $\sum_{n=1}^{\infty}\Big[\int_{n-1}^{n} e^{-\omega s}H(s)^q ds\Big]^{1/q}$ *converges.*

Proof. Let $u \in PAA(\mathbb{X})$. Using the composition of pseudo almost automorphic functions it follows that $F(t) := f(t,u(t))$ belongs to $PAA(\mathbb{Y}) \subset PAA^p_\gamma(\mathbb{Y}) \subset PAA^p_\gamma(\mathbb{X})$. The proof is, up to some slight modifications, similar to the proof of Lemma 4.3. Indeed, write $F = h + \varphi$, where $h^b \in AA\big(L^p((0,1),\gamma ds)\big)$ and $\varphi^b \in PAP_0\big(L^p((0,1),\gamma ds)\big)$. Consider for each $k = 1,2,\dots$, the integral

$$\begin{aligned}
v_k(t) &= \int_{k-1}^{n} A(t-\xi)U(t,t-\xi)g(t-\xi)d\xi\\
&= \int_{k-1}^{k} A(t-\xi)U(t,t-\xi)h(t-\xi)d\xi + \int_{k-1}^{k} A(t-\xi)U(t,t-\xi)\varphi(t-\xi)d\xi
\end{aligned}$$

and set

$$W_k(t) = \int_{k-1}^{k} A(t-\xi)U(t,t-\xi)h(t-\xi)d\xi, \quad Z_k(t) = \int_{k-1}^{k} A(t-\xi)U(t,t-\xi)\varphi(t-\xi)d\xi.$$

Let us show that $W_k \in AA(\mathbb{X})$. For that, letting $r = t-\xi$ one obtains

$$W_k(t) = \int_{t-k}^{t-k+1} A(r)U(t,r)h(r)dr \quad \text{for each } t \in \mathbb{R}.$$

From (H2) it follows that the function $s \to A(r)U(t,r)h(r)$ is integrable over $(-\infty,t)$ for each $t \in \mathbb{R}$. Now using the Hölder's inequality, it follows that

$$\begin{aligned}
\|W_k(t)\| &\le \int_{t-k}^{t-k+1} e^{-\omega(t-r)}H(t-r)\|h(r)\|dr\\
&= \int_{t-k}^{t-k+1} \gamma^{-1/p}(r-t+k)H(t-r)e^{-\omega(t-r)}\|h(r)\|\gamma^{1/p}(r-t+k)dr
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big[\int_{t-k}^{t-k+1} \gamma^{-q/p}(r-t+k)e^{-\omega(t-r)}H^q(t-r)dr\Big]^{1/q}\\
&\quad\times \Big[\int_{t-k}^{t-k+1} \gamma(r-t+k)\|h(r)\|^p dr\Big]^{1/p}\\
&\leq m_0^{-1/p}\Big[\int_{k-1}^{k} e^{-q\omega s}H(s)^q ds\Big]^{1/q}\,\|h\|_{\mathbb{S}^p_\gamma}\\
&\leq m_0^{-1/p}\Big[\int_{k-1}^{n} e^{-q\omega s}H(s)^q ds\Big]^{1/q}\|h\|_{\mathbb{S}^p_\gamma}.
\end{aligned}
$$

Using the fact that the series given by

$$
m_0^{-1/p}\Big[\int_{k-1}^{k} e^{-q\omega s}H(s)^q ds\Big]^{1/q}
$$

converges, we then deduce from the well-known Weirstrass theorem that the series $\sum_{k=1}^{\infty} W_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore,

$$
W(t) := \int_{-\infty}^{t} A(s)U(t,s)h(s)ds = \sum_{k=1}^{\infty} W_k(t),
$$

$W \in C(\mathbb{R},\mathbb{X})$, and

$$
\|W(t)\| \leq \sum_{k=1}^{\infty} \|Y_k(t)\| \leq K_5\ \|h\|_{\mathbb{S}^p_\gamma},
$$

where $K_5 > 0$ is a constant.

Fix $k \in \mathbb{N}$. Let $(s_m)_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $A(s)U(t,s)x \in bAA(\mathbb{R}\times\mathbb{R},\mathbb{X})$ and $h \in AS^p_\gamma(\mathbb{Y}) \subset AS^p_\gamma(\mathbb{X})$, for every sequence $(s_m)_{m\in\mathbb{N}}$ there exists a subsequence $(s_{m_n})_{k\in\mathbb{N}}$ of $(s_m)_{m\in\mathbb{N}}$ and functions Θ_1 and $v \in AS^p_\gamma(\mathbb{Y}) \subset AS^p_\gamma(\mathbb{X})$ such that

$$
\lim_{n\to\infty} A(s+s_{m_n})U(t+s_{m_n}, s+s_{m_n})x = \Theta(t,s)x, \quad t,s\in\mathbb{R},\ x\in\mathbb{X}, \tag{4.6}
$$

$$
\lim_{n\to\infty} \Theta(t-s_{m_n}, s-s_{m_n})x = A(s)U(t,s)x, \quad t,s\in\mathbb{R},\ x\in\mathbb{X}, \tag{4.7}
$$

$$
\lim_{n\to\infty} \|h(t+s_{m_n}+\cdot) - v(t+\cdot)\|_{\mathbb{S}^p_\gamma} = 0, \quad \text{for each } t\in\mathbb{R}, \tag{4.8}
$$

$$
\lim_{n\to\infty} \|v(t-s_{m_n}+\cdot) - h(t+\cdot)\|_{\mathbb{S}^p_\gamma} = 0, \quad \text{for each } t\in\mathbb{R}. \tag{4.9}
$$

Define

$$
\begin{gathered}
T_k(t) = \int_{k-1}^{k} \Theta(t,t-\xi)h(t-\xi)d\xi,\\
Z_k(t) = \int_{k-1}^{k} A(t-\xi))U(t,t-\xi)v(t-\xi)d\xi.
\end{gathered}
$$

Now let

$$
\begin{aligned}
L_n^k(t) := \Big\|\int_{k-1}^{k} & A(t+s_{m_n}-\xi)U(t+s_{m_n}, t+s_{m_n}-\xi)\\
&\times\Big(h(t+s_{m_n}-\xi) - v(t-\xi)\Big)d\xi\Big\|,\\
M_n^k(t) := \Big\|\int_{k-1}^{k} & \Big(A(t+s_{m_n}-\xi)U(t+s_{m_n}, t+s_{m_n}-\xi)
\end{aligned}
$$

$$- A(t-\xi)U(t,t-\xi)\Big)v(t-\xi)d\xi\Big\|.$$

Then

$$\|W_k(t+s_{m_n}) - Z_k(t)\| \le L_n^k(t) + M_n^k(t)$$

Then using the Hölder's inequality we obtain

$$\begin{aligned}
L_n^k &\le \int_{k-1}^{k} e^{-\omega\xi} H(\xi)\|h(t+s_{m_n}-\xi) - v(t-\xi)\|_{\mathbb{Y}} d\xi \\
&\le \int_{k-1}^{k} e^{-\omega\xi} H(\xi)\|h(t+s_{m_n}-\xi) - v(t-\xi)\|_{\mathbb{Y}} d\xi \\
&\le \int_{k-1}^{k} \gamma^{-1/p}(\xi-k) e^{-\omega\xi} H(\xi)\|h(t+s_{m_n}-\xi) - v(t-\xi)\|_{\mathbb{Y}} \gamma^{1/p}(\xi-k) d\xi \\
&\le K_6 \Big[\int_{k-1}^{k} e^{-\omega\xi} H^q(\xi) d\xi\Big]^{1/q} \Big[\int_{k-1}^{k} \gamma(\xi-k)\|h(t+s_{m_n}-\xi) - v(t-\xi)\|_{\mathbb{Y}}^p d\xi\Big]^{1/p}
\end{aligned}$$

where $K_6 > 0$ is a constant. Now using (4.8) it follows that $L_n^k(t) \to 0$ as $n \to \infty$ for each $t \in \mathbb{R}$. Similarly, using the Lebesgue Dominated Convergence theorem and (4.6) it follows that $M_n^k(t) \to 0$ as $n \to \infty$ for each $t \in \mathbb{R}$. Now,

$$\|W_k(t+s_{m_n}) - Z_k(t)\| \to 0 \quad \text{as } n \to \infty.$$

Similarly, using (4.7) and (4.9) it can be shown that

$$\|Z_k(t-s_{m_n}) - W_k(t)\| \to 0 \quad \text{as } n \to \infty.$$

Therefore each $W_k \in AA(\mathbb{X})$ for each k and hence it uniform limit $W(t) \in AA(\mathbb{X})$.

Let us show that each $Z_k \in PAP_0(\mathbb{X})$. For that, note that

$$\begin{aligned}
\|Z_k(t)\| &\le \int_{t-k}^{t-k+1} e^{-\omega(t-r)} H(t-r)\|\varphi(r)\|_{\mathbb{Y}} dr \\
&\le \int_{t-k}^{t-k+1} e^{-\omega(t-r)} H(t-r)\|\varphi(r)\|_{\mathbb{Y}} dr \\
&\le m_0^{-1/p}\Big[\int_{k-1}^{k} e^{-\omega s} H(s)^q ds\Big]^{1/q} \Big[\int_{t-k}^{t-k+1} \gamma(r-t+k)\|\varphi(r)\|_{\mathbb{Y}}^p dr\Big]^{1/p}
\end{aligned}$$

and hence $Z_k \in PAP_0(\mathbb{X})$, as $\varphi^b \in PAP_0(L^p((0,1),\gamma ds))$. Furthermore,

$$Z(t) := \int_{-\infty}^{t} A(s)U(t,s)\varphi(s)ds = \sum_{k=1}^{\infty} Z_k(t),$$

$Z \in C(\mathbb{R},\mathbb{X})$, and

$$\|Z(t)\| \le \sum_{k=1}^{\infty} \|Z_k(t)\| \le K_7 \ \|\varphi\|_{\mathbb{S}^p_\gamma},$$

where $K_7 > 0$ is a constant. Consequently the uniform limit $Z(t) = \sum_{k=1}^{\infty} Z_k(t) \in PAP(\mathbb{X})$, see [9, Lemma 2.5]. Therefore, $\Lambda u(t) = W(t) + Z(t) \in PAA(\mathbb{X})$. $\square$

In addition to the previous assumptions, we suppose that the series

$$\sum_{n=1}^{\infty} \Big[\int_{n-1}^{n} e^{-\omega s} H(s)^q ds\Big]^{1/q}$$

converges.

Theorem 4.5. *Under assumptions* (H1)–(H5), *Equation* (1.2) *has a unique mild solution* $u \in PAA(\mathbb{X})$ *whenever* L *is small enough.*

Proof. Consider the nonlinear operator Γ defined by

$$(\Pi u)(t) = -f(t, u(t)) + \int_{-\infty}^{t} U(t,s) g(s, u(s)) ds - \int_{-\infty}^{t} A(s) U(t,s) f(s, u(s)) ds$$

for each $t \in \mathbb{R}$. Using the proofs of Lemma 4.3 and 4.4 as well as the composition of pseudo almost automorphic function for Lipschitzian function [24, Theorem 2.4], one can easily see that Λ maps $PAA(\mathbb{X})$ into $PAA(\mathbb{X})$. To complete the proof, it suffices to apply the Banach fixed-point theorem to the nonlinear operator Π. For that, note that for all $u, v \in PAA(\mathbb{X})$,

$$\|\Pi u - \Pi v\|_\infty \leq d \|u - v\|_\infty$$

where

$$d := L\Big[M\delta^{-1} + C\Big(1 + \int_0^\infty e^{-\omega s} H(s) ds\Big)\Big].$$

Therefore, (1.2) has a unique fixed-point $u \in PAA(\mathbb{X})$ whenever L is small enough, that is, i.e. $d < 1$, or

$$L < \Big[M\delta^{-1} + C\Big(1 + \int_0^\infty e^{-\omega s} H(s) ds\Big)\Big]^{-1}.$$

□

5. Example

Fix $\gamma \in \mathbb{U}$ and $p > 1$. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open bounded subset with C^2 boundary $\Gamma = \partial\Omega$ and let $\mathbb{X} = L^2(\Omega)$ equipped with its natural topology $\|\cdot\|_2$.

In this section we study the existence and uniqueness of a pseudo almost automorphic solution to the heat equation with a negative time-dependent diffusion coefficient given by

$$\frac{\partial}{\partial t}\Big[u(t,x) + F(t, u(t,x))\Big] = -a(t,x)\Delta u(t,x) + G(t, u(t,x)), \quad \text{in } \mathbb{R} \times \Omega \tag{5.1}$$

$$u = 0, \quad \text{on } \mathbb{R} \times \Gamma \tag{5.2}$$

where $F, G : \mathbb{R} \times L^2(\Omega \to L^2(\Omega)$ are S^p_γ-pseudo almost automorphic and jointly continuous, the function $(t,x) \to a(t,x)$ is jointly continuous, $x \to a(t,x)$ is differentiable for all $t \in \mathbb{R}$, $t \to a(t,x)$ is ω-periodic ($\omega > 0$) in the sense that

$$a(t+\omega, x) = a(t,x)$$

for all $t \in \mathbb{R}$ and $x \in \Omega$, and the following assumptions hold:

- (H6) $\inf_{t \in \mathbb{R}, x \in \Omega} a(t,x) = m_0 > 0$, and
- (H7) there exists $d > 0$ and $0 < \mu \leq 1$ such that $|a(t,x) - a(s,x)| \leq d|s-t|^\mu$ for all $t, s \in \mathbb{R}$ uniformly in $x \in \Omega$.

The problem is quite interesting as the system given by (5.1)-(5.2) models among other things the heat conduction in the domain $\mathbb{R} \times \Omega \subset \mathbb{R} \times \mathbb{R}^N$. Namely, solutions $u(t,x)$ to this system represent the temperature at position $x \in \Omega$ at time $t \in \mathbb{R}$.

Define the linear operators $A(t)$ appearing in (5.1)–(5.2) as follows:

$$A(t)u = -a(t,x)\Delta u \quad \text{for all } u \in D(A(t)) = \mathbb{D} = H_0^1(\Omega) \cap H^2(\Omega).$$

Under previous assumptions, it is clear that the operators $A(t)$ defined above are invertible and satisfy Acquistapace-Terreni conditions. Clearly, the system

$$u'(t) = A(t)u(t), \quad t \geq s,$$
$$u(s) = \varphi \in L^2(\Omega),$$

has an associated evolution family $(U(t,s))_{t\geq s}$ on $L^2(\Omega)$, which satisfies: there exist $\omega_0 > 0$ and $M \geq 1$ such that

$$\|U(t,s)\|_{B(L^2(\Omega))} \leq Me^{-\omega_0(t-s)} \quad \text{for every } t \geq s.$$

Moreover, since $A(t+\omega) = A(t)$ for all $t \in \mathbb{R}$, it follows that

$$U(t+\omega, s+\omega) = U(t,s), \quad A(s+\omega)U(t+\omega, s+\omega) = A(s)U(t,s)$$

for all $t, s \in \mathbb{R}$ with $t \geq s$. Therefore, $(t,s) \mapsto U(t,s)w$ belongs to $bAA(\mathbb{T}, L^2(\Omega))$ uniformly in $w \in L^2(\Omega)$ and $(t,s) \mapsto A(s)U(t,s)w$ belongs to $bAA(\mathbb{T}, \mathbb{D})$ uniformly in $w \in \mathbb{D}$. It is also clear that (H2) holds.

In this section, we take $\mathbb{Y} = (\mathbb{D}, \|\cdot\|_{gr(\Delta))})$ where $\|\cdot\|_{gr(\Delta)}$ is the graph norm of the N-dimensional Laplace operator Δ with domain $\mathbb{D}$ defined by

$$\|u\|_{gr(\Delta)} = \|u\|_2 + \|\Delta u\|_2$$

for all $u \in \mathbb{D}$. Clearly, the bound of the embedding $H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow L^2(\Omega)$ is $C = 1$.

We need the following additional assumption:

(H8) The functions $F \in PAA(\mathbb{R} \times L^2(\Omega), H_0^1(\Omega) \cap H^2(\Omega))$ and $G \in PAA_\gamma^p(\mathbb{R} \times L^2(\Omega), L^2(\Omega)) \cap C(\mathbb{R} \times L^2(\Omega), L^2(\Omega))$. Moreover, there exits $L > 0$ such that

$$\|F(t,u) - F(t,v)\|_{gr(\Delta)} \leq L\|u - v\|_2$$

for all $u, v \in L^2(\Omega)$ and $t \in \mathbb{R}$, and

$$\|G(t,u) - G(t,v)\|_2 \leq L\|u - v\|_2$$

for all $u, v \in L^2(\Omega)$ and $t \in \mathbb{R}$.

Theorem 5.1. *Under assumptions* (H6)–(H8)*, the heat equation* (5.1)-(5.2)*, with time-dependent diffusion coefficient, has a unique solution $u \in PAA(L^2(\Omega))$ whenever L is small enough.*

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