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PERIODIC RANDOM ATTRACTORS FOR STOCHASTIC NAVIER-STOKES EQUATIONS ON UNBOUNDED DOMAINS

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Abstract. This article concerns the asymptotic behavior of solutions to the two-dimensional Navier-Stokes equations with both non-autonomous deterministic and stochastic terms defined on unbounded domains. First we introduce a continuous cocycle for the equations and then prove the existence and uniqueness of tempered random attractors. We also characterize the structures of the random attractors by complete solutions. When deterministic forcing terms are periodic, we show that the tempered random attractors are also periodic. Since the Sobolev embeddings on unbounded domains are not compact, we establish the pullback asymptotic compactness of solutions by Ball's idea of energy equations.

1. Introduction

In this article, we investigate the pullback attractors for the two-dimensional Navier-Stokes equations on unbounded domains with non-autonomous deterministic and stochastic terms. Let Q be an unbounded open set in $\mathbb{R}^2$ with boundary ∂Q. Given $\tau \in \mathbb{R}$, consider the stochastic Navier-Stokes equations with multiplicative noise:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = f(x,t) - \nabla p + \alpha u \circ \frac{dw}{dt}, \quad x \in Q \text{ and } t > \tau, \tag{1.1}$$

$$\operatorname{div} u = 0, \quad x \in Q \text{ and } t > \tau, \tag{1.2}$$

together with homogeneous Dirichlet boundary condition, where $\nu, \alpha \in \mathbb{R}$ with $\nu > 0$, f is a given function defined on $Q \times \mathbb{R}$, and w is a two-sided real valued Wiener process defined in a probability space. The stochastic equation (1.1) is understood in the sense of Stratonovich integration.

The attractors of the Navier-Stokes equations have been extensively studied in the literature; see, e.g. [2, 3, 8, 9, 15, 18, 20, 21] for deterministic equations and [13, 14, 19] for stochastic equations. Particularly, in the deterministic case (i.e., $\alpha = 0$), the autonomous global attractors and the non-autonomous pullback attractors of (1.1)-(1.2) on *unbounded* domains have been studied in [18] and [8, 9], respectively. For the stochastic equations with additive noise and time-independent

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f, the asymptotic compactness of solutions on *unbounded* domains has been investigated in [6]. As far as the author is aware, there is no result available in the literature on the existence of random attractors for the stochastic equations (1.1)-(1.2) with *time-dependent* f even on bounded domains. The purpose of the present article is to investigate this problem and examine the periodicity of random attractors when f is periodic in time.

It is worth mentioning that the concept of pullback attractors for random systems with time-independent f was introduced in [13, 14, 19] and the existence of such attractors for compact systems was proved in [1, 7, 11, 12, 13, 14, 16, 17, 19] and the references therein. For non-compact systems, the existence of pullback attractors was established in [4, 5, 22, 23]. In the present paper, we study pullback attractors for the stochastic equations (1.1)-(1.2) on unbounded domains with time-dependent f. In this case, the random dynamical systems associated with the equations are non-compact.

To deal with the stochastic equations with non-autonomous f, we need to combine the ideas of non-autonomous deterministic dynamical systems and that of random dynamical systems. Particularly, the concept of dynamical systems defined over two parametric spaces, say Ω_1 and Ω_2, is needed, where Ω_1 is a nonempty set used to deal with the non-autonomous deterministic terms, and Ω_2 is a probability space responsible for the stochastic terms. The existence and uniqueness of random attractors for dynamical systems over two parametric spaces have been recently established in [24]. For the stochastic Navier-Stokes equations (1.1)-(1.2), we may take Ω_1 as the set of all translations of f. We can also take Ω_1 as the collection of all initial times; i.e., $\Omega_1 = \mathbb{R}$. In this paper, we will choose $\Omega_1 = \mathbb{R}$. We first define a continuous cocycle for (1.1)-(1.2) over Ω_1 and Ω_2, and then prove the existence of tempered random absorbing sets. Since the Sobolev embeddings on unbounded domains are no longer compact, we have to appeal to the idea of energy equations to establish the pullback asymptotic compactness of solutions. This method was introduced by Ball in [3] for deterministic equations, and used by the authors in [8, 9, 18] for the deterministic Navier-Stokes equations on unbounded domains and in [6] for the stochastic equations with time-independent f. We will adapt this approach to the stochastic equations (1.1)-(1.2) with time-dependent f, and prove the existence of tempered random attractors for the equations. We also consider the random attractors in the case where f is a periodic function in time. If f is periodic, we will show that the tempered random attractors are also periodic in some sense. Following [24], the structures of the tempered random attractors will be characterized by the tempered complete solutions.

In the next section, we will recall some results on pullback attractors for random dynamical systems over two parametric spaces. A continuous cocycle for the stochastic Navier-Stokes equations (1.1)-(1.2) with non-autonomous f is defined in Section 3. We then derive uniform estimates of the solutions in Section 4 and prove the existence and uniqueness of pullback attractors in Section 5.

In the sequel, we will use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and the inner product of $L^2(Q)$, respectively. The norm of a Banach space X is generally written as $\|\cdot\|_X$. The letters c and c_i ($i = 1, 2, \dots$) are used to denote positive constants whose values are not significant in the context.

2. Theory of pullback attractors

In this section, we recall some results on pullback attractors for random dynamical systems with two parametric spaces as presented in [24]. This sort of dynamical systems can be generated by differential equations with both deterministic and stochastic non-autonomous external terms. All results given in this section are not original and they are presented here just for the reader's convenience. We also refer the reader to [4, 12, 13, 14, 19] for the theory of pullback attractors for random dynamical systems with one parametric space.

Let Ω_1 be a nonempty set and $\{\theta_{1,t}\}_{t\in\mathbb{R}}$ be a family of mappings from Ω_1 into itself such that $\theta_{1,0}$ is the identity on Ω_1 and $\theta_{1,s+t} = \theta_{1,t,} \circ \theta_{1,s}$ for all $t, s \in \mathbb{R}$. Let $(\Omega_2, \mathcal{F}_2, P)$ be a probability space and $\theta_2 : \mathbb{R} \times \Omega_2 \to \Omega_2$ be a $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}_2, \mathcal{F}_2)$-measurable mapping such that $\theta_2(0, \cdot)$ is the identity on Ω_2, $\theta_2(s + t, \cdot) = \theta_2(t, \cdot) \circ \theta_2(s, \cdot)$ for all $t, s \in \mathbb{R}$ and $P\theta_2(t, \cdot) = P$ for all $t \in \mathbb{R}$. We usually write $\theta_2(t, \cdot)$ as $\theta_{2,t}$ and call both $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ a parametric dynamical system.

Let (X, d) be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$. Given $r > 0$ and $D \subseteq X$, the neighborhood of D with radius r is written as $\mathcal{N}_r(D)$. Denote by 2^X the collection of all subsets of X. A set-valued mapping $K : \Omega_1 \times \Omega_2 \to 2^X$ is called measurable with respect to $\mathcal{F}_2$ in Ω_2 if the value $K(\omega_1, \omega_2)$ is a closed nonempty subset of X for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, and the mapping $\omega_2 \in \Omega_2 \to d(x, K(\omega_1, \omega_2))$ is $(\mathcal{F}_2,\ \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $\omega_1 \in \Omega_1$. If K is measurable with respect to $\mathcal{F}_2$ in Ω_2, then we say that the family $\{K(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ is measurable with respect to $\mathcal{F}_2$ in Ω_2. We now define a cocycle on X over two parametric spaces.

Definition 2.1. Let $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ be parametric dynamical systems. A mapping Φ: $\mathbb{R}^+ \times \Omega_1 \times \Omega_2 \times X \to X$ is called a continuous cocycle on X over $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ if for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $t, \tau \in \mathbb{R}^+$, the following conditions (i)-(iv) are satisfied:

(i) $\Phi(\cdot, \omega_1, \cdot, \cdot) : \mathbb{R}^+ \times \Omega_2 \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F}_2 \times \mathcal{B}(X),\ \mathcal{B}(X))$-measurable;
(ii) $\Phi(0, \omega_1, \omega_2, \cdot)$ is the identity on X;
(iii) $\Phi(t + \tau, \omega_1, \omega_2, \cdot) = \Phi(t, \theta_{1,\tau}\omega_1, \theta_{2,\tau}\omega_2, \cdot) \circ \Phi(\tau, \omega_1, \omega_2, \cdot)$;
(iv) $\Phi(t, \omega_1, \omega_2, \cdot) : X \to X$ is continuous.

If, in addition, there exists a positive number T such that for every $t \geq 0$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\Phi(t, \theta_{1,T}\omega_1, \omega_2, \cdot) = \Phi(t, \omega_1, \omega_2, \cdot),$$

then Φ is called a continuous periodic cocycle on X with period T.

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty subsets of X:

$$\mathcal{D} = \{D = \{D(\omega_1, \omega_2) \subseteq X :\ D(\omega_1, \omega_2) \neq \emptyset,\ \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\}\}. \tag{2.1}$$

Two elements D_1 and D_2 of $\mathcal{D}$ are said to be equal if $D_1(\omega_1, \omega_2) = D_2(\omega_1, \omega_2)$ for any $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. Sometimes, we require that $\mathcal{D}$ is neighborhood closed which is defined as follows.

Definition 2.2. A collection $\mathcal{D}$ of some families of nonempty subsets of X is said to be neighborhood closed if for each $D = \{D(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$, there exists a positive number ε depending on D such that the family

$$\begin{aligned}\big\{B(\omega_1,\omega_2) : B(\omega_1,\omega_2) \text{ is a nonempty subset of } \mathcal{N}_\varepsilon(D(\omega_1,\omega_2)),\\ \forall\ \omega_1 \in \Omega_1, \forall\ \omega_2 \in \Omega_2\big\}\end{aligned} \tag{2.2}$$

also belongs to $\mathcal{D}$.

Definition 2.3. Let $D = \{D(\omega_1,\omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ be a family of nonempty subsets of X. We say D is tempered in X with respect to $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$ if there exists $x_0 \in X$ such that for every $c > 0$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\lim_{t\to-\infty} e^{ct} d(x_0, D(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2)) = 0.$$

Definition 2.4. *Suppose $T \in \mathbb{R}$ and $\mathcal{D}$ is a collection of some families of nonempty subsets of X as given by* (2.1). *For every $D = \{D(\omega_1,\omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$, we write*

$$D_T = \{D_T(\omega_1,\omega_2) : \quad D_T(\omega_1,\omega_2) = D(\theta_{1,T}\omega_1,\omega_2),\ \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\}.$$

The family D_T is called the T-translation of D. Let $\mathcal{D}_T$ be the collection of T-translations of all elements of $\mathcal{D}$, that is,

$$\mathcal{D}_T = \{D_T : D_T \text{ is the } T\text{-translation of } D,\ D \in \mathcal{D}\}.$$

Then $\mathcal{D}_T$ is called the T-translation of the collection $\mathcal{D}$. If $\mathcal{D}_T \subseteq \mathcal{D}$, we say $\mathcal{D}$ is T-translation closed. If $\mathcal{D}_T = \mathcal{D}$, we say $\mathcal{D}$ is T-translation invariant.

One can check that $\mathcal{D}$ is T-translation invariant if and only if $\mathcal{D}$ is both $-T$-translation closed and T-translation closed. For later purpose, we need the concept of a complete orbit of Φ which is given below.

Definition 2.5. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. A mapping $\psi : \mathbb{R} \times \Omega_1 \times \Omega_2 \to X$ is called a complete orbit of Φ if for every $\tau \in \mathbb{R}$, $t \geq 0$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the following holds:

$$\Phi(t, \theta_{1,\tau}\omega_1, \theta_{2,\tau}\omega_2, \psi(\tau,\omega_1,\omega_2)) = \psi(t+\tau,\omega_1,\omega_2). \tag{2.3}$$

If, in addition, there exists $D = \{D(\omega_1,\omega_2) : \omega_1 \in \Omega, \omega_2 \in \Omega_2\} \in \mathcal{D}$ such that $\psi(t,\omega_1,\omega_2)$ belongs to $D(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2)$ for every $t \in \mathbb{R}$, $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, then ψ is called a $\mathcal{D}$-complete orbit of Φ.

Definition 2.6. Let $B = \{B(\omega_1,\omega_2) : \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\}$ be a family of nonempty subsets of X. For every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, let

$$\Omega(B,\omega_1,\omega_2) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2))}. \tag{2.4}$$

Then the family $\{\Omega(B,\omega_1,\omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ is called the Ω-limit set of B and is denoted by $\Omega(B)$.

Definition 2.7. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $K = \{K(\omega_1,\omega_2) : \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\} \in \mathcal{D}$. Then K is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and for every $B \in \mathcal{D}$, there exists $T = T(B,\omega_1,\omega_2) > 0$ such that

$$\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)) \subseteq K(\omega_1,\omega_2) \quad \text{for all } t \geq T. \tag{2.5}$$

If, in addition, for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, $K(\omega_1,\omega_2)$ is a closed nonempty subset of X and K is measurable with respect to the P-completion of $\mathcal{F}_2$ in Ω_2, then we say K is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ.

Definition 2.8. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the sequence

$$\{\Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \tag{2.6}$$

whenever $t_n \to \infty$, and $x_n \in B(\theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)$ with $\{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\} \in \mathcal{D}$.

Definition 2.9. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathcal{A} = \{\mathcal{A}(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$. Then $\mathcal{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (i)-(iii) are fulfilled:

(i) $\mathcal{A}$ is measurable with respect to the P-completion of $\mathcal{F}_2$ in Ω_2 and $\mathcal{A}(\omega_1, \omega_2)$ is compact for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.

(ii) $\mathcal{A}$ is invariant, that is, for every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\Phi(t, \omega_1, \omega_2, \mathcal{A}(\omega_1, \omega_2)) = \mathcal{A}(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2), \quad \forall\ t \geq 0.$$

(iii) $\mathcal{A}$ attracts every member of $\mathcal{D}$, that is, for every $B = \{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$ and for every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\lim_{t\to\infty} d(\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)), \mathcal{A}(\omega_1, \omega_2)) = 0.$$

If, in addition, there exists $T > 0$ such that

$$\mathcal{A}(\theta_{1,T}\omega_1, \omega_2) = \mathcal{A}(\omega_1, \omega_2), \quad \forall \omega_1 \in \Omega_1, \forall\ \omega_2 \in \Omega_2,$$

then we say $\mathcal{A}$ is periodic with period T.

The following result on the existence and uniqueness of $\mathcal{D}$-pullback attractors for Φ can be found in [24]. The reader is referred to [4, 13, 14, 19] for similar results for random dynamical systems.

Proposition 2.10. *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X, and Φ be a continuous cocycle on X over $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$. Then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable (w.r.t. the P-completion of $\mathcal{F}_2$) $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,*

$$\begin{aligned} \mathcal{A}(\omega_1, \omega_2) &= \Omega(K, \omega_1, \omega_2) = \cup_{B\in\mathcal{D}}\Omega(B, \omega_1, \omega_2) && (2.7)\\ &= \{\psi(0, \omega_1, \omega_2) : \psi \textit{ is a } \mathcal{D}\textit{-complete orbit of } \Phi\}. && (2.8) \end{aligned}$$

The periodicity of $\mathcal{D}$-pullback attractors is proved in [24] as given below.

Proposition 2.11. *Let T be a positive number. Suppose Φ is a continuous periodic cocycle with period T on X over $(\Omega_1, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$. Let $\mathcal{D}$ be a neighborhood closed and T-translation invariant collection of some families of nonempty subsets of X. If Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable (w.r.t. the P-completion of $\mathcal{F}_2$) $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$, then Φ has a unique periodic $\mathcal{D}$-pullback attractor $\mathcal{A} \in \mathcal{D}$ with period T; i.e., $\mathcal{A}(\theta_{1,T}\omega_1, \omega_2) = \mathcal{A}(\omega_1, \omega_2)$.*

3. Cocycles for Navier-Stokes equations on unbounded domains

This section is devoted to the existence of a continuous cocycle for the stochastic Navier-Stokes equations with non-autonomous deterministic terms. Suppose Q is an unbounded open set in $\mathbb{R}^2$ with boundary ∂Q. Then consider the following stochastic equations with multiplicative noise defined on $Q \times (\tau, \infty)$ with $\tau \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u = f(x,t) - \nabla p + \alpha u \circ \frac{dw}{dt}, \quad x \in Q \text{ and } t > \tau, \tag{3.1}$$

$$\operatorname{div} u = 0, \quad x \in Q \text{ and } t > \tau, \tag{3.2}$$

with boundary condition

$$u = 0, \quad x \in \partial Q \text{ and } t > \tau, \tag{3.3}$$

and initial condition

$$u(x, \tau) = u_\tau(x), \quad x \in Q, \tag{3.4}$$

where ν and α are constants, $\nu > 0$, f is a given function defined on $Q \times \mathbb{R}$, and w is a two-sided real valued Wiener process defined in a probability space. Note that equation (3.1) must be understood in the sense of Stratonovich integration.

To reformulate problem (3.1)-(3.4), we recall the standard function space:

$$\mathcal{V} = \{u \in C_0^\infty(Q) \times C_0^\infty(Q) : \operatorname{div} u = 0\}.$$

Let H and V be the closures of $\mathcal{V}$ in $L^2(Q) \times L^2(Q)$ and $H_0^1(Q) \times H_0^1(Q)$, respectively. The dual space of V is denoted by V^* with norm $\| \cdot \|_{V^*}$. The duality pair between V and V^* is denoted by $\langle \cdot, \cdot \rangle$. Given $u, v \in V$, we set

$$(Du, Dv) = \sum_{i,j=1}^{2} \int_Q \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx \quad \text{and} \quad \|Du\| = (Du, Du)^{1/2}.$$

For convenience, we write, for each $u, v, w \in V$,

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_Q u_i \frac{\partial u_j}{\partial x_i} w_j \, dx.$$

Let $\{\theta_{1,t}\}_{t \in \mathbb{R}}$ be a family of shift operators on $\mathbb{R}$ which is given by, for each $t \in \mathbb{R}$,

$$\theta_{1,t}(\tau) = \tau + t, \quad \text{for all } \tau \in \mathbb{R}. \tag{3.5}$$

For the probability space we will use later, we write

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}.$$

Let $\mathcal{F}_1$ be the Borel σ-algebra induced by the compact-open topology of Ω, and P be the corresponding Wiener measure on $(\Omega, \mathcal{F}_1)$. As usual, for each $t \in \mathbb{R}$ and $\omega \in \Omega$, we may write $w_t(\omega) = \omega(t)$. Denote by $\{\theta_{2,t}\}_{t \in \mathbb{R}}$ the standard group on $(\Omega, \mathcal{F}_1, P)$:

$$\theta_{2,t} \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}. \tag{3.6}$$

Then $(\Omega, \mathcal{F}, P, \{\theta_{2,t}\}_{t \in \mathbb{R}})$ is a parametric dynamical system. In addition, there exists a $\theta_{2,t}$-invariant set $\tilde{\Omega} \subseteq \Omega$ of full P measure such that for each $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{3.7}$$

From now on, we only consider the space $\tilde{\Omega}$ instead of Ω, and hence we will write $\tilde{\Omega}$ as Ω for convenience.

Next, we define a continuous cocycle for (3.1)-(3.4) in H over $(\mathbb{R}, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}_1, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$. To this end, we need to transfer the stochastic equation into a deterministic one with random parameters. Given $t \in \mathbb{R}$ and $\omega \in \Omega$, let $z(t,\omega) = e^{-\alpha\omega(t)}$. Then we find that z is a solution of the equation

$$dz = -\alpha z \circ dw. \tag{3.8}$$

Let v be a new variable given by

$$v(t,\tau,\omega,v_\tau) = z(t,\omega)u(t,\tau,\omega,u_\tau) \quad \text{with} \quad v_\tau = z(\tau,\omega)u_\tau. \tag{3.9}$$

Formally, from (3.1)-(3.4) and (3.8) we get that

$$\frac{\partial v}{\partial t} - \nu\Delta v + \frac{1}{z(t,\omega)}(v\cdot\nabla)v = z(t,\omega)\left(f(x,t) - \nabla p\right), \quad x \in Q \text{ and } t > \tau, \tag{3.10}$$

$$\operatorname{div} v = 0, \quad x \in Q \text{ and } t > \tau, \tag{3.11}$$

with boundary condition

$$v = 0, \quad x \in \partial Q \text{ and } t > \tau, \tag{3.12}$$

and initial condition

$$v(x,\tau) = v_\tau(x), \quad x \in Q. \tag{3.13}$$

Let $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $v_\tau \in H$. A mapping $v(\cdot,\tau,\omega,v_\tau)$: $[\tau,\infty) \to H$ is called a solution of problem (3.10)-(3.13) if for every $T > 0$,

$$v(\cdot,\tau,\omega,u_\tau) \in C([\tau,\infty),H) \cap L^2((0,T),V)$$

and v satisfies

$$\begin{aligned}
&(v(t),\zeta) + \nu\int_\tau^t (Dv, D\zeta)ds + \int_\tau^t \frac{1}{z(s,\omega)} b(v,v,\zeta)ds \\
&= (v_\tau,\zeta) + \int_\tau^t z(s,\omega)\langle f(\cdot,s),\zeta\rangle ds,
\end{aligned} \tag{3.14}$$

for every $t \geq \tau$ and $\zeta \in V$. If, in addition, v is $(\mathcal{F}_1, \mathcal{B}(H))$-measurable with respect to $\omega \in \Omega$, we say v is a measurable solution of problem (3.10)-(3.13). Since (3.10) is a deterministic equation, it follows from [21] that for every $\tau \in \mathbb{R}$, $v_\tau \in H$ and $\omega \in \Omega$, problem (3.10)-(3.13) has a unique solution v in the sense of (3.14) which continuously depends on v_τ with the respect to the norm of H. Moreover, the solution v is $(\mathcal{F}_1, \mathcal{B}(H))$-measurable in $\omega \in \Omega$. This enables us to define a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times H \to H$ for problem (3.1)-(3.4) by using (3.9). Given $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in H$, let

$$\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{2,-\tau}\omega,u_\tau) = \frac{1}{z(t+\tau,\theta_{2,-\tau}\omega)} v(t+\tau,\tau,\theta_{2,-\tau}\omega,v_\tau), \tag{3.15}$$

where $v_\tau = z(\tau,\theta_{2,-\tau}\omega)u_\tau$. By (3.15) we have, for every $t \geq 0$, $\tau \geq 0$, $r \in \mathbb{R}$, $\omega \in \Omega$ and $u_0 \in H$,

$$\Phi(t+\tau,r,\omega,u_0) = \frac{1}{z(t+\tau+r,\theta_{2,-r}\omega)} v(t+\tau+r,r,\theta_{2,-r}\omega,v_0), \tag{3.16}$$

where $v_0 = z(r, \theta_{2,-r}\omega)u_0$. Similarly, we have

$$\begin{aligned}
&\Phi\left(t, \tau + r, \theta_{2,\tau}\omega, \Phi(\tau, r, \omega, u_0)\right)\\
&= \frac{1}{z(t+\tau+r, \theta_{2,-r}\omega)} v(t+\tau+r, \tau+r, \theta_{2,-r}\omega, z(\tau+r, \theta_{2,-r}\omega)\Phi(\tau, r, \omega, u_0))\\
&= \frac{1}{z(t+\tau+r, \theta_{2,-r}\omega)} v(t+\tau+r, \tau+r, \theta_{2,-r}\omega, v(\tau+r, r, \theta_{2,-r}\omega, v_0)\\
&= \frac{1}{z(t+\tau+r, \theta_{2,-r}\omega)} v(t+\tau+r, r, \theta_{2,-r}\omega, v_0).
\end{aligned} \tag{3.17}$$

It follows from (3.16)-(3.17) that

$$\Phi(t+\tau, r, \omega, u_0) = \Phi\left(t, \tau+r, \theta_{2,\tau}\omega, \Phi(\tau, r, \omega, u_0)\right). \tag{3.18}$$

Since v is the measurable solution of problem (3.10)-(3.13) which is continuous in initial data in H, we find from (3.18) that Φ is a continuous cocycle on H over $(\mathbb{R}, \{\theta_{1,t}\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}_1, P, \{\theta_{2,t}\}_{t\in\mathbb{R}})$. The rest of this paper is devoted to the existence of pullback attractors for Φ in H. To this end, we assume that the open set Q is a Poincare domain in the sense that there exists a positive number λ such that

$$\int_Q |\nabla\phi(x)|^2 dx \geq \lambda \int_Q |\phi(x)|^2 dx, \quad \text{for all } \ \phi \in H_0^1(Q). \tag{3.19}$$

Given a bounded nonempty subset B of H, we write $\|B\| = \sup_{\phi\in B} \|\phi\|_H$. Suppose $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is a tempered family of bounded nonempty subsets of H, that is, for every $c > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{r\to\infty} e^{-cr} \|D(\tau - r, \theta_{2,-r}\omega)\| = 0. \tag{3.20}$$

Let $\mathcal{D}$ be the collection of all tempered families of bounded nonempty subsets of H; i.e.,

$$\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (3.20)}\}. \tag{3.21}$$

We see that $\mathcal{D}$ is neighborhood closed. For later purpose, we assume that the external term f satisfies the following condition: there exists a number $\delta \in [0, \nu\lambda)$ such that

$$\int_{-\infty}^{\tau} e^{\delta r} \|f(\cdot, r)\|_{V^*}^2 dr < \infty, \quad \forall\ \tau \in \mathbb{R}. \tag{3.22}$$

When proving the existence of tempered pullback absorbing sets for the Navier-Stokes equations, we also assume that there exists $\delta \in [0, \nu\lambda)$ such that for every positive number c,

$$\lim_{r\to-\infty} e^{cr} \int_{-\infty}^{0} e^{\delta s} \|f(\cdot, s+r)\|_{V^*}^2 ds = 0. \tag{3.23}$$

Note that (3.23) implies (3.22) if $f \in L^2_{loc}(\mathbb{R}, V^*)$. It is worth pointing out that both conditions (3.22) and (3.23) do not require that f is bounded in V^* at $\pm\infty$. For instance, for any $\beta \geq 0$ and $f_1 \in V^*$, the function $f(\cdot, t) = t^\beta f_1$ satisfies both (3.22) and (3.23).

4. Uniform estimates of solutions

In this section, we derive uniform estimates on the solutions of problem (3.10)-(3.13) and then prove the $\mathcal{D}$-pullback asymptotic compactness of the solutions by the idea of energy equations as introduced by Ball in [3] for deterministic systems.

Lemma 4.1. *Suppose* (3.19) *and* (3.22) *hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geq T$ and $s \geq \tau - t$, the solution v of problem* (3.10)-(3.13) *with ω replaced by $\theta_{2,-\tau}\omega$ satisfies*

$$\|v(s,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})\|^2 \leq e^{\nu\lambda(\tau-s)} + \frac{2}{\nu}e^{-\nu\lambda s}\int_{-\infty}^{s} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|^2_{V^*}dr,$$

and

$$\begin{aligned}&\int_{\tau-t}^{s} e^{\nu\lambda r}\|Dv(r,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})\|^2dr\\ &\leq \frac{2}{\nu}e^{\nu\lambda\tau} + \frac{4}{\nu^2}\int_{-\infty}^{s} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|^2_{V^*}dr,\end{aligned}$$

where $v_{\tau-t} \in D(\tau-t,\theta_{2,-t}\omega)$.

Proof. Formally, it follows from (3.10)-(3.12) that for each $\tau \in \mathbb{R}$, $t \geq 0$ and $\omega \in \Omega$,

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu\|Dv\|^2 = z(t,\omega)\langle f(\cdot,t),v\rangle. \tag{4.1}$$

The right-hand side of (4.1) is bounded by

$$|z(t,\omega)\langle f(\cdot,t),v\rangle| \leq \frac{1}{4}\nu\|Dv\|^2 + \frac{1}{\nu}z^2(t,\omega)\|f(\cdot,t)\|^2_{V^*}.$$

Therefore, from (4.1) we get

$$\frac{d}{dt}\|v\|^2 + \frac{3}{2}\nu\|Dv\|^2 \leq \frac{2}{\nu}z^2(t,\omega)\|f(\cdot,t)\|^2_{V^*}. \tag{4.2}$$

By (3.19) and (4.2) we have

$$\frac{d}{dt}\|v\|^2 + \nu\lambda\|v\|^2 + \frac{1}{2}\nu\|Dv\|^2 \leq \frac{2}{\nu}z^2(t,\omega)\|f(\cdot,t)\|^2_{V^*}. \tag{4.3}$$

Multiplying (4.3) by $e^{\nu\lambda t}$ and then integrating the inequality on $[\tau-t,s]$, we obtain

$$\begin{aligned}&\|v(s,\tau-t,\omega,v_{\tau-t})\|^2 + \frac{1}{2}\nu\int_{\tau-t}^{s} e^{\nu\lambda(r-s)}\|Dv(r,\tau-t,\omega,v_{\tau-t})\|^2dr\\ &\leq e^{\nu\lambda(\tau-s)}e^{-\nu\lambda t}\|v_{\tau-t}\|^2 + \frac{2}{\nu}\int_{\tau-t}^{s} e^{\nu\lambda(r-s)}z^2(r,\omega)\|f(\cdot,r)\|^2_{V^*}dr.\end{aligned}$$

Replacing ω by $\theta_{2,-\tau}\omega$ in the above, we get that

$$\begin{aligned}&\|v(s,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})\|^2 + \frac{1}{2}\nu\int_{\tau-t}^{s} e^{\nu\lambda(r-s)}\|Dv(r,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})\|^2dr\\ &\leq e^{\nu\lambda(\tau-s)}e^{-\nu\lambda t}\|v_{\tau-t}\|^2 + \frac{2}{\nu}e^{-\nu\lambda s}\int_{\tau-t}^{s} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|^2_{V^*}dr.\end{aligned} \tag{4.4}$$

We now estimate the last term on the right-hand side of (4.4). Let $\tilde{\omega} = \theta_{2,-\tau}\omega$. Then by (3.7) we find that there exists $R < 0$ such that for all $r \le R$,

$$-2\alpha\tilde{\omega}(r) \le -(\nu\lambda - \delta)r,$$

where δ is the positive constant in (3.22). Therefore, for all $r \le R$,

$$z^2(r, \tilde{\omega}) = e^{-2\alpha\tilde{\omega}(r)} \le e^{-(\nu\lambda-\delta)r}. \tag{4.5}$$

By (4.5) we have for all $r \le R$,

$$e^{\nu\lambda r} z^2(r, \theta_{2,-\tau}\omega)\|f(\cdot, r)\|^2_{V^*} = e^{(\nu\lambda-\delta)r} z^2(r, \tilde{\omega}) e^{\delta r}\|f(\cdot, r)\|^2_{V^*} \le e^{\delta r}\|f(\cdot, r)\|^2_{V^*},$$

which along with (3.22) shows that for every $s \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\int_{-\infty}^{s} e^{\nu\lambda r} z^2(r, \theta_{2,-\tau}\omega)\|f(\cdot, r)\|^2_{V^*} dr < \infty. \tag{4.6}$$

On the other hand, since $v_{\tau-t} \in D(\tau-t, \theta_{2,-t}\omega)$, for the first term on the right-hand side of (4.4), we have

$$e^{-\nu\lambda t}\|v_{\tau-t}\|^2 \le e^{-\nu\lambda t}\|D(\tau - t, \theta_{2,-t}\omega)\|^2 \to 0, \quad \text{as } t \to \infty.$$

This shows that there exists $T = T(\tau, \omega, D) > 0$ such that $e^{-\nu\lambda t}\|v_{\tau-t}\|^2 \le 1$ for all $t \ge T$. Thus, the first term on the right-hand side of (4.4) satisfies

$$e^{\nu\lambda(\tau-s)} e^{-\nu\lambda t}\|v_{\tau-t}\|^2 \le e^{\nu\lambda(\tau-s)}, \quad \text{for all } t \ge T. \tag{4.7}$$

From (4.4), (4.6) and (4.7), the lemma follows. □

As an immediate consequence of Lemma 4.1, we have the following estimates on the solutions of problem (3.10)-(3.13).

Lemma 4.2. *Suppose* (3.19) *and* (3.22) *hold. Then for every* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$ *and* $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, there exists* $T = T(\tau, \omega, D) > 0$ *such that for every* $k \ge 0$ *and for all* $t \ge T + k$*, the solution* v *of problem* (3.10)-(3.13) *with* ω *replaced by* $\theta_{2,-\tau}\omega$ *satisfies*

$$\begin{aligned}&\|v(\tau - k, \tau - t, \theta_{2,-\tau}\omega, v_{\tau-t})\|^2\\ &\le e^{\nu\lambda k} + \frac{2}{\nu} e^{\nu\lambda(k-\tau)} \int_{-\infty}^{\tau-k} e^{\nu\lambda r} z^2(r, \theta_{2,-\tau}\omega)\|f(\cdot, r)\|^2_{V^*} dr,\end{aligned}$$

where $v_{\tau-t} \in D(\tau - t, \theta_{2,-t}\omega)$*.*

Proof. Given $\tau \in \mathbb{R}$ and $k \ge 0$, let $s = \tau - k$. Let $T = T(\tau, \omega, D)$ be the positive constant claimed in Lemma 4.1. If $t \ge T + k$, then we have $t \ge T$ and $s \ge \tau - t$. Thus, the desired result follows from Lemma 4.1. □

Next, we prove the $\mathcal{D}$-pullback asymptotic compactness of the solutions of problem (3.10)-(3.13). For this purpose, we need the following weak continuity of solutions in initial data, which can be established by the standard methods as in [18].

Lemma 4.3. *Suppose* (3.19) *holds and* $f \in L^2_{loc}(\mathbb{R}, V^*)$*. Let* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,* v_τ*,* $v_{\tau,n} \in H$ *for all* $n \in \mathbb{N}$*. If* $v_{\tau,n} \rightharpoonup v_\tau$ *in* H*, then the solution* v *of problem* (3.10)-(3.13) *has the properties:*

$$v(r, \tau, \omega, v_{\tau,n}) \rightharpoonup v(r, \tau, \omega, v_\tau) \quad \textit{in } H \quad \textit{for all } r \ge \tau,$$

and

$$v(\cdot,\tau,\omega,v_{\tau,n}) \rightharpoonup v(\cdot,\tau,\omega,v_\tau) \quad \text{in} \;\; L^2((\tau,\tau+T),V) \;\; \text{for every} \;\; T>0.$$

The next lemma is concerned with the pullback asymptotic compactness of problem (3.10)-(3.13).

Lemma 4.4. *Suppose* (3.19) *and* (3.22) *hold. Then for every* $\tau\in\mathbb{R}$, $\omega\in\Omega$, $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathcal{D}$ *and* $t_n\to\infty$, $v_{0,n}\in D(\tau-t_n,\theta_{2,-t_n}\omega)$, *the sequence* $v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})$ *of solutions of problem* (3.10)-(3.13) *has a convergent subsequence in* H.

Proof. It follows from Lemma 4.2 with $k=0$ that, there exists $T=T(\tau,\omega,D)>0$ such that for all $t\ge T$,

$$\|v(\tau,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})\|^2 \le 1+\frac{2}{\nu}e^{-\nu\lambda\tau}\int_{-\infty}^{\tau} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2dr, \tag{4.8}$$

with $v_{\tau-t}\in D(\tau-t,\theta_{2,-t}\omega)$. Since $t_n\to\infty$, there exists $N_0\in\mathbb{N}$ such that $t_n\ge T$ for all $n\ge N_0$. Due to $v_{0,n}\in D(\tau-t_n,\theta_{2,-t_n}\omega)$, we get from (4.8) that for all $n\ge\mathbb{N}_0$,

$$\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2 \le 1+\frac{2}{\nu}e^{-\nu\lambda\tau}\int_{-\infty}^{\tau} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2dr. \tag{4.9}$$

By (4.9) there exists $\tilde v\in H$ and a subsequence (which is not relabeled) such that

$$v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}) \rightharpoonup \tilde v \quad \text{in } H. \tag{4.10}$$

We now prove that the weak convergence of (4.10) is actually a strong convergence, which will complete the proof. Note that (4.10) implies

$$\liminf_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\| \ge \|\tilde v\|. \tag{4.11}$$

So we only need to show

$$\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\| \le \|\tilde v\|. \tag{4.12}$$

We will establish (4.12) by the method of energy equations due to Ball [3]. Given $k\in\mathbb{N}$ we have

$$v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}) = v(\tau,\tau-k,\theta_{2,-\tau}\omega,\; v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})). \tag{4.13}$$

For each k, let N_k be large enough such that $t_n\ge T+k$ for all $n\ge N_k$. Then it follows from Lemma 4.2 that for $n\ge N_k$,

$$\begin{aligned}
&\|v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2\\
&\le e^{\nu\lambda k}+\frac{2}{\nu}e^{\nu\lambda(k-\tau)}\int_{-\infty}^{\tau-k} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2dr,
\end{aligned}$$

which shows that, for each fixed $k\in\mathbb{N}$, the sequence $v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})$ is bounded in H. By a diagonal process, one can find a subsequence (which we do not relabel) and a point $\tilde v_k\in H$ for each $k\in\mathbb{N}$ such that

$$v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}) \rightharpoonup \tilde v_k \quad \text{in } \; H. \tag{4.14}$$

By (4.13)-(4.14) and Lemma 4.3 we get that for each $k\in\mathbb{N}$,

$$v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}) \rightharpoonup v(\tau,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k) \quad \text{in } \; H, \tag{4.15}$$

and

$$v(\cdot,\tau-k,\theta_{2,-\tau}\omega,\ v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}))\rightharpoonup v(\cdot,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k) \tag{4.16}$$

in $L^2((\tau-k,\tau),V)$. By (4.10) and (4.15) we have

$$v(\tau,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k)=\tilde v. \tag{4.17}$$

Note that (4.1) implies that

$$\frac{d}{dt}\|v\|^2+\nu\lambda\|v\|^2+\psi(v)=2z(t,\omega)\langle f(\cdot,t),v\rangle, \tag{4.18}$$

where ψ is a functional on V given by

$$\psi(v)=2\nu\|Dv\|^2-\nu\lambda\|v\|^2,\quad \text{for all } v\in V.$$

By (3.19) we see that

$$\nu\|Dv\|^2\le\psi(v)\le 2\nu\|Dv\|^2,\quad \text{for all } v\in V.$$

This indicates that $\psi(\cdot)$ is an equivalent norm of V. It follows from (4.18) that for each $\omega\in\Omega$, $s\in\mathbb{R}$ and $\tau\ge s$,

$$\begin{aligned}\|v(\tau,s,\omega,v_s)\|^2=e^{\nu\lambda(s-\tau)}\|v_s\|^2-\int_s^\tau e^{\nu\lambda(r-\tau)}\psi(v(r,s,\omega,v_s))dr\\ +2\int_s^\tau e^{\nu\lambda(r-\tau)}z(r,\omega)\langle f(\cdot,r),v(r,s,\omega,v_s)\rangle dr.\end{aligned} \tag{4.19}$$

By (4.17) and (4.19) we find that

$$\begin{aligned}\|\tilde v\|^2&=\|v(\tau,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k)\|^2\\ &=e^{-\nu\lambda k}\|\tilde v_k\|^2-\int_{\tau-k}^\tau e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k))dr\\ &\quad+2\int_{\tau-k}^\tau e^{\nu\lambda(r-\tau)}z(r,\theta_{2,-\tau}\omega)\langle f(\cdot,r),v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k)\rangle dr.\end{aligned} \tag{4.20}$$

Similarly, by (4.13) and (4.19) we obtain that

$$\begin{aligned}&\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2\\ &=\|v(\tau,\tau-k,\theta_{2,-\tau}\omega,\ v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}))\|^2\\ &=e^{-\nu\lambda k}\|v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2\\ &\quad-\int_{\tau-k}^\tau e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})))dr\\ &\quad+2\int_{\tau-k}^\tau e^{\nu\lambda(r-\tau)}z(r,\theta_{2,-\tau}\omega)\\ &\quad\times\langle f(\cdot,r),v(r,\tau-k,\theta_{2,-\tau}\omega,v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}))\rangle dr.\end{aligned} \tag{4.21}$$

We now consider the limit of each term on the right-hand side of (4.21) as $n\to\infty$. For the first term, by (4.4) with $s=\tau-k$ and $t=t_n$ we get that

$$\begin{aligned}&e^{-\nu\lambda k}\|v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2\\ &\le e^{-\nu\lambda t_n}\|v_{0,n}\|^2+\frac{2}{\nu}e^{-\nu\lambda\tau}\int_{-\infty}^{\tau-k}e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2dr.\end{aligned} \tag{4.22}$$

Since $v_{0,n}\in D(\tau-t_n,\theta_{2,-t_n}\omega)$ we have

$$e^{-\nu\lambda t_n}\|v_{0,n}\|^2\le e^{-\nu\lambda t_n}\|D(\tau-t_n,\theta_{2,-t_n}\omega)\|^2\to 0\quad\text{as } n\to\infty,$$

which along with (4.22) shows that

$$\begin{aligned}
&\limsup_{n\to\infty} e^{-\nu\lambda k}\|v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2 \\
&\le \frac{2}{\nu}e^{-\nu\lambda\tau}\int_{-\infty}^{\tau-k} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2\,dr.
\end{aligned} \tag{4.23}$$

By (4.16) we find that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}z(r,\theta_{2,-\tau}\omega) \\
&\quad\times\langle f(\cdot,r),v(r,\tau-k,\theta_{2,-\tau}\omega,v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n}))\rangle dr \\
&=\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}z(r,\theta_{2,-\tau}\omega)\langle f(\cdot,r),v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k)\rangle dr,
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
&\liminf_{n\to\infty}\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})))dr \\
&\ge\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k))dr.
\end{aligned} \tag{4.25}$$

Note that (4.25) implies that

$$\begin{aligned}
&\limsup_{n\to\infty} -\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,v(\tau-k,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})))dr \\
&\le -\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k))dr.
\end{aligned} \tag{4.26}$$

Taking the limit of (4.21) as $n\to\infty$, by (4.23), (4.24) and (4.26) we obtain that

$$\begin{aligned}
&\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2 \\
&\le \frac{2}{\nu}e^{-\nu\lambda\tau}\int_{-\infty}^{\tau-k} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2\,dr \\
&\quad-\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}\psi(v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k))dr \\
&\quad+2\int_{\tau-k}^{\tau} e^{\nu\lambda(r-\tau)}z(r,\theta_{2,-\tau}\omega)\langle f(\cdot,r),v(r,\tau-k,\theta_{2,-\tau}\omega,\tilde v_k)\rangle dr.
\end{aligned} \tag{4.27}$$

It follows from (4.20) and (4.27) that

$$\begin{aligned}
&\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2 \\
&\le\|\tilde v\|^2+\frac{2}{\nu}e^{-\nu\lambda\tau}\int_{-\infty}^{\tau-k} e^{\nu\lambda r}z^2(r,\theta_{2,-\tau}\omega)\|f(\cdot,r)\|_{V^*}^2\,dr.
\end{aligned} \tag{4.28}$$

Let $k\to\infty$ in (4.28) to yield

$$\limsup_{n\to\infty}\|v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})\|^2\le\|\tilde v\|^2. \tag{4.29}$$

By (4.10)-(4.11) and (4.29) we find that

$$\lim_{n\to\infty} v(\tau,\tau-t_n,\theta_{2,-\tau}\omega,v_{0,n})=\tilde v \quad\text{in } H.$$

This completes the proof. □

5. Existence of pullback attractors

In this section, we establish the existence of $\mathcal{D}$-pullback attractors for the Navier-Stokes equations (3.1)-(3.2). Based on the uniform estimates on the solutions of problem (3.10)-(3.13), we first show that the cocycle Φ associated with the stochastic system (3.1)-(3.4) has a measurable $\mathcal{D}$-pullback absorbing set in H, and then prove the $\mathcal{D}$-pullback asymptotic compactness of Φ.

Lemma 5.1. *Suppose* (3.19) *and* (3.22) *hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$, the solution u of problem* (3.1)-(3.4) *with ω replaced by $\theta_{2,-\tau}\omega$ satisfies*

$$\begin{aligned}
&\|u(\tau, \tau - t, \theta_{2,-\tau}\omega, u_{\tau-t})\|^2 \\
&\leq z^{-2}(\tau, \theta_{2,-\tau}\omega) + \frac{2}{\nu} z^{-2}(\tau, \theta_{2,-\tau}\omega) \int_{-\infty}^{\tau} e^{\nu\lambda(r-\tau)} z^2(r, \theta_{2,-\tau}\omega) \|f(\cdot, r)\|_{V^*}^2 dr,
\end{aligned}$$

where $u_{\tau-t} \in D(\tau - t, \theta_{2,-t}\omega)$.

Proof. Given $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$\tilde{D}(\tau, \omega) = \{v \in H : \|v\| \leq |z(\tau, \theta_{2,-\tau}\omega)| \; \|D(\tau, \omega)\|\}. \tag{5.1}$$

Let $\tilde{D}$ be a family corresponding to D which consists of the sets given by (5.1); i.e.,

$$\tilde{D} = \{\tilde{D}(\tau, \omega) : \tilde{D}(\tau, \omega) \text{ is defined by } (5.1), \tau \in \mathbb{R}, \omega \in \Omega\}. \tag{5.2}$$

We now prove $\tilde{D}$ is tempered in H for $D \in \mathcal{D}$. Given $c > 0$, by (3.7) we find that for each $\omega \in \Omega$, there exists $R > 0$ such that for all $r \geq R$,

$$| - \alpha\omega(-r)| \leq \frac{1}{2} cr. \tag{5.3}$$

Since $D \in \mathcal{D}$, from (5.3) it follows that

$$\begin{aligned}
e^{-cr} \|\tilde{D}(\tau - r, \theta_{2,-r}\omega)\| &= e^{-cr} |z(\tau - r, \theta_{2,-\tau}\omega)| \; \|D(\tau - r, \theta_{2,-r}\omega)\| \\
&\leq e^{\alpha\omega(-\tau)} e^{-\frac{1}{2}cr} \|D(\tau - r, \theta_{2,-r}\omega)\| \to 0, \quad \text{as } r \to \infty,
\end{aligned}$$

which shows that $\tilde{D} \in \mathcal{D}$. Since $u_{\tau-t} \in D(\tau - t, \theta_{2,-t}\omega)$, by (3.9) we know that

$$\|v_{\tau-t}\| = \|z(\tau - t, \theta_{2,-\tau}\omega) \; u_{\tau-t}\| \leq |z(\tau - t, \theta_{2,-\tau}\omega)| \, \|D(\tau - t, \theta_{2,-t}\omega)\|,$$

which along with (5.1) implies that $v_{\tau-t} \in \tilde{D}(\tau - t, \theta_{2,-t}\omega)$. Since $\tilde{D}$ is tempered, it follows from Lemma 4.2 with $k = 0$ that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$\|v(\tau, \tau - t, \theta_{2,-\tau}\omega, v_{\tau-t})\|^2 \leq 1 + \frac{2}{\nu} \int_{-\infty}^{\tau} e^{\nu\lambda(r-\tau)} z^2(r, \theta_{2,-\tau}\omega) \|f(\cdot, r)\|_{V^*}^2 dr,$$

which along with (3.9) completes the proof. □

Lemma 5.2. *Suppose* (3.19) *and* (3.23) *hold. Then the continuous cocycle Φ associated with problem* (3.1)-(3.4) *has a closed measurable $\mathcal{D}$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Proof. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$K(\tau, \omega) = \{u \in H : \|u\|^2 \le M(\tau, \omega)\}, \tag{5.4}$$

where $M(\tau, \omega)$ is given by

$$\begin{aligned} M(\tau, \omega) &= z^{-2}(\tau, \theta_{2,-\tau}\omega) \\ &\quad + \frac{2}{\nu} z^{-2}(\tau, \theta_{2,-\tau}\omega) \int_{-\infty}^{\tau} e^{\nu\lambda(r-\tau)} z^2(r, \theta_{2,-\tau}\omega) \|f(\cdot, r)\|_{V^*}^2 dr. \end{aligned} \tag{5.5}$$

Since for each $\tau \in \mathbb{R}$, $M(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathcal{F}_1, \mathcal{B}(\mathbb{R}))$-measurable, we know that $K(\tau, \cdot) : \Omega \to 2^H$ is a measurable set-valued mapping. It follows from Lemma 5.1 that, for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \ge T$,

$$\Phi(t, \tau - t, \theta_{2,-t}\omega, D(\tau - t, \theta_{2,-t}\omega)) = u(\tau, \tau - t, \theta_{2,-\tau}\omega, D(\tau - t, \theta_{2,-t}\omega)) \subseteq K(\tau, \omega).$$

Therefore, $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ will be a closed measurable $\mathcal{D}$-pullback absorbing set of Φ in H if one can show that K belongs to $\mathcal{D}$. For each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $r > 0$, by (5.4) we have

$$\begin{aligned} &\|K(\tau - r, \theta_{2,-r}\omega)\| \\ &\le \frac{1}{z(\tau - r, \theta_{2,-\tau}\omega)} \Big(1 + \frac{2}{\nu} \int_{-\infty}^{\tau - r} e^{\nu\lambda(s-\tau+r)} z^2(s, \theta_{2,-\tau}\omega) \|f(\cdot, s)\|_{V^*}^2 ds\Big)^{1/2} \\ &\le e^{-\alpha\omega(-\tau)} e^{\alpha\omega(-r)} \\ &\quad \times \Big(1 + \frac{2}{\nu} \int_{-\infty}^{0} e^{\nu\lambda s} z^2(s + \tau - r, \theta_{2,-\tau}\omega) \|f(\cdot, s + \tau - r)\|_{V^*}^2 ds\Big)^{1/2} \\ &\le e^{-\alpha\omega(-\tau)} e^{\alpha\omega(-r)} \Big(1 \\ &\quad + \Big(\frac{2}{\nu} \int_{-\infty}^{0} e^{(\nu\lambda-\delta)s} z^2(s + \tau - r, \theta_{2,-\tau}\omega) e^{\delta s} \|f(\cdot, s + \tau - r)\|_{V^*}^2 ds\Big)^{1/2}\Big). \end{aligned} \tag{5.6}$$

Let c be an arbitrary positive number and $\varepsilon = \min\{\nu\lambda - \delta,\ \frac{1}{2}c\}$. By (3.7) we see that there exists $N_1 > 0$ such that

$$|-2\alpha\ \omega(p)| \le -\varepsilon p \quad \text{for all } p \le -N_1. \tag{5.7}$$

Let $s \le 0$ and $r \ge N_1$. Then $p = s - r \le -N_1$ and hence it follows from (5.7) that

$$-2\alpha\ \omega(s - r) \le -\varepsilon(s - r), \quad \text{for all } s \le 0 \text{ and } r \ge N_1. \tag{5.8}$$

By (5.8) we have, for all $s \le 0$ and $r \ge N_1$,

$$e^{(\nu\lambda-\delta)s} z^2(s + \tau - r, \theta_{2,-\tau}\omega) \le e^{(\nu\lambda-\delta)s} e^{2\alpha\omega(-\tau)} e^{-2\alpha\omega(s-r)} \le e^{2\alpha\omega(-\tau)} e^{\varepsilon r}. \tag{5.9}$$

From (5.6), (5.7) and (5.9) we have that, for all $r \ge N_1$,

$$\begin{aligned} &\|K(\tau - r, \theta_{2,-r}\omega)\| \\ &\le e^{\varepsilon r - \alpha\omega(-\tau)} + \sqrt{\frac{2}{\nu}} e^{\frac{3}{2}\varepsilon r} \Big(\int_{-\infty}^{0} e^{\delta s} \|f(\cdot, s + \tau - r)\|_{V^*}^2 ds\Big)^{1/2} \\ &\le e^{\frac{1}{2}cr - \alpha\omega(-\tau)} + \sqrt{\frac{2}{\nu}} e^{\frac{3}{4}cr} \Big(\int_{-\infty}^{0} e^{\delta s} \|f(\cdot, s + \tau - r)\|_{V^*}^2 ds\Big)^{1/2}, \end{aligned} \tag{5.10}$$

where we have used the fact $\varepsilon \le c/2$. It follows from (5.10) that, for all $r \ge N_1$,

$$e^{-cr} \|K(\tau - r, \theta_{2,-r}\omega)\|$$

$$\leq e^{-\frac{1}{2}cr-\alpha\omega(-\tau)} + \sqrt{\frac{2}{\nu}}e^{-\frac{1}{4}cr}\Big(\int_{-\infty}^{0} e^{\delta s}\|f(\cdot, s+\tau-r)\|_{V^*}^2 ds\Big)^{1/2}$$
$$\leq e^{-\frac{1}{2}cr-\alpha\omega(-\tau)} + \sqrt{\frac{2}{\nu}}e^{-\frac{1}{4}c\tau}\left(e^{\frac{1}{2}c(\tau-r)}\int_{-\infty}^{0} e^{\delta s}\|f(\cdot, s+\tau-r)\|_{V^*}^2 ds\right)^{1/2},$$

which along with (3.23) shows that for every positive constant c,

$$\lim_{r\to\infty} e^{-cr}\|K(\tau-r, \theta_{2,-r}\omega)\| = 0,$$

and hence $K = \{K(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is tempered. This completes the proof. □

We now prove the $\mathcal{D}$-pullback asymptotic compactness of solutions of the stochastic equations (3.1)-(3.2).

Lemma 5.3. *Suppose* (3.19) *and* (3.23) *hold. Then the continuous cocycle* Φ *associated with problem* (3.1)-(3.4) *is* $\mathcal{D}$*-pullback asymptotically compact in* H*, that is, for every* $\tau \in \mathbb{R}$*,* $\omega \in \Omega$*,* $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$*, and* $t_n \to \infty$*,* $u_{0,n} \in D(\tau - t_n, \theta_{2,-t_n}\omega)$*, the sequence* $\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$ *has a convergent subsequence in* H*.*

Proof. Since $D \in \mathcal{D}$ and $u_{0,n} \in D(\tau - t_n, \theta_{2,-t_n}\omega)$, by the proof of Lemma 5.1 we find that for each $n \in \mathbb{N}$, $v_{0,n} = z(\tau - t_n, \theta_{2,-\tau}\omega)u_{0,n} \in \tilde{D}(\tau - t_n, \theta_{2,-t_n}\omega)$, where $\tilde{D} \in \mathcal{D}$ is the family defined by (5.2). Then it follows from Lemma 4.4 that the sequence $v(\tau, \tau - t_n, \theta_{2,-\tau}\omega, v_{0,n})$ of solutions of problem (3.10)-(3.13) has a convergent subsequence in H. By (3.9) we have

$$u(\tau, \tau - t_n, \theta_{2,-\tau}\omega, u_{0,n}) = \frac{1}{z(\tau, \theta_{2,-\tau}\omega)} v(\tau, \tau - t_n, \theta_{2,-\tau}\omega, v_{0,n}),$$

and hence the sequence $u(\tau, \tau - t_n, \theta_{2,-\tau}\omega, u_{0,n})$ has a convergent subsequence in H. This implies $\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$ has a convergent subsequence in H. □

We are now in a position to present the main result of the paper, that is, the existence of tempered pullback attractors for the stochastic Navier-Stokes equations.

Theorem 5.4. *Suppose* (3.19) *and* (3.23) *hold. Then the continuous cocycle* Φ *associated with problem* (3.1)-(3.4) *has a unique* $\mathcal{D}$*-pullback attractor* $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R},\ \omega \in \Omega\} \in \mathcal{D}$ *in* H*. Moreover, for each* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

$$\mathcal{A}(\tau,\omega) = \Omega(K,\tau,\omega) = \cup_{B\in\mathcal{D}}\Omega(B,\tau,\omega) \tag{5.11}$$
$$= \{\psi(0,\tau,\omega) : \psi \text{ is any } \mathcal{D}\text{-complete orbit of } \Phi\}. \tag{5.12}$$

Proof. By Lemma 5.2 we know that Φ has a closed measurable $\mathcal{D}$-pullback absorbing set in H. On the other hand, by Lemma 5.3 we know that Φ is $\mathcal{D}$-pullback asymptotically compact. Then it follows from Proposition 2.10 that Φ has a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ in H and the structure of $\mathcal{A}$ is given by (5.11)-(5.12). □

We now discuss the existence of periodic pullback attractors for problem (3.1)-(3.4). Suppose $f : \mathbb{R} \to V^*$ is a periodic function with period $T > 0$. If, in addition, $f \in L^2_{loc}(\mathbb{R}, V^*)$, then one can verify that f satisfies (3.23) for any $\delta > 0$. In this case, for every $\tilde{u} \in H$, $t \geq 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have that

$$\begin{aligned}\Phi(t, \tau + T, \omega, \tilde{u}) &= u(t + \tau + T, \tau + T, \theta_{2,-\tau-T}\omega, \tilde{u})\\ &= u(t + \tau, \tau, \theta_{2,-\tau}\omega, \tilde{u}). = \Phi(t, \tau, \omega, \tilde{u}).\end{aligned}$$

By Definition 2.1, we find that Φ is periodic with period T. Let $D \in \mathcal{D}$ and D_T be the T-translation of D. Then for every $c > 0$, $s \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{r\to\infty} e^{-cr}\|D(s-r,\theta_{2,-r}\omega)\|^2 = 0. \tag{5.13}$$

In particular, for $s = \tau + T$ with $\tau \in \mathbb{R}$, we get from (5.13) that

$$\lim_{r\to\infty} e^{-cr}\|D_T(\tau-r,\theta_{2,-r}\omega)\|^2 = \lim_{r\to\infty} e^{-cr}\|D(\tau+T-r,\theta_{2,-r}\omega)\|^2 = 0. \tag{5.14}$$

From (5.14) we see that $D_T \in \mathcal{D}$, and hence $\mathcal{D}$ is T-translation closed. Similarly, one may check that $\mathcal{D}$ is also $-T$-translation closed. Therefore, we find that $\mathcal{D}$ is T-translation invariant. By Proposition 2.11, the periodicity of the $\mathcal{D}$-pullback attractor of problem (3.1)-(3.4) follows.

Theorem 5.5. *Let $f : \mathbb{R} \to V^*$ be a periodic function with period $T > 0$ and $f \in L^2((0,T),V^*)$. If* (3.19) *holds, then the continuous cocycle Φ associated with problem* (3.1)-(3.4) *has a unique $\mathcal{D}$-pullback attractor $\mathcal{A} \in \mathcal{D}$ in H, which is periodic with period T.*

In the present article, we have discussed the pullback attractors of the two-dimensional stochastic Navier-Stokes equations with non-autonomous deterministic force. It is also interesting to consider the same problem for the three-dimensional Navier-Stokes equations, where the uniqueness of solutions does not hold anymore. In this case, the author believes that the idea of multivalued dynamical systems developed in [10] can be extended to study the pullback attractors of the three-dimensional equations with non-autonomous deterministic force. The author will pursue this line of research in the future.

References

[1] L. Arnold, *Random Dynamical Systems*; Springer-Verlag, 1998.

[2] A. V. Babin, M. I. Vishik; *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[3] J. M. Ball; Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, *J. Nonl. Sci.*, **7** (1997), 475-502.

[4] P. W. Bates, H. Lisei, K. Lu; Attractors for stochastic lattice dynamical systems, *Stoch. Dyn.*, **6** (2006), 1-21.

[5] P. W. Bates, K. Lu, B. Wang; Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **246** (2009), 845-869.

[6] Z. Brzezniak, Y. Li; Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains, *Transactions of American Mathematical Society*, **358** (2006), 5587-5629.

[7] T. Caraballo, J. Real, I. D. Chueshov; Pullback attractors for stochastic heat equations in materials with memory, *Discrete Continuous Dynamical Systems B*, **9** (2008), 525-539.

[8] T. Caraballo, G. Lukaszewicz, J. Real; Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Analysis, TMA*, **64** (2006), 484-498.

[9] T. Caraballo, G. Lukaszewicz, J. Real; Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains, *C. R. Acad. Sci. Paris I*, **342** (2006), 263-268.

[10] T. Caraballo, J. A. Langa, V. S. Melnik, J. Valero; Pullback attractors of non-autonomous and stochastic multivalued dynamical systems, *Set-Valued Analysis*, **11** (2003), 153-201.

[11] I. Chueshov, M. Scheutzow; On the structure of attractors and invariant measures for a class of monotone random systems, *Dynamical Systems*, **19** (2004), 127-144.

[12] H. Crauel, A. Debussche, F. Flandoli; Random attractors, *J. Dyn. Diff. Eqns.*, **9** (1997), 307-341.

[13] H. Crauel, F. Flandoli; Attractors for random dynamical systems, *Probab. Th. Re. Fields*, **100** (1994), 365-393.

[14] F. Flandoli, B. Schmalfuß; Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, *Stoch. Stoch. Rep.*, **59** (1996), 21-45.
[15] J. K. Hale; Asymptotic Behavior of Dissipative Systems, American Mathematical Society, Providence, RI, 1988.
[16] J. Huang, W. Shen; Pullback attractors for nonautonomous and random parabolic equations on non-smooth domains, *Discrete and Continuous Dynamical Systems*, **24** (2009), 855-882.
[17] P. E. Kloeden, J. A. Langa; Flattening, squeezing and the existence of random attractors, *Proc. Royal Soc. London Serie A.*, **463** (2007), 163-181.
[18] R. Rosa; The global attractor for the 2D Navier-Stokes flow on some unbounded domains, *Nonlinear Analysis, TMA*, **32** (1998), 71-85.
[19] B. Schmalfuß; Backward cocycles and attractors of stochastic differential equations, *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*, 1992, 185-192.
[20] R. Sell, Y. You; Dynamics of Evolutionary Equations, Springer-Verlag, New York, 2002.
[21] R. Temam; Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1997.
[22] B. Wang; Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Transactions of American Mathematical Society*, **363** (2011), 3639-3663.
[23] B. Wang, Random Attractors for the Stochastic Benjamin-Bona-Mahony Equation on Unbounded Domains, *J. Differential Equations*, **246** (2009), 2506-2537.
[24] B. Wang; Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, arXiv:1202.2390v1 [math.AP], 2012.

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