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EXISTENCE AND UNIFORM ASYMPTOTIC STABILITY FOR AN ABSTRACT DIFFERENTIAL EQUATION WITH INFINITE DELAY

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Abstract. Using the Contraction Mapping Principle, we study the existence, uniqueness, and uniform asymptotic stability of solutions to an abstract differential equation with infinite delay of the form $du(t)/dt + Au(t) = B(t, u_t)$, where A is a positive sectorial operator and the nonlinear part B is Lipschitz continuous with respect to a fractional power of A in the second variable and the Lipschitz coefficient may depend on time t. Some special cases and examples are provided to illustrate the results obtained.

1. Introduction

The study of functional differential equations is motivated by the fact that when one wants to model some evolution phenomena arising in physics, biology, engineering, etc., some hereditary characteristics such as aftereffect, time lag and time delay can appear in the variables. Typical examples arise from the researches of materials with thermal memory, biochemical reactions, population models, etc (see e.g. [14, 29]). One of the most important and interesting problem in the analysis of functional differential equations is to study the stability of solutions. This theory has been greatly developed over the previous years for both ordinary differential equations (ODEs) with delay and partial differential equations (PDEs) with delay.

PDEs with delay are often considered in the model such as maturation time for population dynamics in mathematical biology and other fields. Such equations are naturally more difficult than ODEs with delay since they are infinite dimensional both in time and space variables. As mentioned in [12], the stability analysis of PDEs with delay is essentially complicated. In recent years, the existence and stability of solutions to partial functional differential equations with delay has attracted widespread attraction. The development was initiated for equations with finite delay by Travis and Webb [25, 26], and later by many other authors (see the monograph [29] and references therein). The problem for equations with infinite delay was discussed recently by Henriquez, Adimy et. al. (see e.g. [15, 1, 2, 3]). It is noticed that in these works, the delay term $B(t, u_t)$ is usually assumed to be Lipschitz continuous with respect to u_t, with the Lipschitz coefficient independent

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of time t. Moreover, when studying the stability of the zero solution in the case of infinite delay, the authors required that B does not depend on t explicitly and is differentiable with respect to u_t. Then the stability of the origin equation is deduced from the stability of the linearized equation $du(t)/dt + Au(t) = B'(u_t)$. Another approach for the stability problem for PDEs with delay is using Lyapunov functions, see for instance [7, 8, 9, 10, 12, 21, 27] for some recent works, but the later approach seems to be difficult to use in the case of infinite delay.

In this article, we are concerned with the existence, uniqueness and uniform asymptotic stability of global solutions for the partial functional differential equation with infinite delay,

$$\begin{gathered} \frac{du(t)}{dt} + Au(t) = B(t, u_t), \quad t > 0 \\ u_0(t) = \phi(t), \quad t \le 0. \end{gathered} \tag{1.1}$$

We shall make the following assumptions on the operator A and the nonlinearity B:

- (H1) A is a positive sectorial operator on a Banach space $(E, \|.\|)$ with associated analytic semigroup $T(t)$ and a family of fractional power spaces $D(A^\alpha)$ (see Sect. 2.1 for more details).
- (H2) The nonlinear term $B : \mathbb{R}^+ \times D(A^\alpha) \to E$ satisfies
$$\|B(t, \phi) - B(t, \psi)\| \le L(t)\|\phi - \psi\|_{\mathcal{B}}, \quad \forall \phi, \psi \in \mathcal{B},\ t \ge 0,$$
where $L(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative measurable function in $L^p_{\rm loc}(\mathbb{R}^+)$ with $p > \frac{1}{1-\alpha}$, $0 \le \alpha < 1$.

Here ϕ is an element in a phase space $\mathcal{B}$ of functions from $(-\infty, 0]$ into $D(A^\alpha)$, which will be specified later. For each $u : (-\infty, T] \to D(A^\alpha)$, $T > 0$, and $t \in [0, T]$, u_t denotes, as usual, the element of $\mathcal{B}$ defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in (-\infty, 0]$.

It is known that there are numerous technical difficulties in dealing with partial differential equations with infinite delay due to the unboundedness of the delay involved. To overcome these difficulties, in this paper we exploit the fixed point method to prove the existence and asymptotic stability of the solution. The idea of using the fixed point method to study the stability problem for ordinary and functional differential equations was initiated by Burton and Furumochi [5] and developed later by other authors for many types of (functional) differential and integro-differential equations (see, for example, [6, 16, 18, 22, 23, 30]). A new feature in our paper is that we are able to use this method to prove the existence of a mild solution when the Lipschitz coefficient depending on time t, and study the stability of the zero solution of partial differential equations with infinite delay.

The article is organized as follows. In Section 2, for convenience of readers, we recall axioms and some examples of the phase space $\mathcal{B}$, and some properties of fractional power spaces and fractional power operators generated by the operator A. The existence and uniqueness of a mild solution to (1.1) are proved in Section 3, and the uniform asymptotic stability of the zero solution is studied in Section 4 by using the Contraction Mapping Principle. In the last section, we provide some special cases and examples to illustrate the results obtained.

2. Preliminaries

2.1. **Phase space.** In the literature devoted to equations with finite delay, the state space is the space of all continuous functions on $[-r, 0]$, $r > 0$, endowed with the uniform norm topology. When the delay is infinite, the selection of the state space plays an important role in the study of both qualitative and quantitative theory. A usual choice is a seminormed space satisfying suitable axioms introduced by Hale and Kato [13], and considered later by Kappel and Schappacher [19], and Schumacher [24]. For a detailed discussion on this topic, we refer the reader to the book by Hino et. al. [17]. In what follows we introduce Axioms and some examples of the phase space $\mathcal{B}$ which will be used in the paper.

Assume that E is a real Banach space with a norm $\|\cdot\|_E$. We will assume that the phase space $\mathcal{B}$ is a linear space of maps from $(-\infty, 0]$ into E, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and satisfying the following fundamental axioms:

- (A1) If $x : (-\infty, \sigma + a) \to E$, $a > 0$, such that $x_\sigma \in \mathcal{B}$ and $x(\cdot)$ is continuous on $[\sigma, \sigma + a)$, then for all t in $[\sigma, \sigma + a)$ the following conditions hold:
  - (1) $x_t \in \mathcal{B}$,
  - (2) $\|x(t)\|_E \leq H\|x_t\|_{\mathcal{B}}$,
  - (3) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\|_E : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

  where H is a constant, and the functions $K(\cdot)$, $M(\cdot) : [0, +\infty) \to [0, +\infty)$, with K continuous and M locally bounded, and they are independent of x.
- (A2) For the function $x(\cdot)$ in (A1), $t \mapsto x_t$ is a $\mathcal{B}$-valued continuous function for t in $[\sigma, \sigma + a)$.
- (B1) The space $\mathcal{B}$ is complete.

Remark 2.1 ([17])**.** From above Axioms, we note that

- Axiom $(A1)(2)$ is equivalent to
$$|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}, \quad \text{for every } \phi \in \mathcal{B}. \tag{2.1}$$
- Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$. But, from (2.1), we see that $\phi, \psi \in \mathcal{B}$ and $\|\phi - \psi\|_{\mathcal{B}} = 0$ implies that $\phi(0) = \psi(0)$.
- Axiom (B1) is equivalent to saying that the space of equivalence classes
$$\hat{\mathcal{B}} = \mathcal{B}/\|\cdot\|_{\mathcal{B}} = \{\hat{\phi} : \phi \in \mathcal{B}\}$$
is a Banach space.

Let us give some examples of concrete functional spaces that verify Axioms (A1), (A2), and (B1).

Example 2.2. For any continuous function $g : (-\infty, 0] \to (0, +\infty)$, let
$$C_g^0 := \big\{\phi \in C((-\infty, 0]; E) : \lim_{\theta \to -\infty} \frac{|\phi(\theta)|}{g(\theta)} = 0\big\},$$
endowed with the norm
$$\|\phi\|_g := \sup_{-\infty < \theta \leq 0} \frac{|\phi(\theta)|}{g(\theta)}.$$
It was proved in [17, Theorems 1.3.2 and 1.3.6] that if g is nonincreasing, then $\big(C_g^0, \|\cdot\|_g\big)$ satisfies Axioms (A1), (A2), and (B1).

In the special case $g(\theta) = e^{-\gamma\theta}$, $\gamma > 0$, we have the following example.

Example 2.3. The above axioms are satisfied by the space
$$C_\gamma^0 = \big\{\phi \in C((-\infty, 0]; E) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) = 0\big\}, \quad \gamma > 0,$$

with the norm $\|\phi\|_\gamma = \sup_{\theta\le 0} e^{\gamma\theta}|\phi(\theta)|$, $\phi \in C^0_\gamma$. It is satisfied, in general, by the space

$$C_\gamma = \big\{\phi \in C((-\infty,0];E) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } E\big\},\ \gamma > 0,$$

and set

$$\|\phi\|_\gamma = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}|\phi(\theta)|, \quad \text{for } \phi \text{ in } C_\gamma.$$

For this space, as shown in [17, Theorem 3.7, p. 23], we can take $H = 1$, $K(t) = 1$, and $M(t) = e^{-\gamma t}$.

Remark 2.4. For the space C_γ, instead of Axiom (A1)(3), we have

$$\|x_t\|_{\mathcal{B}} \le \max\big\{K(t)\sup_{0\le s\le t}\|x(s)\|_E, M(t)\|x_0\|_{\mathcal{B}}\big\} \quad \text{for all } t \ge 0,$$

where $K(t) = 1$ and $M(t) = e^{-\gamma t}$. Indeed, for all $t \ge 0$,

$$\begin{aligned}
\|x_t\|_\gamma &= \sup_{\theta\le 0} e^{\gamma\theta}\|x(t+\theta)\| = e^{-\gamma t}\sup_{\theta\le t} e^{\gamma\theta}\|x(\theta)\| \\
&= e^{-\gamma t}\max\big\{\sup_{\theta\le 0} e^{\gamma\theta}\|x(\theta)\|, \sup_{0\le\theta\le t} e^{\gamma\theta}\|x(\theta)\|\big\} \\
&= \max\big\{\sup_{0\le\theta\le t} e^{-\gamma(t-\theta)}\|x(\theta)\|, e^{-\gamma t}\|x_0\|_{\mathcal{B}}\big\} \\
&\le \max\big\{\sup_{0\le\theta\le t}\|x(\theta)\|, e^{-\gamma t}\|x_0\|_{\mathcal{B}}\big\}.
\end{aligned}$$

2.2. **Operator.** Let A be a positive sectorial operator on a Banach space E. We now recall some results in [20].

Let $T(t)$ be the analytic semigroup generated by $-A$. It is known that there exists a positive number $\lambda > 0$ such that

$$\|T(t)x\| \le Ce^{-\lambda t}\|x\|, \quad \text{for } t \ge 0,\ x \in E. \tag{2.2}$$

For $\alpha > 0$, we define

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T(t)dt.$$

We have, $A^{-\alpha}$ is one-to-one, hence, we can define $A^\alpha = (A^{-\alpha})^{-1}$. For $\alpha = 0$, we also define $A^0 = I$, where I is the identity of E.

Proposition 2.5. *We have*

(1) *The operator A^α is a densely defined closed linear operator with the domain $D(A^\alpha) = R(A^{-\alpha})$, the range of the operator $A^{-\alpha}$;*
(2) *$D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A^\alpha x\|$, $x \in D(A^\alpha)$;*
(3) *For $\alpha \ge \beta$, one has $D(A^\alpha) \subset D(A^\beta)$ and $D(A^\alpha)$ is dense in $D(A^\beta)$. If in addition, A has compact resolvent, then one has $D(A^\alpha) \subset\subset D(A^\beta)$, whenever $\alpha > \beta$;*
(4) *One has*
$$A^\alpha A^\beta x = A^\beta A^\alpha x = A^{\alpha+\beta}x,$$
for every $x \in D(A^\gamma)$, where $\gamma = \max(\alpha,\beta,\alpha+\beta)$;
(5) *$T(t) : E \to D(A^\alpha)$ for every $t > 0$ and $\alpha \ge 0$;*
(6) *For any $\alpha \ge 0$, we have*
$$\|A^\alpha T(t)x\| \le C_\alpha e^{-\lambda t}t^{-\alpha}\|x\|, \quad \textit{for all } t > 0, x \in E.$$

We now give a typical example of the operator A (see, e.g., [11]). Assume A is a densely-defined self-adjoint positive linear operator and with compact resolvent in a separable Hilbert space E (for example, $A = -\Delta_D$ with the homogeneous Dirichlet condition).

It is known that A has a discrete spectrum that only contains positive eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ satisfying

$$0 < \lambda_1 \le \lambda_2 \le \dots, \quad \lambda_k \to \infty, \quad \text{as } k \to \infty,$$

and the corresponding eigenfunctions $\{e_k\}_{k=1}^{\infty}$ compose an orthonormal basis of the Hilbert space E such that

$$(e_j, e_k) = \delta_{jk} \text{ and } Ae_k = \lambda_k e_k, \quad k = 1, 2, \dots$$

Hence we can define the fractional power spaces and operators as

$$D(A^\alpha) = \{u = \sum_{k=1}^{\infty} c_k e_k \in E : \sum_{k=1}^{\infty} c_k^2 \lambda_k^{2\alpha} < \infty\},$$
$$A^\alpha u = \sum_{k=1}^{\infty} c_k \lambda_k^\alpha e_k, \text{ where } u = \sum_{k=1}^{\infty} c_k e_k.$$

It is known that if $\alpha > \beta$ then the space $D(A^\alpha)$ is compactly embedded into $D(A^\beta)$. We have, $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge 0}$. Furthermore, we have the following estimates for all $t > 0$ (see [11]),

$$\|T(t)x\| \le e^{-\lambda_1 t}\|x\|,$$
$$\|A^\alpha T(t)x\| \le \Big[\big(\frac{\alpha}{t}\big)^\alpha + \lambda_1^\alpha\Big] e^{-\lambda_1 t}\|x\|,$$
$$\|A^\alpha T(t)x\| \le \alpha^\alpha e^{-\alpha} t^{-\alpha}\|x\|.$$

3. Existence of solutions

Definition 3.1. We say that a function $u : (-\infty, T] \to D(A^\alpha)$, $T > 0$, is a mild solution (in $D(A^\alpha)$) of the Cauchy problem (1.1) on the interval $[0, T]$ if $u_0 = \phi$ and the restriction $u : [0, T] \to D(A^\alpha)$ is continuous and satisfies the integral equation:

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)B(s, u_s)ds, \quad 0 \le t \le T.$$

In the rest of this work we will abbreviate our terminology calling solutions to the mild solutions.

Theorem 3.2. *Suppose* (H1)–(H2) *hold. Then for each* $\phi \in \mathcal{B}$ *and* $T > 0$ *given, there is a unique mild solution of* (1.1) *on the interval* $[0, T]$.

The proof of the above theorem is based on the following lemma, whose proof is straightforward so we omit it.

Lemma 3.3. *For* $L(\cdot)$ *and* p *as in* (H2), $K(\cdot)$ *as in* (A1)(3). *Let*

$$\eta(t) = [L(t)K(t)]^p, \ t \ge 0.$$

For each real number $0 < \kappa < 1$, *set*

$$g_\kappa(t) = \exp\big\{\frac{1}{\kappa}\int_0^t \eta(s)ds\big\}, \quad t \ge 0. \tag{3.1}$$

Then the following statements hold:

(1) *The function* η *is a non-negative function in* $L^1_{\rm loc}(\mathbb{R}^+)$*;*
(2) *The function* $g_\kappa(\cdot)$ *is monotonically increasing and greater than or equal to* 1 *on interval* $[0,+\infty)$*;*
(3) *For all* $t \geq 0$*, we have*

$$\int_0^t \eta(s)[g_\kappa(s)]^p ds \leq \frac{\kappa}{p}[g_\kappa(t)]^p;$$

(4) *For all* $T > 0$*,* $\mathcal{C} = C([0,T];E)$ *is a Banach space with the norm*

$$|||x||| = \sup_{0\leq t\leq T}\big\{\frac{\|x(t)\|_E}{g_\kappa(t)}\big\}, \quad \text{for all } x \in \mathcal{C}, \tag{3.2}$$

and this norm is equivalent to the usual supremum norm.

Remark 3.4. The introduction of the function g_κ for global existence problems was due to Bielecki [4]. It plays the same role as Gronwall's inequality.

Proof of Theorem 3.2. Define

$$\mathcal{C}_\phi = \{x \in C([0,T];E) : x(0) = A^\alpha\phi(0)\},$$

then $\mathcal{C}_\phi$ is a closed subset of the Banach space $\mathcal{C} = C([0,T];E)$ with the norm $|||\cdot|||$. For $x \in \mathcal{C}_\phi$, define the map

$$(Fx)(t) = A^\alpha T(t)\phi(0) + \int_0^t A^\alpha T(t-s)B(s, A^{-\alpha}x_s)ds, \quad 0 \leq t \leq T.$$

Clearly, $Fx \in \mathcal{C}_\phi$. We now prove that F is a contracting map on $\mathcal{C}_\phi$. Given x and y in $\mathcal{C}_\phi$, we have for all $t \in [0,T]$,

$$\begin{aligned}
&\|(Fx)(t) - (Fy)(t)\| \\
&\leq \int_0^t \|A^\alpha T(t-s)\|.\|B(s, A^{-\alpha}x_s) - B(s, A^{-\alpha}y_s)\|ds \\
&\leq \int_0^t C_\alpha (t-s)^{-\alpha}e^{-\lambda(t-s)}L(s)\|A^{-\alpha}x_s - A^{-\alpha}y_s\|_{\mathcal{B}}ds \\
&\leq C_\alpha \int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}L(s)K(s)\sup_{0\leq\tau\leq s}\|A^{-\alpha}x(\tau) - A^{-\alpha}y(\tau)\|_\alpha ds \\
&= C_\alpha \int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}L(s)K(s)\sup_{0\leq\tau\leq s}\|x(\tau) - y(\tau)\|ds \\
&= C_\alpha \int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}L(s)K(s)g_\kappa(s)\sup_{0\leq\tau\leq s}\frac{\|x(\tau) - y(\tau)\|}{g_\kappa(s)}ds \\
&\leq C_\alpha \sup_{0\leq t\leq T}\frac{\|x(t) - y(t)\|}{g_\kappa(t)}\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}L(s)K(s)g_\kappa(s)ds \\
&\leq C_\alpha|||x-y|||\Big(\int_0^t (t-s)^{-q\alpha}e^{-\lambda q(t-s)}ds\Big)^{1/q}\Big(\int_0^t \eta(s)[g_\kappa(s)]^p ds\Big)^{1/p},
\end{aligned} \tag{3.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. We use the Gamma function formula

$$\Gamma(1-\alpha)k^{\alpha-1} = \int_0^\infty e^{-ks}s^{-\alpha}ds$$

to see that

$$\begin{aligned}\int_0^t (t-s)^{-q\alpha}e^{-\lambda q(t-s)}ds &= \int_0^t s^{-q\alpha}e^{-q\lambda s}ds \le \int_0^\infty s^{-q\alpha}e^{-q\lambda s}ds\\ &= \Gamma(1-q\alpha)(q\lambda)^{q\alpha-1}\end{aligned}$$

and

$$\int_0^t \eta(s)[g_\kappa(s)]^p ds \le \frac{\kappa}{p}[g_\kappa(t)]^p.$$

Plug them into (3.3), we obtain

$$\|(Fx)(t)-(Fy)(t)\| \le C_\alpha\Gamma(1-q\alpha)^{1/q}(q\lambda)^{\frac{q\alpha-1}{q}}p^{-1/p}\kappa^{1/p}g_\kappa(t)|||x-y|||.$$

So, we can choose κ small enough so that

$$C_\alpha\Gamma(1-q\alpha)^{1/q}(q\lambda)^{\frac{q\alpha-1}{q}}p^{-1/p}\kappa^{1/p} = k < 1$$

to obtain

$$|||Fx-Fy||| \le k|||x-y|||.$$

By the Contraction Mapping Theorem, the map F has a unique fixed point $x \in \mathcal{C}_\phi$. This fixed point satisfies the integral equation

$$x(t) = A^\alpha T(t)\phi(0) + \int_0^t A^\alpha T(t-s)B(s, A^{-\alpha}x_s)ds, \quad \text{for } 0 \le t \le T.$$

Define

$$u(t) = \begin{cases} A^{-\alpha}x(t), & 0 \le t \le T\\ \phi(t), & t \le 0.\end{cases}$$

Obviously, $u(t) \in D(A^\alpha)$ for all $t \in (-\infty, T]$, and $u \in C([0,T]; D(A^\alpha))$. Furthermore, since $A^{-\alpha} \in B(E,E)$, $u(t)$ is the unique solution of (1.1). $\square$

4. Uniform asymptotic stability of solutions

In this section we assume $B(t,0)=0$ for all $t \ge 0$, so that $u(t) \equiv 0$ is a solution of problem (1.1) with zero initial condition.

Theorem 4.1. *Assume that conditions* (H1)–(H2) *and the following conditions hold:*

(H3)
$$\sup_{t\ge 0}\int_0^t C_\alpha e^{-\lambda(t-s)}(t-s)^{-\alpha}L(s)[K(s)+\frac{M(s)}{H}]ds \le \kappa < 1;$$

(H4) *for all $\epsilon > 0$ and $t_1 \ge 0$, there exists a $t_2 > t_1$ such that*
$$\|B(t,u_t)\| \le L(t)\Big(\epsilon + \sup_{s\in[t_1,t]}\|u(s)\|_\alpha\Big), \quad \text{for all } t \ge t_2.$$

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

Proof. Let $\ell > 0$, $\epsilon > 0$ be given, we can find $\delta > 0$ such that $\delta + \kappa\ell \le \ell$ and $\delta < \epsilon$. Let $\phi \in \mathcal{B}$ be a given function with $\|\phi\|_{\mathcal{B}} < \min\{\frac{\delta}{CH}, \frac{\delta}{H}\}$ (C is the constant in (2.2)) and let

$$\mathcal{S}_{\ell,\phi} = \{u : \mathbb{R} \to D(A^\alpha) : u \in C(\ell),\ u(t) = \phi(t) \text{ if } t \le 0,\ u(t) \to 0 \text{ as } t \to +\infty\},$$

where

$$C(\ell) = \{u : \mathbb{R}^+ \to D(A^\alpha) \text{ is continuous and } \|u(t)\|_\alpha \le \ell\}.$$

Define $P : \mathcal{S}_{\ell,\phi} \to \mathcal{S}_{\ell,\phi}$ by

$$(Pu)(t) = \begin{cases} T(t)\phi(0) + \int_0^t T(t-s)B(s,u_s)ds & \text{if } t \ge 0 \\ \phi(t) & \text{if } t \le 0. \end{cases}$$

Clearly, $(Pu) : \mathbb{R}^+ \to D(A^\alpha)$ is continuous and $(Pu)(t) = \phi(t)$ for all $t \le 0$. We now show that $\|(Pu)(t)\|_\alpha \le \ell$ for all $t \ge 0$ and $(Pu)(t) \to 0$ as $t \to \infty$. First, noting that for all $t \ge 0$,

$$\begin{aligned}
&\|(Pu)(t)\|_\alpha \\
&\le \|T(t)\phi(0)\|_\alpha + \int_0^t \|T(t-s)B(s,u_s)\|_\alpha ds \\
&\le Ce^{-\lambda t}\|\phi(0)\|_\alpha + \int_0^t C_\alpha e^{-\lambda(t-s)}(t-s)^{-\alpha}\|B(s,u_s)\|ds \\
&\le CHe^{-\lambda t}\|\phi\|_{\mathcal{B}} + \int_0^t C_\alpha e^{-\lambda(t-s)}(t-s)^{-\alpha}L(s)\|u_s\|_{\mathcal{B}}ds \\
&\le CH\|\phi\|_{\mathcal{B}} + \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}L(s)\Big[K(s)\sup_{0\le\tau\le s}\|u(s)\|_\alpha + M(s)\|u_0\|_{\mathcal{B}}\Big]ds \\
&\le \delta + \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}L(s)[K(s)\ell + M(s)\frac{\delta}{H}]ds \quad (\text{since } u_0 = \phi \text{ on } (-\infty,0]) \\
&\le \delta + \ell\int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}L(s)[K(s) + \frac{M(s)}{H}]ds \quad (\text{since } \delta \le \ell) \\
&\le \delta + \ell\kappa \le \ell.
\end{aligned}$$

Next, we show that $(Pu)(t) \to 0$ as $t \to +\infty$. Since $u(t) \to 0$ as $t \to +\infty$, there exists $t_1 > 0$ such that $\|u(t)\|_\alpha < \epsilon$ for all $t \ge t_1$. Since $\|u(t)\|_\alpha \le \ell$ for all $t \in \mathbb{R}^+$, by (H4), there is a $t_2 > t_1$ such that for all $t \ge t_2$,

$$\|B(t,u_t)\| \le L(t)\Big(\epsilon + \sup_{s\in[t_1,t]}\|u(s)\|_\alpha\Big) \le L(t)(\epsilon+\epsilon) = 2\epsilon L(t).$$

Therefore, for $t \ge t_2$, we have

$$\begin{aligned}
&\int_0^t \|T(t-s)B(s,u_s)\|_\alpha ds \\
&\le \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\|B(s,u_s)\|ds \\
&\le \int_0^{t_2} C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\|B(s,u_s)\|ds + \int_{t_2}^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\|B(s,u_s)\|ds \\
&\le \int_0^{t_2} C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}L(s)\|u_s\|_{\mathcal{B}}ds + 2\epsilon\int_{t_2}^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}L(s)ds \\
&\le \ell\int_0^{t_2} C_\alpha \frac{e^{-\lambda(t_2-s)}e^{-\lambda(t-t_2)}}{(t_2-s)^\alpha}L(s)\big(K(s) + \frac{M(s)}{H}\big)ds + 2\epsilon\kappa \\
&\le \ell e^{-\lambda(t-t_2)}\int_0^{t_2} C_\alpha \frac{e^{-\lambda(t_2-s)}}{(t_2-s)^\alpha}L(s)\big(K(s) + \frac{M(s)}{H}\big)ds + 2\epsilon\kappa \\
&\le \kappa\ell e^{-\lambda(t-t_2)} + 2\epsilon\kappa,
\end{aligned}$$

where we have used the fact that

$$\int_{t_2}^{t} C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} L(s) ds \le \kappa.$$

There exists $t_3 > t_2$ such that for all $t \ge t_3$, we have

$$\delta e^{-\lambda t} + \kappa \ell e^{-\lambda(t-t_2)} < \epsilon.$$

Thus, for $t \ge t_3$, we have

$$\|(Pu)(t)\|_\alpha \le \epsilon(1+2\kappa).$$

This implies that $(Pu)(t) \to 0$ as $t \to +\infty$, and hence $(Pu) \in \mathcal{S}_{\ell,\phi}$.

To prove that P is a contraction mapping, observe for $t \ge 0$,

$$\begin{aligned}
&\|(Pu)(t) - (Pv)(t)\|_\alpha \\
&\le \int_0^t \|T(t-s)B(s,u_s)\|_\alpha ds \\
&\le \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} \|B(s,u_s)\| ds \\
&\le \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} L(s) \|u_s - v_s\|_{\mathcal{B}} ds \\
&\le \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} L(s) K(s) \sup_{0\le\tau\le s} \|u(\tau) - v(\tau)\|_\alpha ds \\
&\le \sup_{0\le\tau\le T} \|u(\tau) - v(\tau)\|_\alpha \int_0^t C_\alpha \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} L(s) \big[K(s) + \frac{M(s)}{H}\big] ds \\
&\le \kappa \sup_{0\le\tau\le T} \|u(\tau) - v(\tau)\|_\alpha,
\end{aligned}$$

or

$$\sup_{0\le s\le T} \|(Pu)(s) - (Pv)(s)\|_\alpha \le \kappa \sup_{0\le s\le T} \|u(s) - v(s)\|_\alpha.$$

By the Contraction Mapping Principle, P has a unique fixed point u in $S_{\ell,\phi}$ which is a solution of (1.1) with $\phi \in \mathcal{B}$, $\|\phi\|_{\mathcal{B}} \le \min\{\frac{\delta}{CH}, \frac{\delta}{H}\}$, and $u(t) = u(t,\phi) \to 0$ as $t \to +\infty$.

To obtain the uniform asymptotic stability, we need to show that the zero solution of (1.1) is uniformly stable. Let $\epsilon > 0$ (with $\epsilon < \ell$) be given. Choose $\delta > 0$ such that $\delta + \kappa\epsilon < \epsilon$. If $u(t) = u(t,\phi)$ is a solution of (1.1) with $\|\phi\|_{\mathcal{B}} < \min\{\frac{\delta}{CH}, \frac{\delta}{H}\}$, then

$$u(t) = T(t)\phi(0) + \int_0^t T(t-s)B(s,u_s) ds.$$

We claim that $\|u(t)\|_\alpha < \epsilon$ for all $t > 0$. Notice that

$$\|u(0)\|_\alpha = \|\phi(0)\|_\alpha \le H\|\phi\|_{\mathcal{B}} < H.\frac{\delta}{H} = \delta < \epsilon.$$

If there exists $t^* > 0$ such that $\|u(t^*)\|_\alpha = \epsilon$ and $\|u(s)\|_\alpha < \epsilon$ for $0 \le s < t^*$, then

$$\begin{aligned}
\|u(t^*)\|_\alpha &\le Ce^{-\lambda t^*} \|\phi(0)\|_\alpha + \int_0^{t^*} C_\alpha \frac{e^{-\lambda(t^*-s)}}{(t^*-s)^\alpha} L(s) \|u_s\|_{\mathcal{B}} ds \\
&\le \delta + \kappa\epsilon < \epsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, $\|u(t)\|_\alpha < \epsilon$ for all $t > 0$. This shows that the zero solution of (1.1) is uniformly asymptotically stable. $\square$

Remark 4.2. In the case $\mathcal{B} = C_\gamma$ with $\gamma > 0$, by using Remark 2.4, condition (H3) becomes

(H3b) $\sup_{t\geq 0} \int_0^t C_\alpha e^{-\lambda(t-s)}(t-s)^{-\alpha} L(s)ds \leq \kappa < 1$.

5. Some special cases and examples

5.1. **An abstract differential equation without delay.** Consider the abstract semilinear differential equation in a Banach space E,

$$\begin{gathered} \frac{du}{dt} + Au = f(t,u), \quad t > 0 \\ u(0) = x \in D(A^\alpha), \end{gathered} \tag{5.1}$$

where A is a positive sectorial operator on E (see Sect. 2.1), and $f(\cdot,\cdot) : \mathbb{R}^+ \times D(A^\alpha) \to E$, $0 \leq \alpha < 1$, satisfies

$$\|f(t,u_1) - f(t,u_2)\| \leq L(t)\|A^\alpha(u_1 - u_2)\| \quad \text{and} \quad f(t,0) = 0$$

for all u_1 and u_2 from the domain $D(A^\alpha)$, where $L(\cdot) \in L^p_{\rm loc}(\mathbb{R}^+)$ with $p > \frac{1}{1-\alpha}$.

Consider the phase space C_γ with the norm $\|.\|_\gamma$. Since

$$f(t,u(t)) = f(t,u(t+0)) = f(t,u_t(0)),$$

we set

$$B(t,\phi) = f(t,\phi(0)) \text{ and } \phi(t) = x \quad \text{for all } t \leq 0.$$

Then, (5.1) can be rewritten as follows

$$\begin{gathered} \frac{du(t)}{dt} + Au(t) = B(t,u_t), \quad t \geq 0 \\ u_0 = \phi \in C_\gamma. \end{gathered} \tag{5.2}$$

Noting that for all ϕ, $\psi \in C_\gamma$, we have

$$\begin{aligned} \|B(t,\phi) - B(t,\psi)\| &= \|f(t,\phi(0)) - f(t,\psi(0))\| \\ &\leq L(t)\|\phi(0) - \psi(0)\|_\alpha \leq L(t)\|\phi - \psi\|_\gamma. \end{aligned}$$

Thus, Theorem 3.2 ensures the existence of a unique mild solution of (5.1) on the interval $[0,+\infty)$.

Moreover, using Theorem 4.1 and Remark 4.2, one can see that if

$$\sup_{t\geq 0} C_\alpha \int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha} L(s)ds = \kappa < 1, \tag{5.3}$$

then the zero solution of (5.1) is uniformly asymptotically stable. In particular, if $L(t) \equiv L$, then condition (5.3) holds provided that

$$\begin{aligned} \sup_{t\geq 0} C_\alpha \int_0^t e^{-\lambda(t-s)}(t-s)^{-\alpha} L(s)ds &\leq C_\alpha L \int_0^{+\infty} e^{-\lambda(t-s)}(t-s)^{-\alpha} ds \\ &= C_\alpha L \Gamma(1-\alpha)\lambda^{\alpha-1} < 1, \end{aligned}$$

that is,

$$0 < L < \frac{\lambda^{1-\alpha}}{C_\alpha \Gamma(1-\alpha)}.$$

This is exactly the result derived by Webb in [28].

5.2. **An ordinary differential equation with infinite delay.** In this section, we consider the following Volterra equation with infinite delay

$$x'(t) + ax(t) = \int_{-\infty}^{t} g(t, s, x(s))ds, \tag{5.4}$$

where a is a positive number, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, where $\Omega = \{(t, s) \in \mathbb{R}^2 : t \geq s\}$. Suppose there exists a continuous function $m : \Omega \to \mathbb{R}^+$ such that

$$|g(t, s, x) - g(t, s, y| \leq m(t, s)|x - y| \text{ and } g(t, s, 0) = 0 \text{ for all } (t, s) \in \Omega,$$

and for all $\epsilon > 0$, $t_1 \geq 0$, there exists a $t_2 > t_1$ such that $t \geq t_2$ implies

$$\int_{-\infty}^{t_1} m(t, s)ds \leq \epsilon \int_{-\infty}^{t} m(t, s)ds. \tag{5.5}$$

We will prove that if

$$\sup_{t\geq 0} \int_{0}^{t} e^{-a(t-s)} \int_{-\infty}^{0} e^{-\gamma\tau} m(s, s + \tau)d\tau ds < 1, \tag{5.6}$$

for some $\gamma > 0$, then the zero solution of (5.4), considered as an equation on $[0, \infty) \times C_\gamma$, is uniformly asymptotically stable. Indeed, we only need to verify conditions (H2), (H3b) and (H4). Put

$$\begin{aligned} B(t, x_t) &= \int_{-\infty}^{t} g(t, s, x(s))ds = \int_{-\infty}^{0} g(t, t + s, x(t + s))ds \\ &= \int_{-\infty}^{0} g(t, t + s, x_t(s))ds, \end{aligned}$$

we have

$$\begin{aligned} |B(t, x_t) - B(t, y_t)| &\leq \int_{-\infty}^{0} |g(t, t + s, x_t(s)) - g(t, t + s, y_t(s))|ds \\ &\leq \int_{-\infty}^{0} m(t, t + s)|x_t(s) - y_t(s)|ds \\ &= \int_{-\infty}^{0} e^{-\gamma s} m(t, t + s)e^{\gamma s}|x_t(s) - y_t(s)|ds \\ &= \sup_{s\leq 0} e^{\gamma s}|x_t(s) - y_t(s)| \int_{-\infty}^{0} e^{-\gamma s} m(t, t + s)ds \\ &= L(t)\|x_t - y_t\|_{C_\gamma}, \end{aligned}$$

where $L(t) = \int_{-\infty}^{0} e^{-\gamma s} m(t, t+s)ds$. This implies (H2). By Remark 4.2 and noting that $\alpha = 0$ and $\lambda = a$, condition (H3b) follows directly from assumption (5.6). Next, let $\epsilon > 0$ and $t_1 \geq 0$ be given. By (5.5), there exists a $t_2 > t_1$ such that

$$\sup_{s\leq t_1} |x(s)| \int_{-\infty}^{t_1} m(t, s)ds < \epsilon \int_{-\infty}^{t} m(t, s)ds$$

for all $t \geq t_2$. Then

$$|B(t, x_t)| \leq \int_{-\infty}^{t_1} |g(t, s, x(s))|ds + \int_{t_1}^{t} |g(t, s, x(s))|ds$$

$$
\begin{aligned}
&\le \sup_{s\le t_1}|x(s)|\int_{-\infty}^{t_1} m(t,s)ds + \sup_{s\in[t_1,t]}|x(s)|\int_{t_1}^{t} m(t,s)ds\\
&\le \epsilon\int_{-\infty}^{t} m(t,s)ds + \sup_{s\in[t_1,t]}|x(s)|\int_{-\infty}^{t} m(t,s)ds\\
&\le L(t)(\epsilon + \sup_{s\in[t_1,t]}|x(s)|).
\end{aligned}
$$

This implies that (H4) is satisfied.

5.3. **A partial differential equation with infinite delay.** In this section, we apply our abstract results to the reaction-diffusion equation with infinite delay,

$$
\begin{gathered}
\frac{\partial}{\partial t}w(t,x) = a\frac{\partial^2}{\partial x^2}w(t,x) + bw(t,x) + c\int_{-\infty}^{0} G(t+s)w(t+s,x)ds,\\
t\ge 0,\ 0<x<\pi,\\
w(t,0)=w(t,\pi)=0,\quad t\ge 0,\\
w(t,x)=w_0(t,x),\quad -\infty<t\le 0,\ 0<x<\pi,
\end{gathered} \tag{5.7}
$$

where a,b and c are positive constants, for each $t\ge 0$, $G(t+\cdot)$ is a positive integrable function on $(-\infty,0]$ and $w_0:(-\infty,0]\times[0,\pi]\to\mathbb{R}$ is an appropriate continuous function.

We choose $E=L^2(0,\pi)$, $A=-a\frac{\partial^2}{\partial x^2}$ with the Dirichlet boundary condition is a positive linear operator with discrete spectrum consisting of the simple eigenvalues $\lambda_n = an^2$. Its domain is $D(A)=H^2(0,\pi)\cap H_0^1(0,\pi)$.

Set

$$
\begin{gathered}
u(t)(x) = w(t,x),\quad t\ge 0, x\in[0,\pi],\\
\phi(s)(x) = w_0(s,x),\quad s\le 0, x\in[0,\pi],\\
B(t,\phi)(x) = b\phi(0)(x) + c\int_{-\infty}^{0} G(t+s)\phi(s)(x)ds,\quad x\in[0,\pi],\phi\in C_\gamma.
\end{gathered}
$$

Then problem (5.7) can be transformed as follows

$$
\begin{gathered}
\frac{du}{dt} + Au(t) = B(t,u_t),\quad t\ge 0\\
u_0=\phi\in C_\gamma.
\end{gathered}
$$

We assume that

(1) For all $t\ge 0$, $s\mapsto G(t+s)e^{-\gamma s}$ is integrable on $(-\infty,0]$,
(2) $\lim_{\theta\to-\infty}\big(e^{\gamma\theta}\|w_0(\theta,\cdot)\|\big)$ exists, and $w_0(0,0)=w_0(0,\pi)=0$.

We have, for every $\phi,\psi\in C_\gamma$,

$$
\begin{aligned}
&\int_{-\infty}^{0} G(t+\theta)\|\phi(\theta)(\cdot)-\psi(\theta)(\cdot)\|d\theta\\
&= \int_{-\infty}^{0} e^{-\gamma\theta}G(t+\theta)\big(e^{\gamma\theta}\|\phi(\theta)(\cdot)-\psi(\theta)(\cdot)\|\big)d\theta\\
&\le \Big(\int_{-\infty}^{0} e^{-\gamma\theta}G(t+\theta)d\theta\Big)\sup_{\theta\le 0}e^{\gamma\theta}\|\phi(\theta)(\cdot)-\psi(\theta)(\cdot)\|\\
&= \Big(\int_{-\infty}^{0} e^{-\gamma\theta}G(t+\theta)d\theta\Big)\|\phi-\psi\|_\gamma.
\end{aligned}
$$

Hence, we obtain
$$\|B(t,u_t)-B(t,v_t)\| \le L(t)\|\phi-\psi\|_\gamma,$$
where
$$L(t)=\Big(b+c\int_{-\infty}^{0} e^{-\gamma\theta}G(t+\theta)d\theta\Big).$$
Therefore, assumptions (1) and (2) imply that B is Lipschitz continuous. Consequently, Theorem 3.2 ensures the existence and uniqueness of an integral solution $w(t,x)$ on $\mathbb{R}\times[0,\pi]$.

To obtain the uniform asymptotic stability, we further suppose that

(3) $\sup_{t\ge 0}\int_0^t e^{-a(t-s)}\Big(c\int_{-\infty}^0 e^{-\gamma\theta}G(s+\theta)d\theta+b\Big)ds<1$,

and for all $\epsilon>0$, $t_1\ge 0$, there exists a $t_2>t_1$ such that for all $t\ge t_2$,
$$\int_{-\infty}^{t_1} G(s)ds \le \epsilon\int_{-\infty}^{t} G(s)ds.$$
This condition implies that condition (H3b) and (H4) hold (with $\alpha=0$). Thus, by Theorem 4.1 we conclude that the zero solution of (5.7) is uniformly asymptotically stable.

In the case $G(t,s)\equiv G(s)$ is independent of t, $L(t)\equiv\Big(b+c\int_{-\infty}^0 e^{-\gamma\theta}G(\theta)d\theta\Big)$, and the stability condition (3) becomes

(3') $c\int_{-\infty}^0 e^{-\gamma\theta}G(\theta)d\theta+b<a$.

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