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EXISTENCE OF POSITIVE SOLUTIONS AND EIGENVALUES INTERVALS FOR NONLINEAR STURM LIOUVILLE PROBLEMS WITH A SINGULAR INTERFACE

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Abstract. In this article, we define the Green's matrix for a nonlinear Sturm Liouville problem associated with a pair of dynamic equations on time scales with a singularity at the point of interface. Then using iterative techniques, we obtain eigenvalue intervals for which there exist positive solutions. Then we present iterative schemes for approximating the solutions, and discus an example that illustrates the the results obtained.

1. Introduction

Solving boundary-value problems with different types of singularities has remained a challenge for mathematicians over the ages. While "regular" problems, those over finite intervals with well-behaved coefficients pose no difficulties. There are applications where either the domain of the problem is not well defined, or the continuity and/or smoothness of the functions, coefficients involved are not guaranteed in some parts of the domain, sometimes in the boundary or parts of the boundary. In all such cases the problem is considered to be a "singular" problem. The definition of the problem and therefore the description of the solution becomes a highly difficult task. Here are quite a number of different approaches that we come across in the literature to tackle these singular problems [3, 17, 18, 19, 20, 22, 25].

In the literature we find a class of interface problems, termed as mixed pair of equations, discussed in the papers [4, 5, 6, 10, 11, 12, 13, 14, 31, 32, 33, 34, 36, 35, 15] where two different differential equations are defined on two adjacent intervals and the solutions satisfy a matching condition at the point of interface. These problems are called as matching interface problems. If the boundary is well defined then we call the problem to be a regular interface problem. These interface problems with singularities in the domain are always of great interest.

We see that these interface problems for regular case has been discussed in [4, 6, 32, 33, 34, 36, 35, 15] and the problem of having singularity at the boundary is discussed in [5, 10, 11, 12, 13, 14, 31].

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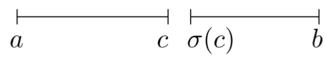
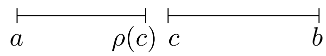
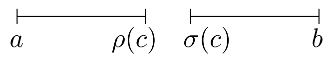
From the above we see that the regular interface problems and interface problems with singularity at the boundary are dealt in detail. But the problem of having a singularity at the point of interface seems to be less explored. Study of these problems using classical analytical tools is tedious. We term these problems as singular interface problems.

The singularity at the point of interface in the domain of definition of the mixed pair of equations could be of the following three types satisfying certain matching conditions at the singular interface.

Interface 1: $[a,c]\cup[\sigma(c),b]$ 

Interface 2: $[a,\rho(c)]\cup[c,b]$ 

Interface 3: $[a,\rho(c)]\cup[\sigma(c),b]$ 

To describe the singularities in the domain of definition we take help of the terminology used on Time Scale [16]. The new framework of the dynamic equations on time scale with facilities of the two jump operators with various definitions of continuity and derivatives make one's job simple to study the interface problems with mixed operators along with a singular interface. Recently we have worked on the linear singular interface problems as seen in [7, 8, 9], [28, 29]. Here we discuss the corresponding nonlinear problem.

The method of lower and upper solutions is one of the commonly used methods for dealing with the second order initial and boundary value problems. It has its origin as early as 1893 [24]. Also this method of lower and upper solutions clubbed with the monotone iterative technique is used in the existence theory for nonlinear problems. A good introduction covering different aspects for the monotone iterative methods is given by Lakshmikantham and others in [21].

Lower and upper solutions give bounds for solutions which are improved iteratively using monotone iterative process. This method of lower and upper solutions for separated BVPs on time scales was developed recently by Akin in [1].

Off late iterative methods have been used to prove the existence of positive solutions of nonlinear boundary value problems for ordinary differential equations [23, 26, 27, 37]. By applying iterative methods, we not only obtain the existence of positive solutions, but also establish iterative schemes for approximating the solutions.

In this paper we define the Green's matrix for a nonlinear non-homogenous Sturm Liouville boundary value problem associated with singular interface problems(NN-SL-BVP-SIP) on time scales. Using the Green's matrix we obtain eigenvalue intervals for which positive solutions exist for the NN-SL-BVP-SIP on time scales using iterative methods. We also establish iterative schemes for approximating the solutions. We present an example that illustrates the results obtained.

2. Preliminaries

An introduction on Time scale and Dynamic equations can be found in [16]. In the following section we introduce few definitions for our usage.

Definition 2.1. Let $\mathbb{T}$ be a time scale(an arbitrary closed subset of real numbers). For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered, while $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated.

Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense.

Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$.

Definition 2.2.

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} - \{m\} & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases}$$

where m is the left scattered maximum of $\mathbb{T}$.

Definition 2.3. Let f be a function defined on $\mathbb{T}$. We say that f is delta differentiable at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in \mathcal{N}.$$

Definition 2.4. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa 2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta$: $\mathbb{T}^{\kappa 2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa n} \to \mathbb{R}$.

Theorem 2.5 (Arzela-Ascoli Theorem)**.** *A subset M of $\mathcal{C}([a, b], \mathbb{R}^n)$ is relatively compact if and only if it is bounded and equicontinuous.*

Theorem 2.6 ([2])**.** *Let K be a normal cone of a Banach space E and $v_0 \leq w_0$. Let us suppose that*

- (A1) *$T : [v_0, w_0] \to E$ is completely continuous;*
- (A2) *T is monotone increasing on $[v_0, w_0]$;*
- (A3) *v_0 is a lower solution of T, that is, $v_0 \leq Tv_0$;*
- (A4) *w_0 is an upper solutions of T, that is, $Tw_0 \leq w_0$.*

Then the iterative sequences $v_n = Tv_{n-1}$ and $w_n = Tw_{n-1}$ ($n = 1, 2, 3 \dots$) satisfy

$$v_0 \leq v_1 \leq \dots \leq v_n \leq \dots \leq w_n \dots \leq w_1 \leq w_0$$

and converge to v and $w \in [v_0, w_0]$, respectively, which are fixed points of T.

3. Definition of Problem

Let $\mathbb{T}_1 = [a, \rho(c)]_\mathbb{T}$, $\mathbb{T}_2 = [\sigma(c), b]_\mathbb{T}$ where $-\infty < a, \rho(c), \sigma(c), b < +\infty$. Also let (f_1, f_2) be nonlinear function tuple in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{T}_1, \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{T}_2, \mathbb{R})$. Let $\lambda \in \mathbb{R}$. The nonlinear nonhomogenous Sturm Liouville boundary-value problem associated with singular interface problems (NN-SL-BVP-SIP) is defined by

$$y_1^{\Delta\Delta}(t) = \lambda f_1(t, y_1^\sigma), \quad t \in \mathbb{T}_1^{\kappa^2} \tag{3.1}$$

$$y_2^{\Delta\Delta}(t) = \lambda f_2(t, y_2^\sigma), \quad t \in {\mathbb{T}_2}^{\kappa^2} \tag{3.2}$$

with the boundary conditions

$$y_1(a) = 0 = y_2(b) \tag{3.3}$$

followed by the matching interface conditions

$$y_1(\rho(c)) = y_2(\sigma(c)) \tag{3.4}$$

$$y_1^\Delta(\rho(c)) = y_2^\Delta(\sigma(c)). \tag{3.5}$$

4. Green's Matrix associated with NN-SL-BVP-SIP

A proof for the following theorem can be found in [30].

Theorem 4.1. *Let $Y = (y_1, y_2)$, $F = (f_1, f_2)$. Then the NN-SL-BVP-SIP has a unique solution $Y(t)$ for which the formula*

$$Y(t) = \lambda \int_a^b G(t,s)F(s, y^\sigma)\Delta s$$

holds, where $G(t,s)$ is the Green's matrix associated with NN-SL-BVP-SIP given by $\begin{pmatrix} G_{11}(t,s) & G_{12}(t,s) \\ G_{21}(t,s) & G_{22}(t,s) \end{pmatrix}$ where

$$G_{11}(t,s) = \begin{cases} u_1 = a - t, & a \le t \le s \le \rho(c) \\ v_1 = a - s, & a \le s \le t \le \rho(c) \end{cases}$$

$$G_{22}(t,s) = \begin{cases} u_2 = s - b, & \sigma(c) \le t \le s \le b \\ v_2 = t - b, & \sigma(c) \le s \le t \le b \end{cases}$$

$$G_{12}(t,s) = \begin{cases} (a-t)(b-s), & a \le t \le \rho(c),\ \sigma(c) \le s \le b \end{cases}$$

$$G_{21}(t,s) = \begin{cases} (a-s)(b-t), & a \le s \le \rho(c),\ \sigma(c) \le t \le b \end{cases}$$

provided f_1 and f_2 satisfy the following conditions:

$$\int_a^{\rho(c)} ((a+1) - s) f_1(s, y_1^\sigma)\Delta s = \int_{\sigma(c)}^b (s - (b+1)) f_2(s, y_2^\sigma)\Delta s \tag{4.1}$$

$$[(\sigma(c)+1) - b] \int_a^{\rho(c)} (a-s) f_1(s, y_1^\sigma)\Delta s = [(a+1) - \rho(c)] \int_{\sigma(c)}^b (s-b) f_2(s, y_2^\sigma)\Delta s. \tag{4.2}$$

By $Y(t) = \lambda \int_a^b G(t,s)F(s,y^\sigma)\Delta s$, we mean

$$\begin{aligned} y_1(t) &= \lambda \Big[\int_a^b G_{11}(t,s) f_1(s, y_1^\sigma)\Delta s + \int_a^b G_{12}(t,s) f_2(s, y_2^\sigma)\Delta s \Big] \\ &= \lambda \int_a^{\rho(c)} G_{11}(t,s) f_1(s, y_1^\sigma)\Delta s + \lambda \int_{\sigma(c)}^b G_{12}(t,s) f_2(s, y_2^\sigma)\Delta s, \end{aligned}$$

for $t \in \mathbb{T}_1$; and

$$y_2(t) = \lambda \Big[\int_a^b G_{21}(t,s) f_1(s, y_1^\sigma)\Delta s + \int_a^b G_{22}(t,s) f_2(s, y_2^\sigma)\Delta s \Big]$$

$$= \lambda \int_a^{\rho(c)} G_{21}(t,s) f_1(s, y_1^\sigma)\Delta s + \lambda \int_{\sigma(c)}^b G_{22}(t,s) f_2(s, y_2^\sigma)\Delta s,$$

for $t \in \mathbb{T}_2$.

5. Preliminary Results

We define the integral operator $T : \mathcal{C}(\mathbb{T}_1 \cup \mathbb{T}_2, \mathbb{R}) \to \mathcal{C}(\mathbb{T}_1 \cup \mathbb{T}_2, \mathbb{R})$ by

$$(Ty)(t) = \begin{cases} (Ty_1)(t) = \lambda\big[\int_a^b G_{11}(t,s) f_1(s, y_1^\sigma)\Delta s + \int_a^b G_{12}(t,s) f_2(s, y_2^\sigma)\Delta s\big], & t \in \mathbb{T}_1 \\ (Ty_2)(t) = \lambda\big[\int_a^b G_{21}(t,s) f_1(s, y_1^\sigma)\Delta s + \int_a^b G_{22}(t,s) f_2(s, y_2^\sigma)\Delta s\big], & t \in \mathbb{T}_2. \end{cases}$$

We define the Banach space $E = \mathcal{C}(\mathbb{T}_1 \cup \mathbb{T}_2, \mathbb{R})$ with the supremum norm

$$\|y\| = \sup_{t\in\mathbb{T}_1} |y_1(t)| + \sup_{t\in\mathbb{T}_2} |y_2(t)|$$

and the cone $K \subset E$ as

$$K = \{y \geq 0 : y \in E\}.$$

Lemma 5.1. *Let f_1 be positive on $\mathbb{T}_1$ and f_2 be positive on $\mathbb{T}_2$. Also let $\lambda \in \mathbb{R}^-$. Then the operator $T : K \to K$ is completely continuous.*

Proof. We first show that T is continuous. We prove it by showing that T preserves convergence. Indeed let $y_n(= (y_{n1}, y_{n2}))$ be a sequence of functions in $\mathcal{C}(\mathbb{T}_1 \cup \mathbb{T}_2, \mathbb{R})$ such that they converge to $y(= (y_1, y_2))$. In other words

$$\lim_{n\to\infty} \|y_n - y\| \to 0$$

i.e., $\lim_{n\to\infty} \|(y_{n1}, y_{n2}) - (y_1, y_2)\| \to 0$. The above equation implies

$$\lim_{n\to\infty} \|(y_{n1} - y_1, y_{n2} - y_2)\| \to 0;$$

i.e., $\lim_{n\to\infty} \sup_{t_1\in\mathbb{T}_1} |(y_{n1} - y_1)(t_1)| \to 0$ and $\lim_{n\to\infty} \sup_{t_2\in\mathbb{T}_2} |(y_{n2} - y_2)(t_2)| \to 0$. Now with $\|T(y_n) - T(y)\| = \sup_{t\in\mathbb{T}_1} |T(y_{n1} - y_1)(t)| + \sup_{t\in\mathbb{T}_2} |T(y_{n2} - y_2)(t)|$, we see that

$$\begin{aligned}
&\sup_{t\in\mathbb{T}_1} |T(y_{n1} - y_1)(t)| \\
&\leq \sup_{t\in\mathbb{T}_1} \lambda \Big| \int_a^b G_{11}(t,s) f_1(s, y_{n1})\Delta s - \int_a^b G_{11}(t,s) f_1(s, y_1)\Delta s \Big| \\
&+ \quad \sup_{t\in\mathbb{T}_1} \lambda \Big| \int_a^b G_{12}(t,s) f_2(s, y_{n2})\Delta s - \int_a^b G_{12}(t,s) f_2(s, y_2) \Big| \\
&\leq \sup_{t\in\mathbb{T}_1} \lambda \int_a^b G_{11}(t,s) |f_1(s, y_{n1}) - f_1(s, y_1^\sigma)|\Delta s \\
&+ \quad \sup_{t\in\mathbb{T}_1} \lambda \int_a^b G_{12}(t,s) |f_2(s, y_{n2}) - f_2(s, y_2^\sigma)|\Delta s.
\end{aligned}$$

Similarly it can be sown that

$$\begin{aligned}
\sup_{t\in\mathbb{T}_2} |T(y_{n2} - y_2)(t)| &\leq \sup_{t\in\mathbb{T}_2} \lambda \int_a^b G_{21}(t,s) |f_1(s, y_{n1}) - f_1(s, y_1^\sigma)|\Delta s \\
&\quad + \sup_{t\in\mathbb{T}_2} \lambda \int_a^b G_{22}(t,s) |f_2(s, y_{n2}) - f_2(s, y_2^\sigma)|\Delta s.
\end{aligned}$$

Since (f_1,f_2) is continuous on $\mathcal{C}(\mathbb{T}_1\times\mathbb{T}_1,\mathbb{R})\times\mathcal{C}(\mathbb{T}_2\times\mathbb{T}_2,\mathbb{R})$ we have

$$\lim_{n\to\infty}|f_1(s,y_{n1})-f_1(s,y_1^\sigma)|\to 0,$$
$$\lim_{n\to\infty}|f_2(s,y_{n2})-f_1(s,y_2)|\to 0.$$

Hence, $\lim_{n\to\infty}\|T(y_n)-T(y)\|\to 0$ proving that T is continuous. Let

$$f_1(s,y_1^\sigma)\le M_1,\ \text{for some } M_1>0,\forall s\in\mathbb{T}_1,$$
$$f_2(s,y_2^\sigma)\le M_2,\ \text{for some } M_2>0,\forall s\in\mathbb{T}_2.$$

We now show that $T(\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R}))$ is bounded and equicontinuous subset of $\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R})$. Let us assume that $y(=(y_1,y_2))\in\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R})$ and $\|y(=(y_1,y_2))\|\le M'$. Then

$$\begin{aligned}\|Ty\|\le \sup_{t_1\in\mathbb{T}_1}\lambda\Big[\int_a^b|G_{11}(t,s)||f_1(s,y_1^\sigma)|\Delta s+\int_a^b|G_{12}(t,s)||f_2(s,y_2^\sigma)|\Delta s\Big]\\ +\sup_{t_2\in\mathbb{T}_2}\lambda\Big[\int_a^b|G_{21}(t,s)||f_1(s,y_1^\sigma)|\Delta s+\int_a^b|G_{22}(t,s)||f_2(s,y_2^\sigma)|\Delta s\Big]\end{aligned}$$

Since (f_1,f_2) is bounded we can conclude that there exists a $K'>0$ independent of choice of $y(=(y_1,y_2))$ such that $\|Ty(=(y_1,y_2))\|\le K'$. Hence, $T(\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R}))$ is bounded. We next show that $T(\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R}))$ is equicontinuous subset of $\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R})$. We need to show that for all $\epsilon>0$ there exists $\delta>0$ such that whenever $\|t-t'\|<\delta$ we have $\|Ty(t)-Ty(t')\|<\epsilon$.

Now

$$\begin{aligned}\|Ty(t)-Ty(t')\|&=\sup_{t\in\mathbb{T}_1}|Ty_1(t)-Ty_1(t')|+\sup_{t_2\in\mathbb{T}_2}|Ty_2(t)-Ty_2(t')|\\ &\le\sup_{t_1\in\mathbb{T}_1}\Big[|\lambda\int_a^b(G_{11}(t,s)-G_{11}(t',s))f_1(s,y_1^\sigma)\Delta s|\Big]\\ &\quad+\sup_{t\in\mathbb{T}_1}\Big[|\lambda\int_a^b(G_{12}(t,s)-G_{12}(t',s)f_2(s,y_2^\sigma))\Delta s|\Big]\\ &\quad+\sup_{t\in\mathbb{T}_2}\Big[|\lambda\int_a^b(G_{21}(t,s)-G_{21}(t',s)f_1(s,y_1^\sigma))\Delta s|\Big]\\ &\quad+\sup_{t\in\mathbb{T}_2}\Big[|\lambda\int_a^b(G_{22}(t,s)-G_{22}(t',s)f_2(s,y_2^\sigma))\Delta s|\Big]\end{aligned}$$

Let $M=\max\{M_1,M_2\}$. Then we have

$$\begin{aligned}\|Ty(t)-Ty(t')\|&\le M\sup_{t\in\mathbb{T}_1}\Big[|\lambda\int_a^b(G_{11}(t,s)-G_{11}(t',s))\Delta s|\Big]\\ &\quad+M\sup_{t\in\mathbb{T}_1}\Big[|\lambda\int_a^b(G_{12}(t,s)-G_{12}(t',s))\Delta s|\Big]\\ &\quad+M\sup_{t\in\mathbb{T}_2}\Big[|\lambda\int_a^b(G_{21}(t,s)-G_{21}(t',s))\Delta s|\Big]\\ &\quad+M\sup_{t\in\mathbb{T}_2}\Big[|\lambda\int_a^b(G_{22}(t,s)-G_{22}(t',s))\Delta s|\Big]\end{aligned}$$

We see that

$$
\begin{aligned}
&M \sup_{t\in\mathbb{T}_1} \Big[|\lambda \int_a^b (G_{11}(t,s) - G_{11}(t',s))\Delta s|\Big] \\
&= M \sup_{t\in\mathbb{T}_1} \Big[|\lambda \int_a^t (a-s)\Delta s - \lambda \int_a^{t'} (a-s)\Delta s + \int_t^{\rho(c)} (a-t)\Delta s \\
&\quad - \lambda \int_{t'}^{\rho(c)} (a-t')\Delta s|\Big] \\
&= M \sup_{t\in\mathbb{T}_1} \Big(|\lambda(t-t')\Big[a - \frac{1}{2}(t+t')\Big] + \lambda(t-t')[(t+t') - (a+\rho(c))]|\Big) \\
&\le M \sup_{t\in\mathbb{T}_1} \Big(|t-t'||\lambda\Big[\frac{1}{2}(t+t') - \rho(c)\Big]|\Big)
\end{aligned}
$$

Also

$$
\begin{aligned}
&M \sup_{t\in\mathbb{T}_1} \Big[|\lambda \int_a^b (G_{12}(t,s) - G_{12}(t',s))\Delta s|\Big] \\
&= M \sup_{t\in\mathbb{T}_1} \Big[|\lambda \int_{\sigma(c)}^b (a-t)(b-s)\Delta s - \lambda \int_{\sigma(c)}^b (a-t')(b-s)\Delta s|\Big] \\
&\le M|\lambda| \sup_{t\in\mathbb{T}_1} \Big[\int_{\sigma(c)}^b |t-t'|(b-s)\Delta s\Big] \\
&= M|\lambda| \sup_{t\in\mathbb{T}_1} |t-t'| \int_{\sigma(c)}^b \Delta s
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&M \sup_{t\in\mathbb{T}_2} \Big[|\lambda \int_a^b (G_{21}(t,s) - G_{21}(t',s))\Delta s|\Big] \\
&= M \sup_{t\in\mathbb{T}_2} \Big[|\lambda \int_a^{\rho(c)} (a-s)(b-t)\Delta s - \lambda \int_a^{\rho(c)} (a-s)(b-t')\Delta s \\
&\le M|\lambda| \sup_{t\in\mathbb{T}_2} |t-t'| \int_a^{\rho(c)} (a-s)\Delta s
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
&M \sup_{t\in\mathbb{T}_2} \Big[|\lambda \int_a^b (G_{22}(t,s) - G_{22}(t',s))\Delta s|\Big] \\
&\le M|\lambda| \sup_{t\in\mathbb{T}_2} \Big[|\lambda \int_a^b (G_{22}(t,s) - G_{22}(t',s))\Delta s|\Big] \\
&= M|\lambda| \sup_{t\in\mathbb{T}_2} \Big[|\int_{\sigma(c)}^t (t-b)\Delta s - \int_{\sigma(c)}^{t'} (t'-b)\Delta s \\
&\quad + \int_t^b (s-b)\Delta s - \int_{t'}^b (s-b)\Delta s|\Big] \\
&\le M|\lambda| \sup_{t\in\mathbb{T}_2} \Big[|t-t'|[t+t'-(b+\sigma(c))] + \int_t^{t'} |s-b|\Delta s\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\le M|\lambda| \sup_{t\in\mathbb{T}_2}\Big[|t-t'|[t+t'-(b+\sigma(c))]+|t'-t|b\Big]\\
&= M|\lambda| \sup_{t\in\mathbb{T}_2}\Big[|t-t'|((t+t')-\sigma(c))\Big].
\end{aligned}
$$

From the above it is clear that $\|Ty(t)-Ty(t')\|<\epsilon$ whenever $\|t-t'\|<\delta$. Hence $T(\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R}))$ is equi-continuous subset of $\mathcal{C}(\mathbb{T}_1\cup\mathbb{T}_2,\mathbb{R})$. Hence from the Arzela-Ascoli theorem we see that T is completely continuous. $\square$

6. Eigenvalue Intervals

Theorem 6.1. *Let f_1 be positive on $\mathbb{T}_1$ and f_2 be positive on $\mathbb{T}_2$. Also let $\lambda\in\mathbb{R}^-$. Let us assume that there exists $\Omega, K>0$ such that for $u=(u_1,u_2), v=(v_1,v_2)$ we have*

$$
\begin{aligned}
f_1(t,u_1^\sigma)\le f_1(t,v_1^\sigma)\le \Omega K,\quad t\in\mathbb{T}_1,\\
f_2(t,u_2^\sigma)\le f_2(t,u_2^\sigma)\le \Omega K,\quad t\in\mathbb{T}_2,
\end{aligned}
$$

whenever $0\le u^\sigma\le v^\sigma\le\Omega K$; i.e.,

$$0\le u_1^\sigma\le v_1^\sigma<\Omega K,\quad 0\le u_2^\sigma\le v_2^\sigma<\Omega K.$$

Then for all λ satisfying

$$
\begin{aligned}
\lambda &\le \frac{1}{\Omega\big[\int_a^b G_{11}(t,s)\Delta s+\int_a^b G_{12}(t,s)\Delta s\big]}\\
\lambda &\le \frac{1}{\Omega\big[\int_a^b G_{11}(t,s)\Delta s+\int_a^b G_{12}(t,s)\Delta s\big]},
\end{aligned}
$$

there exists positive solutions for NN-SL-BVP-SIP.

Proof. Let $v_0(t)=0$ and $w_0(t)=K$ for all $t\in\mathbb{T}_1\cup\mathbb{T}_2$. Then from Lemma 5.1 it is clear that $T:[v_0,w_0]\to K$ is completely continuous.

- We claim that T is monotone increasing on $[v_0,w_0]$. Let us suppose that $u=(u_1,u_2), v=(v_1,v_2)\in[v_0,w_0]$ such that $u^\sigma\le v^\sigma$. Then clearly $0\le u^\sigma(t)\le v^\sigma(t)\le\Omega K$, for all $t\in\mathbb{T}_1\cup\mathbb{T}_2$. We have

$$
\begin{aligned}
&(Tu)(t)\\
&=\begin{cases}(Tu_1)(t)=\lambda\big[\int_a^b G_{11}(t,s)f_1(s,u_1^\sigma)\Delta s+\int_a^b G_{12}(t,s)f_2(s,u_2^\sigma)\Delta s\big], & t\in\mathbb{T}_1\\ (Tu_2)(t)=\lambda\big[\int_a^b G_{21}(t,s)f_1(s,u_1^\sigma)\Delta s+\int_a^b G_{22}(t,s)f_2(s,u_2^\sigma)\Delta s\big], & t\in\mathbb{T}_2.\end{cases}
\end{aligned}
$$

From the hypothesis it is clear that $(Tu)(t)\le(Tv)(t)$ where

$$
\begin{aligned}
&(Tv)(t)\\
&=\begin{cases}(Tv_1)(t)=\lambda\big[\int_a^b G_{11}(t,s)f_1(s,v_1^\sigma)\Delta s+\int_a^b G_{12}(t,s)f_2(s,v_2^\sigma)\Delta s\big], & t\in\mathbb{T}_1\\ (Tv_2)(t)=\lambda\big[\int_a^b G_{21}(t,s)f_1(s,v_1^\sigma)\Delta s+\int_a^b G_{22}(t,s)f_2(s,v_2^\sigma)\Delta s\big], & t\in\mathbb{T}_2.\end{cases}
\end{aligned}
$$

Hence T is monotone increasing on $[v_0,w_0]$.

- We claim that v_0 is an lower solution of T. We see that

$$
\begin{aligned}
&(Tv_0)(t)\\
&=\begin{cases}(Tv_{01})(t)=\lambda\big[\int_a^b G_{11}(t,s)f_1(s,0)\Delta s+\int_a^b G_{12}(t,s)f_2(s,0)\Delta s\big]\ge 0, & t\in\mathbb{T}_1\\ (Tv_{02})(t)=\lambda\big[\int_a^b G_{21}(t,s)f_1(s,0)\Delta s+\int_a^b G_{22}(t,s)f_2(s,0)\Delta s\big]\ge 0, & t\in\mathbb{T}_2.\end{cases}
\end{aligned}
$$

which implies that $v_0\le Tv_0$.

• We claim that w_0 is an upper solution of T. We see that

$$
\begin{aligned}
&(Tw_0)(t)\\
&= \begin{cases} (Tw_{01})(t) = \lambda\big[\int_a^b G_{11}(t,s)f_1(s,w_0)\Delta s + \int_a^b G_{12}(t,s)f_2(s,w_0)\Delta s\big], & t\in\mathbb{T}_1\\ (Tw_{02})(t) = \lambda\big[\int_a^b G_{21}(t,s)f_1(s,w_0)\Delta s + \int_a^b G_{22}(t,s)f_2(s,w_0)\Delta s\big], & t\in\mathbb{T}_2. \end{cases}
\end{aligned}
$$

Let $t\in\mathbb{T}_1$. Then

$$
\begin{aligned}
&(Tw_{01})(t)\\
&\le \Omega K\lambda\Big[\int_a^b G_{11}(t,s)\Delta s\Big] + \Omega K\lambda\Big[\int_a^b G_{12}(t,s)\Delta s\Big]\\
&= K\Omega\lambda\Big[\int_a^b G_{11}(t,s)\Delta s\Big] + K\Omega\lambda\Big[\int_a^b G_{12}(t,s)\Delta s\Big]\\
&\le K\Omega\Big[\int_a^b G_{11}(t,s)\Delta s + \int_a^b G_{12}(t,s)\Delta s\Big]\frac{1}{\Omega\big[\int_a^b G_{11}(t,s)\Delta s + \int_a^b G_{12}(t,s)\Delta s\big]}\\
&= K = w_0.
\end{aligned}
$$

We now let $t\in\mathbb{T}_2$. Then

$$
\begin{aligned}
&(Tw_{02})(t)\\
&\le \Omega K\lambda\Big[\int_a^b G_{21}(t,s)\Delta s\Big] + \Omega K\lambda\Big[\int_a^b G_{22}(t,s)\Delta s\Big]\\
&= K\Omega\lambda\Big[\int_a^b G_{21}(t,s)\Delta s\Big] + K\Omega\lambda\Big[\int_a^b G_{22}(t,s)\Delta s\Big]\\
&\le K\Omega\Big[\int_a^b G_{21}(t,s)\Delta s + \int_a^b G_{22}(t,s)\Delta s\Big]\frac{1}{\Omega\big[\int_a^b G_{21}(t,s)\Delta s + \int_a^b G_{22}(t,s)\Delta s\big]}\\
&= K = w_0.
\end{aligned}
$$

Hence $Tw_0 \le w_o$ proving that w_0 is an upper solution of T. We now construct sequences $\{v_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ as follows:

$$
v_n = Tv_{n-1}, \quad w_n = Tw_{n-1}, \quad \text{for } n = 1,2,3,\dots
$$

Then from theorem 2.6 we have that

$$
v_0 \le v_1 \le \dots \le v_n \le \dots w_n \le \dots \le w_1 \le w_0,
$$

and $\{v_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge to, v and w in $[v_0,w_0]$, which are the fixed points of the operator T. In other words v and w are the positive solutions of the NN-SL-BVP-SIP. □

7. An example

In this section, an example is given to illustrate the main result of this paper. Let $\mathbb{T}_1 = [1,5]_{\mathbb{T}}$, $\mathbb{T}_2 = [6,10]_{\mathbb{T}}$. Let us consider the NN-SL-BVP-SIP-A

$$
\begin{gathered}
y_1^{\Delta\Delta}(t) = \lambda\frac{1}{4}y_1^{\Delta^2}(\sigma(t)), \quad t\in{\mathbb{T}_1}^{\kappa^2},\\
y_2^{\Delta\Delta}(t) = \lambda\Big(\frac{1}{2}y_2^{\Delta^2}(\sigma(t)) + \frac{1}{4}y_2^2(t)\Big), \quad t\in{\mathbb{T}_2}^{\kappa^2},
\end{gathered}
$$

along with the boundary and matching interface conditions

$$y_1(1) = 0 = y_2(10)$$
$$y_1(5) = y_2(6)$$
$$y_1^{\Delta}(5) = y_2^{\Delta}(6).$$

Let $\Omega = 100$, $K = 10$. Also let $(u_1, u_2) = (t, t)$, $(v_1, v_2) = (t^2, t^2)$. We have

$$u_1^{\Delta}(\sigma(t)) = 1, u_2^{\Delta}(\sigma(t)) = 1,$$
$$v_1^{\Delta}(\sigma(t)) = 2\sigma(t), v_2^{\Delta}(\sigma(t)) = 2\sigma(t).$$

Clearly

$$u_1(\sigma(t)) = \sigma(t) \le \sigma(t)^2 = v_1(\sigma(t)) < \Omega K,$$
$$u_2(\sigma(t)) = \sigma(t) \le \sigma(t)^2 = v_2(\sigma(t)) < \Omega K.$$

Also

$$f_1(t, u_1^{\sigma}) = \frac{1}{4} \le \sigma^2(t) = f_1(t, v_1^{\sigma}),$$
$$f_2(t, u_2^{\sigma}) = \frac{1}{2} + \frac{1}{4}\sigma^2(t) \le 2\sigma^2(t) + \frac{1}{4}\sigma^4(t) = f_2(t, v_2^{\sigma}).$$

Hence from theorem (6.1), for all λ satisfying

$$\lambda \le \frac{1}{\Omega\big[\int_a^b G_{11}(t,s)\Delta s + \int_a^b G_{12}(t,s)\Delta s\big]}$$
$$\lambda \le \frac{1}{\Omega\big[\int_a^b G_{21}(t,s)\Delta s + \int_a^b G_{22}(t,s)\Delta s\big]},$$

there exists positive solutions for NN-SL-BVP-SIP-A. That is, for all λ satisfying

$$\lambda \le \frac{1}{\Omega\Big[\int_1^t (1-t)\Delta s + \int_t^5 (1-s)\Delta s + \int_1^5 (1-t)(10-s)\Delta s + \int_6^{10} (1-t)(10-s)\Delta s\Big]}$$

and

$$\lambda \le \frac{1}{\Omega\big[\int_1^5 (1-s)(10-t)\Delta s + \int_6^{10} (1-s)(10-t)\Delta s + \int_6^t (t-10)\Delta s + \int_t^{10} (s-10)\Delta s\big]}.$$

$$\lambda \le \frac{1}{100}\Big(-\frac{t^2}{2} - 35t + \frac{55}{2}\Big), \quad t \in \mathbb{T}_1$$
$$\lambda \le \frac{1}{100}\Big(\frac{t^2}{2} + 30t - 350\Big), \quad t \in \mathbb{T}_2.$$

So for all $\lambda \le -1/800$ there exists positive solutions for the NN-SL-BVP-SIP-A.

Remark 7.1. We also note that the type of results embodied in [4, 5, 6, 10, 11, 12, 13, 14, 31, 32, 33, 34, 36, 35, 15] when worked for second order are special cases of this work whenever $\rho(c) = c = \sigma(c)$. Also the interfaces I and II explained in introduction can be studied as special cases of the results presented in this work.

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