Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 60, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

HARNACK INEQUALITIES, A PRIORI ESTIMATES, AND SUFFICIENT STATISTICS FOR NONLINEAR ELLIPTIC SYSTEMS IN QUANTUM MECHANICS

CARLOS C. ARANDA

Abstract. In this article, we consider systems of nonlinear elliptic problems and their relations with minimal sufficient statistics, which is a fundamental tool in classics statistics. This allows us to introduce new experimental tools in quantum physics.

1. The Hamiltonian operator and the Schrödinger equation

In [7] it is stated that "The wave function Ψ completely determines the states of a physical system in quantum mechanics". Thus we let q represent the coordinates of a particle and $|\Psi|^2 dq$ the probability that a measurement performed on the system will find the values of the coordinates to be in the element dq of the space. The Hamilton equation is

$$\imath\hbar \frac{d\Psi}{dt} = \hat{H}\Psi, \tag{1.1}$$

where $\hat{H}$ is the linear operator given by

$$\hat{H} = -\frac{1}{2}\hbar^2 \sum_a \frac{\Delta_a}{m_a} + U(r_1, r_2, \dots), \tag{1.2}$$

where a is the number of the particle, Δ_a is the Laplacian operator in which the differentiation is with respect to the coordinate of the ath particle and $U(r_1, r_2, \dots)$ is the potential energy of the interaction in function only of the coordinates of the particle. The eigenvalue equation for one particle

$$\hat{H}\psi = E\psi,$$

represents the eigenvalue E energy and ψ represents the stationary waves. The stationary wave corresponding to the smallest energy is called the normal or ground state of the system. Finally an arbitrary wave function has a expansion given by

$$\Psi = \sum_n a_n e^{-\frac{\imath}{\hbar} E_n t} \psi_n(q).$$

2000 *Mathematics Subject Classification.* 35A20, 35B05, 35B40.
Key words and phrases. A priori bounds; integral relation method; quantum mechanics; celestial mechanics.

Submitted August 10, 211. Published April 12, 2012.

2. A review on statistics

In this section, we recall some basics on mathematical statistics. Our main goal is to join theories that usually are studied in a no related approach.

Definition 2.1. The characteristic S_1 is subordinate to S_2 if S_1 is a measurable function of $S_2 : S_1 = \varphi(S_2)$. In the σ-algebra representation the statistic S_1 is subordinate to S_2 if $\sigma(S_1) \subseteq \sigma(S_2)$, where $\sigma(S)$ is the σ-algebra generated by the measurable function $S : (\Omega, \sigma, P_\theta) \to \mathbb{R}$. We assume that there exists a measure P such that $\frac{dP_\theta}{dP}$ the Radon Nykodim derivative is well defined for all θ in the parameter space Θ.

Definition 2.2. If S_1 is subordinate to S_2 and S_2 is subordinate to S_1, we denominate this statistics equivalents.

Definition 2.3 (Fisher [8], Neyman [20] and Zacks [25]). *The statistic S is sufficient if for any measurable function $T : (\Omega, \sigma) \to \mathbb{R}$, the conditional distribution $P_\theta(T \in B|\sigma(S))$ is independent of θ.*

Definition 2.4. The sufficient statistic S_0 is a minimal statistic if it is subordinate to any S sufficient statistic.

Theorem 2.5 (Lehmann-Scheffe [14]). *The σ-algebra $\mathbb{U} = \sigma(r(x,\theta) \equiv \frac{f_\theta}{f_P}(x) = \frac{dP_\theta}{dP}(x),\ \theta \in \Theta)$ is a minimal sufficient σ-algebra.*

Remark 2.6. If we define $S(x)$ such that $S(x) = S(x_0)$ if and only if the relation $h(x, x_0) = \frac{f_\theta(x)}{f_\theta(x_0)}$ no depend on θ. Then S is a sufficient minimal statistic.

Theorem 2.7. *(Blackwell [3], Rao [21]) If we assume $E_\theta(T) = \int T(x) f_\theta(x) dP = \theta$, $\theta \in \Theta$. Then*

$$\int (E(T|S)(x) - \theta)^2 f_\theta(x) dP \le \int (T(x) - \theta)^2 f_\theta(x) dP,$$

for all $\theta \in \Theta$.

3. A priori estimates

This section is concerned with the study of a priori estimates for the equation

$$-\Delta u = f(x, u, \nabla u),\ u(x) \ge 0,\ x \in \Omega \subset \mathbb{R}^N, \tag{3.1}$$

and the fully coupled systems

$$\begin{gathered} -\Delta u = f_1(x, u, v), \\ -\Delta v = f_2(x, u, v), \\ u(x), v(x) \ge 0, \quad x \in \Omega \subset \mathbb{R}^N, \end{gathered}$$

where Ω is a open set in $\mathbb{R}^N$, $N > 2$. A priori estimate means: we obtain bounds independent of any given solution and any given boundary condition. This is central because in Schrödinger equations we have no boundary and therefore for a statistical analysis we need this class of a priori estimates. Another central topic in this article is the derivation of weakly coupled Harnack inequalities for fully coupled systems. In [7] it is stated that

> If the integral $|\Psi|^2$ converges, then by choosing an appropriate constant coefficient the function ψ can always be, as we say, normalized. However, we shall see later that the integral of $|\Psi|^2$ may diverge, and then ψ cannot be normalized by the condition... In such cases $|\Psi|^2$ does not, of course, determine the absolute values of the probability of the coordinates, but the ratio of the values of $|\Psi|^2$ at two different points of configuration space determines the relative probability of the corresponding values of the coordinates.

Existence of a priori estimates for the equation (3.1) is a question with few known results, moreover it seems there are not previous results for systems. In [10] the authors obtain a priori estimates for (3.1) with $f(x, u, \nabla u) = u^r$, in the neighborhood of an isolated singularity. In [6] we have the same result:

Theorem 3.1 (Dancer [6]). *Assume $N > 2$ and $1 < r < \frac{N+2}{N-2}$. Let u be a non-negative solution of the equation*

$$-\Delta u = u^r,$$

in a domain $\Omega \not\equiv \mathbb{R}^N$. Then for every $x \in \Omega$ we have

$$u(x) \le C(N, p)[\operatorname{dist}(x, \partial\Omega)]^{-2/(r-1)}.$$

In particular, u is bounded on any compact subset Ω' of Ω, the bound being independent of the solution. The range $1 < r < \frac{N+2}{N-2}$ and the exponent are both optimal.

Now, we recall two results.

Theorem 3.2 (Ladyzhenskaya and Ural'tseva [13]). *Suppose that $u(x)$ is a generalized solution in $W^{1,2}(\Omega)$ of the equation*

$$\frac{\partial}{\partial x_i}(a_{ij}u_{x_j} + a_i u) + b_i u_{x_i} + au = f + \frac{\partial f_i}{\partial x_i},$$

with the conditions

$$\nu \sum_{i=1}^{N} \xi_i^2 \le a_{ij}\xi_i\xi_j \le \mu \sum_{i=1}^{N} \xi_i^2, \quad \nu > 0,$$

$$\| \sum_{i=1}^{N} a_i^2, \quad \sum_{i=1}^{N} b_i^2, \quad a\|_{L^{\frac{q}{2}}(\Omega)} \le \mu, \quad q > N.$$

Then, for an arbitrary $\Omega' \subset \Omega$, the quantity $\operatorname{ess,max}_{\Omega'} |u|$ is finite and bounded from above by a constant depending only on $\nu, \mu, q, \|u\|_{L^2(\Omega)}$ and the distance from Ω' to $\partial\Omega$.

Remark 3.3. The conditions in Theorem 3.2 are necessary and sufficient for uniqueness "in the small", and more important in our context, to have bounded solutions [13].

Lemma 3.4 (Bidaut Veron and Pohozaev [1]). *If u is a non-negative weak solution of*

$$-\Delta u \ge C_1 u^r \quad \text{in } \Omega, \tag{3.2}$$

where Ω is an open set in $\mathbb{R}^N$. Then

$$\int_{B_R} u^\gamma dx \le C(r, \gamma, N, C_1) R^{N - \frac{2\gamma}{r-1}}, \tag{3.3}$$

where $B_{2R} \subset \Omega$ *is a ball of radius* $2R$ *and* $r-1<\gamma<r$.

A careful examination of Theorem 3.2 and Lemma 3.4 suggest that it is possible to obtain a priori estimates for a class of nonlinear elliptic problems with convection terms and optimal perturbations. Our first result is as follows.

Theorem 3.5. *If* u *is a non-negative* $W^{1,2}(\Omega)$ *solution of the equation*

$$-\Delta u = du + f(u) + G(\nabla u) + g \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{3.4}$$

where

(i) $d, g \in L^{\frac{q}{2}}(\Omega)$, $q > N$, $d \geq 0$ *and* $g \geq 0$.
(ii) $0 \leq G(\nabla u) \leq \mathcal{C}|\nabla u| + \mathcal{C}_0$.
(iii) $C_1 u^r \leq f(u) \leq C_2 u^r$, $1 < r < \frac{N}{N-2} = 2_* - 1$.

Then for any ball $B_{3R}(y) \subset \Omega$,

$$\sup_{B_R(y)} \{u+1\} \leq H_1(R), \tag{3.5}$$

$$\int_{B_R(y)} |\nabla u|^2 dx \leq H_2(R), \tag{3.6}$$

where $lim_{R\searrow 0} H_i(R) = \infty$, *for* $i = 1,2$ *and the dependence is given by*

$$H_1(R) = H_1(R, \|d,g\|_{L^{q/2}_{\rm loc}(\Omega)}, \mathcal{C}, \mathcal{C}_0, C_1, C_2),$$
$$H_2(R) = H_2(R, \operatorname{dist}(B_R(y), \partial\Omega), \|d,g\|_{L^{q/2}_{\rm loc}(\Omega)}, \mathcal{C}, \mathcal{C}_0, C_1, C_2).$$

Remark 3.6. In [22, Lemma 2.4] is stated that: if $-\Delta u \geq u^r$ in Ω where $u \geq 0$ and $r > 1$, then $\int_{B_R} |\nabla u|^\mu dx \leq C$ for all $\mu \in (0, \frac{2r}{r+1})$. This result does not cover our estimate (3.6). Moreover in [22], there are not optimal condition like $d, g \in L^{q/2}(\Omega)$ on the perturbation.

Remark 3.7. Matukuma [17] proposed the equation

$$-\Delta u = \frac{u^r}{1+|x|^2} \quad \text{in } \mathbb{R}^3 \tag{3.7}$$

where u is the gravity potential and $\rho = (2\pi)^{-1}(1+|x|^2)^{-1}u^r$ is the density, to study the gravitational potential u of a globular cluster of stars. For the same problem Hénon [12] suggested

$$-\Delta u = |x|^l u^r \quad \text{in } \Omega \subset \mathbb{R}^3. \tag{3.8}$$

Our work is well suited to establish bounds for the gravitational potential u, particularly in the presence of "black holes", that means situations where u becomes singular, Ω is punctured or if in (3.8) we have $l < 0$. For example we analyze equation (3.8): if $0 \in \Omega$, $1 < sq < 3$ and $0 < -l\frac{s}{s-1}\frac{q}{2} < 1$ for $s > 1$, $q > 3$, using the Young inequality $ab \leq \frac{a^q}{q} + \frac{q-1}{q} b^{\frac{q}{q-1}}$, we deduce that the problem (3.8) has not "black holes" solutions. Black holes solutions means that the gravitational potential of the cluster behaves like $\frac{1}{r}$ $(r = |x|)$ near the center. In 1972 Peebles gives for the first time a derivation of the steady state distribution of the star near a massive collapsed object, the same year Peebles motivated the observer and theoretician with the title of his paper "Black holes are where you find them"([15]-[16]. The question of the existence of black hole in a globular cluster is still open (1995). Core collapse does occur, for instance using Hubble Space Telescope, Bendinelli documented the first detection of a collapsed cluster in M31 [2]

> Tous ces corps devenus invisibles son à le même place où ils on été observés, puisqu'ils n' en ont point changé, durant leur apparation il existe donc dans les spaces célestes, des corps obscurs aussi considérables et peut être en aussi grand nombre que les etoiles. Un astre lumineux de meme densite que la terre, et dont le diametre serait deux cents cinquante fois plus grand que celui du soleil, ne laisserant en vertu de son attraction parvenr aucun de ses rayons jusqu'à nous, il es done possible que les plus grands corps lumineux de l'univers, soient par cela même invisibles. Une étoile qui sans être de cette grandeur, surpaserait considerablement le soileil, affaiblirait sensiblemente la vîtesse de la lumiere et augmenterait ainsi l'étendue de la lumière.

This quotation belongs to Pierre Simon de Laplace in Exposition du systeme du Monde 1796, second volume p348.

Harnack inequality for weak solutions is a classical property in the study of linear elliptic equations [19]. For nonlinear elliptic equations, we have

Theorem 3.8 (Schoen [23]). *For $N \geq 3$, let B_{3R} be a ball of radius $3R$ in $\mathbb{R}^N$, and let $u \in C^2(B_{3R})$ be a positive solution of*

$$-\Delta u = N(N-2)u^{\frac{N+2}{N-2}}.$$

Then

$$(\max_{B_R} u)(\min_{B_{2R}} u) \leq C(N)R^{2-N}.$$

For the p-Laplacian operator, we have

Definition 3.9 (Serrin and Zou [22]). *We say that f is subcritical if $N > p$ and there exists a number $1 < \alpha < p^* = \frac{Np}{N-p}$ such that*

$$f(u) \geq 0,\ (\alpha-1)f(u) - uf'(u) \geq 0,\ \text{for } u \geq 0.$$

Note in particular that the function $f(u) = u^{r-1}$ is subcritical when $1 < r < p^*$. Let $\Omega \in \mathbb{R}^N$ and assume $N > p$. Let u be a non-negative weak solution of the two-sided differential inequality

$$u^{r-1} - u^{p-1} \leq -\Delta_p u \leq \Lambda(u^{r-1} + 1),\ x \in \Omega, \tag{3.9}$$

where $\Lambda > 1$ and $p < r < p_* = \frac{p(N-1)}{N-p}$. Let u be a solution of

$$-\Delta_p u = f(u),\ u \geq 0,\ x \in \Omega. \tag{3.10}$$

Suppose that f is subcritical and that, for some $\Lambda > 1$ and $r > p$, it satisfies the power-like condition

$$u^{r-1} \leq f(u) \leq \Lambda(u^{r-1} + 1). \tag{3.11}$$

Theorem 3.10 (Serrin and Zou [22]). *Let R and x_0 be such that $B_R \equiv B_R(x_0) \subset B_{2R}(x_0) \subset \Omega$, and assume $N > p$. Then we have the following conclusions.*

(a) Let u be a non-negative weak solution of the differential inequality (3.9). *Then for every $R_0 > 0$ there exists $C = C(N, p, r, \Lambda, R_0) > 0$ such that*

$$\sup_{B_R} u \leq C \inf_{B_R} u, \tag{3.12}$$

provided $R \leq R_0$. If the terms u^{p-1} and 1 are dropped in (3.9), *then* (3.12) *holds with $C = C(N, p, r, \Lambda)$ and with no further restriction on R.*

(b) Let u be a solution of (3.10)*, where f is subcritical. Suppose either* $N = 2$ *and* $p > \frac{1}{4}(1+\sqrt{17})$ *, or* $N \in [3, 2p)$*,* $p > \frac{3}{2}$*. Then* (3.12) *holds with* $C = C(N, p, \alpha) > 0$*.*

(c) Let u be a solution of (3.10)*, where f is subcritical, and suppose that* (3.11) *is satisfied for some* $r > p$*. Then* (3.12) *holds with* $C = C(N, p, r, \alpha, \Lambda)$*.*

The second statement of this work is as follows.

Theorem 3.11. *If u is a non-negative* $W^{1,2}(\Omega)$ *solution of the equation*

$$-\Delta u = du + f(u) + G(\nabla u) + g \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{3.13}$$

where:

(i) $d, g \in L^{\frac{q}{2}}(\Omega)$*,* $q > N$ *and* $d, g \geq 0$*.*
(ii) $0 \leq G(\nabla u) \leq \mathcal{C}|\nabla u| + \mathcal{C}_0$*.*
(iii) $C_1 u^r \leq f(u) \leq C_2 u^r$*,* $1 < r < \frac{N}{N-2}$*.*

Then for any ball $B_{3R}(y) \subset \Omega$

$$\sup_{B_R(y)} \{u + 1\} \leq C(R) \inf_{B_R(y)} \{u + 1\},$$

where the dependence is given by

$$C(R) = C(R, \|d, g\|_{L^{q/2}_{\rm loc}(\Omega)}, \mathcal{C}, \mathcal{C}_0, C_1, C_2).$$

The above theorem is valid for $f(u) \equiv 0$. This statement of classical Harnack inequality is different from the usual ones where $d \in L^\infty(\Omega)$ and generalizes the usual Harnack inequality for homogeneous elliptic equations for the nonhomogeneous situation. The reader can see that in [11]. With the Moser iteration technique developed in this paper, we obtain a priori estimates for a class of fully coupled systems:

Theorem 3.12. *If u,v are* $W^{1,2}(\Omega)$ *non-negative solutions of*

$$-\Delta u = d_1 u + f_{1,1}(u) + f_{1,2}(v) + g_1 \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{3.14}$$

$$-\Delta v = d_2 v + f_{2,1}(u) + f_{2,2}(v) + g_2 \quad \text{in } \Omega \subset \mathbb{R}^N,, \tag{3.15}$$

where:

(i) $C_{1,i,j} u^r \leq f_{i,j}(u) \leq C_{2,i,j} u^r$ *for all* $i, j = 1, 2$ *and* $1 < r < \frac{N}{N-2}$*.*
(ii) $0 \leq g_i, d_i \in L^{\frac{q}{2}}(\Omega)$*, for all* $i = 1, 2$ *and* $q > N$*.*

Then for any ball $B_{3R}(y) \subset \Omega$*,*

$$\sup_{B_R(y)} \{u + v + 1\} \leq H(R), \tag{3.16}$$

where $\lim_{R \searrow 0} H(R) = \infty$ *and the dependence is given by*

$$H(R) = H(R, \|d_i, g_i\|_{L^{q/2}_{\rm loc}(\Omega)}, C_{1,i,j}, C_{2,i,j}).$$

From this last result, we infer the following result.

Theorem 3.13. *If* u_i*,* $i = 1, \dots, n$ *are* $W^{1,2}(\Omega)$ *non-negative solutions of*

$$\begin{aligned}
-\Delta u_1 &= d_1 u_1 + f_{1,1}(u_1) + f_{1,2}(u_2) + \dots + g_1 \quad \text{in } \Omega \subset \mathbb{R}^N, \\
-\Delta u_2 &= d_2 u_2 + f_{2,1}(u_1) + f_{2,2}(u_2) + \dots + g_2 \quad \text{in } \Omega \subset \mathbb{R}^N, \\
&\dots, \\
-\Delta u_n &= d_n u_2 + f_{2,1}(u_1) + f_{2,2}(u_2) + \dots + g_n \quad \text{in } \Omega \subset \mathbb{R}^N,
\end{aligned} \tag{3.17}$$

where:

(i) $C_{1,i,j}u^r \le f_{i,j}(u) \le C_{2,i,j}u^r$ *for all* $i,j = 1,2\dots n$ *and* $1 < r < \frac{N}{N-2}$.

(ii) $0 \le g_i, d_i \in L^{\frac{q}{2}}(\Omega)$, *for all* $i = 1,2$ *and* $q > N$.

Then for any ball $B_{3R}(y) \subset \Omega$

$$\sup_{B_R(y)} \{1 + \sum_{i=1}^{n} u_i\} \le H(R),$$

where $\lim_{R\searrow 0} H(R) = \infty$ *and the dependence is given by*

$$H(R) = H(R, \|d_i, g_i\|_{L^{q/2}_{\rm loc}(\Omega)}, C_{1,i,j}, C_{2,i,j}).$$

To our knowledge, for fully coupled elliptic systems, one of the most representative results on Harnack inequality is as follows.

Theorem 3.14 (Busca and Sirakov [4]). *Let us consider the problem*

$$-\Delta u = f_1(u,v) \quad \text{in } \Omega, \tag{3.18}$$
$$-\Delta v = f_2(u,v) \quad \text{in } \Omega. \tag{3.19}$$

Assume $f_1(u,v)$, $f_2(u,v)$ *are globally Lipschitz continuous functions, with Lipschitz constant* A, *which satisfy the cooperativeness assumption:*

$$\frac{\partial f_1}{\partial v} \ge 0, \quad \frac{\partial f_2}{\partial u} \ge 0.$$

Let (u,v) *be a nonnegative solution of* (3.18), *3.19 in* Ω. *We suppose that the system is fully coupled, in the sense that* $f_1(0,v) > 0$ *for all* $v > 0$, *and* $f_2(u,0) > 0$ *for* $u > 0$. *Then for any compact subset* K *of* Ω *there exists a function* $\Phi(t)$ *(depending on* A, K *and* Ω*), continuous on* $[0,\infty)$, *such that* $\Phi(0) = 0$ *and*

$$\sup \max\{u,v\} \le \Phi(\inf_{x\in K} \min\{u,v\}). \tag{3.20}$$

In particular, if any of u,v *vanishes at one point in* Ω *then both* u *and* v *vanish identically in* Ω

We call (3.20) a fully coupled Harnack inequality because in that inequality both u and v appear. In this frame, we derive weakly coupled Harnack inequalities, where global Lipschitz continuity is not assumed. The authors are not aware of a more advanced result in this direction.

Theorem 3.15. *If* $u,v \in W^{1,2}(\Omega)$ *are non-negative solutions of*

$$-\Delta u = d_1 u + f_{1,1}(u) + f_{1,2}(v) + g_1 \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{3.21}$$
$$-\Delta v = d_2 v + f_{2,1}(u) + f_{2,2}(v) + g_2 \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{3.22}$$

with the same conditions as in Theorem 3.12. Then for any ball $B_{3R}(y) \subset \Omega$,

$$\sup_{B_R(y)} \{u+1\} \le C_1(R) \inf_{B_R(y)} \{u+1\}, \tag{3.23}$$
$$\sup_{B_R(y)} \{v+1\} \le C_2(R) \inf_{B_R(y)} \{v+1\}, \tag{3.24}$$

where the dependence is given by

$$C_1(R) = C_1(R, \|d_i, g_i\|_{L^{q/2}_{\rm loc}(\Omega)}, C_{1,i,j}, C_{2,i,j}),$$
$$C_2(R) = C_2(R, \|d_i, g_i\|_{L^{q/2}_{\rm loc}(\Omega)}, C_{1,i,j}, C_{2,i,j}).$$

We call (3.23), (3.24) a weakly coupled Harnack inequalities because the coupling is located in $C_1(R)$ and $C_2(R)$. This is a new phenomenon: the fully coupled equation (3.21), 3.22 satisfies a weakly coupled Harnack inequalities. A nice survey is [9] for more details on elliptics systems, critical hyperbola, hamiltonian elliptic systems etc.

Using the integral relations method to the problem of local properties of weak solutions (see [11, Theorem 8.17 section 8.6]), we have the following result.

Theorem 3.16 (Morrey [18], Stampacchia [24]). *If u is a $W^{1,2}(\Omega)$ solution of the problem*

$$-\Delta u + cu = g \quad \text{in } \Omega, \tag{3.25}$$

where $c \in L^\infty(\Omega)$, $g \in L^{\frac{q}{2}}(\Omega)$ for some $q > N$. Then we have, for any ball $B_{2R}(y) \subset \Omega$ and $p > 1$:

$$\sup_{B_R(y)} u \le C(R^{-\frac{N}{p}} \|u^+\|_{L^p(B_{2R}(y))} + R^{2-\frac{2N}{q}} \|g\|_{L^{\frac{q}{2}}(\Omega)}), \tag{3.26}$$

where $C = C(N, \|c\|_{L^\infty(\Omega)} R, q, p)$.

Therefore, if we want to establish a priori estimates for the simplified problem $-\Delta u = u^r + g$, $u \ge 0$ in Ω, this is equivalent to (3.25) with $c = -u^{r-1}$, but we cannot apply directly Theorem 3.16 because it is not ensured $u^{r-1} \in L^\infty(\Omega)$. With this in mind, we prove the following theorem.

Theorem 3.17. *If u is a non-negative $W^{1,2}(\Omega)$ solution of*

$$-\Delta u = cu + g + G(|\nabla u|) \quad \text{in } \Omega, \tag{3.27}$$

where Ω is a open set in $\mathbb{R}^N$, c and g belongs to $L^{\frac{q}{2}}(\Omega)$, $q > N$, $|G(\nabla u)| \le \mathcal{C}|\nabla u| + \mathcal{C}_0$. Then, for any ball $B_{\frac{3R}{2}}(y) \subset \Omega$,

$$\sup_{B_R(y)} \{u+1\} \le \Big(\int_{B_{\frac{3R}{2}}(y)} |u+1|^p dx \Big)^{1/p} C R^{-N/(2p)} (1 + \frac{R}{2})^{N/(2p)}, \tag{3.28}$$

where

$$C = C(p, N) \exp \sum_{s=0}^{\infty} \frac{\log \max\{C_s^{\frac{N}{q-N}}, C_s\}}{\chi^s p},$$
$$C_s = 2\Big(\max\{ \frac{(\chi^s p)^2}{2|\chi^s p - 1|} \max\{\|\hat{b}_s\|_{L^{\frac{q}{2}}(B_{(1+2^{-s-1})R})}, \frac{16}{|\chi^s p - 1|}\}, 1\}\Big)^{1/2},$$
$$\hat{b}_s = |c| + |g| + \frac{4\mathcal{C}^2}{|\chi^s p|} + \mathcal{C}_0,$$

and $p \in \mathbb{R}_{>0} - \{\chi^{-s}\}_{s=0}^\infty$, $\chi = \frac{N}{N-2}$. Moreover for $p < 0$, with the same constants,

$$\inf_{B_R} \{u+1\} \ge \Big(\int_{B_{\frac{3R}{2}}(y)} |u+1|^p dx \Big)^{1/p} C R^{-N/(2p)} (1 + \frac{R}{2})^{N/(2p)}. \tag{3.29}$$

Our main results - Theorems 3.5, 3.11, 3.12 and 3.15 - follow by combining Theorem 3.17 and Lemma 3.4. In section 5, we collect some preliminary results: we recall a lemma essentially proved in [1] and we state variants of classical Lemmas by Morrey and Trudinger (see [11]).

4. The statistical procedure

First at all, we remember the fundamentals of quantum mechanics:
Postulate 1. For all time t a quantum system is determinated for a vector $|\psi(t)\rangle$ of states belonging to $\mathcal{E}$ the space of quantum states.
Postulate 2. For all measurable value $\mathcal{A}$, it is possible to built an operator A with domain $\mathcal{E}$. This operator A is an observable.
Postulate 3. The eigenvalues of the observable A are the unique measurable amount.
Postulate 4. The Hamiltonian operator $\hat{H}(t)$ of a quantum system is the observable associated to the total energy of the system. The vector of state is given by $\hat{H}(t)|\psi(t)\rangle = i\hbar\dfrac{d}{dt}|\psi(t)\rangle$.

This section es dedicated to the construction of statistical estimators for high dimensional problems in quantum physics. We consider another real problem: financial markets. We deal with the problem of the estimate the eigenvalue energy E. We begin with a unique particle because this problem is easy to understood in this frame. The equation

$$\begin{gathered} -\Delta u_{\sigma,\varphi} + U(r)u_{\sigma,\varphi} = Eu \quad \text{in } \Omega\subset\mathbb{R}^3, \\ \Big(\frac{\partial u_{\sigma,\varphi}}{\partial n} + \sigma(x)u_{\sigma,\varphi}\Big)|_{\partial\Omega} = \varphi(x), \end{gathered} \tag{4.1}$$

where $0 \le \sigma(x) \in C(\partial\Omega)$ has a unique solution in a classical weak representation. We recall that the boundary condition represents combinations of absorption and reflection of quantum particles. Moreover because the Hamiltonian is a linear operator $\varphi(x) \equiv 0$, we are restricted to the class $u_\sigma = u_{\sigma,\varphi\equiv 0}$. The corresponding eigenvalue energy equation is

$$\begin{gathered} -\Delta\psi + U(r)\psi = E\psi \quad \text{in } \Omega\subset\mathbb{R}^3, \\ \Big(\frac{\partial \psi}{\partial n} + \sigma(x)\psi\Big)|_{\partial\Omega} = 0. \end{gathered} \tag{4.2}$$

In our quantum representation $qdq = udP_\sigma = \frac{u_\sigma(x)}{\int_\Omega u_\sigma(x)dx}dx$.

4.1. **Quantum mechanics and minimal statistics.**

We present here the connection on quantum mechanics and minimal statistics. First at all, we note that

$$\begin{aligned} E_\sigma(X_1) &= \int_{\mathbb{R}} x_1\Big\{\int_{\Omega\bigcap X_1=x_1} \frac{u_\sigma(x_1,x_2,x_3)}{\int_\Omega u_\sigma(x)dx} dx_2dx_3\Big\}dx_1 = \theta_1(\sigma), \\ E_\sigma(X_2) &= \int_{\mathbb{R}} x_2\Big\{\int_{\Omega\bigcap X_2=x_2} \frac{u_\sigma(x_1,x_2,x_3)}{\int_\Omega u_\sigma(x)dx} dx_1dx_3\Big\}dx_2 = \theta_1(\sigma), \\ E_\sigma(X_3) &= \int_{\mathbb{R}} x_3\Big\{\int_{\Omega\bigcap X_3=x_3} \frac{u_\sigma(x_1,x_2,x_3)}{\int_\Omega u_\sigma(x)dx} dx_1dx_2\Big\}dx_3 = \theta_3(\sigma). \end{aligned}$$

Therefore, we use Theorems 2.5, 2.7 and Remark 2.6 to obtain unbiased estimators of minimum variance. This connection allow to use all the classical mathematical statistical procedure like the Bayesian approach on the parameter space. Of fundamental importance is our a priori estimates without dependence on the distance to the boundary. In fact this provides a law of nature on the energy level on non-relativistic quantum mechanics.

Using Theorem 3.5, we obtain a quasilinear version of quantum mechanics, with unbiased estimators of minimum variance and the Bayesian approach on the parameter space. This procedure es new in quantum mechanics because usually de hamiltonian linear operator $\hat{H}$ is perturbed with another linear operator, in our setting the perturbation is nonlinear.

If $r \in (1, 1+\epsilon)$ then equation 3.4 in Theorem 3.5 has an almost linear behaviour, moreover is a good aproximation for equation 1.2 for $a = 1$. If $r \leq \frac{N}{N-2} = 2_* - 1$, we can use our results to derive a priori bounds for the equations 3.7 and 3.8 and the corresponding statistical tools.

In section 5, we calculate all the constants because this is fundamental for numerical computations.

4.2. **High dimensional problems in quantum mechanics and in financial mathematics.** This is a central problem in mathematics. Again our a priori estimates without dependence on the distance to the boundary plays a central role. The Schrödinger equation

$$\imath\hbar\frac{d\Psi}{dt} = -\frac{1}{2}\hbar^2\sum_a \frac{\Delta_a}{m_a} + U(r_1, r_2, \dots), \tag{4.3}$$

is in real situations a high dimension problem because a is a usually big number. The stationary equation in the weak formulation for the eigenvalue energy and the stationary wave for the particle a is given by

$$-\frac{1}{2}\hbar^2\frac{\Delta_a}{m_a}\psi_a + U(r_1, r_2, \dots)\psi_a = E\psi_a, \quad \text{in } \Omega \subseteq \mathbb{R}^3, \tag{4.4}$$

but this equation was studied mainly in the last subsection. The potential U has a big number of terms, this term when it is formulated in a weak sense, it is calculated with great accuracy using modern Montecarlo methods. Using Theorem 3.13, we derive a quasilinear quantum statistic theory with the corresponding unbiased estimators of minimum variance and the Bayesian approach.

5. Local properties of weak solutions of nonlinear elliptic problems with gradient term

Proof of Lemma 3.4. Let $\hat{\eta}$ be a radially symmetric C^2 cut-off function on $B_2(0)$ that is

(a) $\hat{\eta}(x) = 1$ for $|x| \leq 1$.
(b) $\hat{\eta}$ has compact support in $B_2(0)$ and $0 \leq \hat{\eta} \leq 1$.
(c) $|\nabla\hat{\eta}| \leq 2$.

We consider $\eta(x) = \hat{\eta}(\frac{x}{R})$. We use the classical test function $\eta^k u^\beta$. From $-\Delta u \geq C_1 u^r$, taking $\beta < 0$, we have

$$k\int \eta^{k-1}u^\beta\nabla\eta\cdot\nabla u dx + \beta\int \eta^k u^{\beta-1}|\nabla u|^2 dx \geq C_1\int \eta^k u^{\beta+r}dx.$$

Therefore,

$$C_1\int \eta^k u^{\beta+r}dx + |\beta|\int \eta^k u^{\beta-1}|\nabla u|^2 dx \leq k\int \eta^{k-1}u^\beta\nabla\eta\cdot\nabla u\, dx. \tag{5.1}$$

Observe that

$$u^\beta\nabla\eta^k\cdot\nabla u \leq \eta^k\frac{2k}{R\eta}|\nabla u|u^\beta.$$

Using $u^\beta = u^{\frac{\beta-1}{2}} u^{\frac{\beta+1}{2}}$, $\eta^{k-1} = \eta^{\frac{k-2}{2}}\eta^{k/2}$ and the Young inequality $ab \le \epsilon a^2 + \epsilon^{-1} b^2$, $a, b \ge 0$, we compute

$$\begin{aligned}
\eta^{k-1}\frac{2k}{R}|\nabla u| u^\beta &= (\eta^{k/2} u^{\frac{\beta-1}{2}}|\nabla u|)(\frac{2k}{R}\eta^{\frac{k-2}{2}} u^{\frac{\beta+1}{2}}) \\
&\le \epsilon(\eta^{k/2} u^{\frac{\beta-1}{2}}|\nabla u|)^2 + \epsilon^{-1}(\frac{2k}{R}\eta^{\frac{k-2}{2}} u^{\frac{\beta+1}{2}})^2 \\
&= \epsilon(\eta^k u^{\beta-1}|\nabla u|^2) + \epsilon^{-1}(\frac{4k^2}{R^2}\eta^{k-2}u^{\beta+1}).
\end{aligned}$$

If we choose $\epsilon = \frac{|\beta|}{2}$ using (5.1), we find

$$\frac{|\beta|}{2}\int \eta^k u^{\beta-1}|\nabla u|^2 dx + C_1\int \eta^k u^{\beta+r} \le \frac{\epsilon^{-1}4k^2}{R^2}\int \eta^{k-2}u^{\beta+1}dx. \tag{5.2}$$

With the assumption $\gamma > r-1$, we fix $\beta = \gamma - r$ and $k = \frac{2\gamma}{r-1}$. Therefore

$$\eta^{k-2}u^{\beta+1} = \eta^{\frac{2\gamma}{r-1}-2}u^{\gamma-r+1} = \eta^{2\frac{\gamma-r+1}{r-1}}u^{\gamma-r+1}.$$

Using Young's inequality $ab \le \epsilon_0 a^q + \epsilon_0^{\frac{1}{1-q}} b^{\frac{q}{q-1}}$ with $q = \frac{\gamma}{\gamma-r+1}$ (and consequently $\frac{q}{q-1} = \frac{\gamma}{r-1}$), we have

$$R^{-2}\eta^{k-2}u^{\beta+1} = \eta^{2\frac{\gamma-r+1}{r-1}}u^{\gamma-r+1} \le \epsilon_0\eta^{\frac{2\gamma}{r-1}}u^\gamma + \epsilon_0^{\frac{1}{1-q}}R^{-\frac{2\gamma}{r-1}}.$$

It follows that

$$\begin{aligned}
\int R^{-2}\eta^{k-2}u^{\beta+1}dx &\le \epsilon_0\int \eta^{\frac{2\gamma}{r-1}}u^\gamma dx + \epsilon_0^{\frac{1}{1-q}}R^{-\frac{2\gamma}{r-1}}\int_{B_{2R}} dx \\
&= \epsilon_0\int \eta^k u^\gamma dx + \epsilon_0^{\frac{1}{1-q}}\omega_N R^{N-\frac{2\gamma}{r-1}},
\end{aligned}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$. From (5.2), we find

$$\begin{aligned}
C_1\int \eta^k u^{\beta+r} &\le \epsilon^{-1}4k^2\int R^{-2}\eta^{k-2}u^{\beta+1}dx \\
&\le \epsilon^{-1}4k^2\epsilon_0\int \eta^k u^\gamma dx + \epsilon^{-1}4k^2\epsilon_0^{\frac{1}{1-q}}\omega_N R^{N-\frac{2\gamma}{r-1}}.
\end{aligned}$$

Choosing

$$\epsilon^{-1}4k^2\epsilon_0 = \frac{2}{r-\gamma}4(\frac{2\gamma}{r-1})^2\epsilon_0 = \frac{C_1}{2}.$$

We find

$$\begin{aligned}
\frac{C_1}{2}\int_{B_R} u^\gamma dx &\le \int \eta^k u^\gamma dx \\
&\le 2\epsilon_0^{\frac{q}{1-q}}\omega_N R^{N-\frac{2\gamma}{r-1}} \\
&= 2(\frac{(r-\gamma)(r-1)^2 C_1}{2^6\gamma^2})^{\frac{\gamma}{1-r}}\omega_N R^{N-\frac{2\gamma}{r-1}}.
\end{aligned}$$

This completes the proof. □

Next, we prove our main tool.

Lemma 5.1. *If u is a $W^{1,2}(\Omega)$ solution of*

$$-\Delta u = cu + g + G(\nabla u) \quad \text{in } \Omega, \tag{5.3}$$

where Ω is a open set in $\mathbb{R}^N$, $|G(\nabla u)| \le \mathcal{C}|\nabla u| + \mathcal{C}_0$, c and g belongs to $L^{\frac{q}{2}}(\Omega)$, $q > N$. Then, for any ball $B_{R_1}(y) \subset B_{R_2}(y) \subset \Omega$ and $\beta \neq -1$,

$$\begin{aligned} &\Big(\|u+1\|_{L^{\frac{N(\beta+1)}{N-2}}(B_{R_1})}\Big)^{(\beta+1)/2} \\ &\le \Big\{\frac{1}{\epsilon_1}\max\{\epsilon_1^{\frac{N}{N-q}}, \sqrt{2}\}(1+\frac{1}{R_2-R_1})\Big\} \times (\|u+1\|_{L^{\beta+1}(B_{R_2})})^{(\beta+1)/2}, \end{aligned} \tag{5.4}$$

where

$$\epsilon_1 = \frac{\frac{1}{2}\sqrt{C(N)}}{\sqrt{\max\big\{\frac{(\beta+1)^2}{2(|\beta|)}\max\{\||c|+|g|+\frac{4\mathcal{C}^2}{|\beta|}+\mathcal{C}_0\|_{L^{\frac{q}{2}}(B_{R_2})}, \frac{16}{|\beta|}\}, 1\big\}}}. \tag{5.5}$$

and $C(N)$ is the Sobolev embedding constant ($C(N)\|w\|^2_{\frac{2N}{N-2}} \le \int |\nabla w|^2 dx$ for all $w \in H^1_0(\Omega)$).

Proof. We define $\hat{u} = u + 1$. We use a classical test function $\eta^2\hat{u}^\beta$ where η satisfies for $0 < R_1 < R_2$, $\eta \equiv 1$ in B_{R_1}, $\eta \equiv 0$ in $\Omega - B_{R_2}$ with $|\nabla\eta| \le \frac{1}{R_2-R_1}$

$$\nabla(\eta^2\hat{u}^\beta) = 2\eta\hat{u}^\beta\nabla\eta + \beta\eta^2\hat{u}^{\beta-1}\nabla u.$$

From (5.3), we have

$$\begin{aligned} \beta\int|\nabla u|^2\eta^2\hat{u}^{\beta-1}dx &= \int g\eta^2\hat{u}^\beta dx + \int G(\nabla u)\eta^2\hat{u}^\beta dx \\ &\quad - 2\int\eta\hat{u}^\beta\nabla u\cdot\nabla\eta dx + \int cu\eta^2\hat{u}^\beta dx. \end{aligned} \tag{5.6}$$

Using $\hat{u}^\beta = \hat{u}^{\frac{\beta-1}{2}}\hat{u}^{\frac{\beta+1}{2}}$, we have

$$2\eta\hat{u}^\beta|\nabla u||\nabla\eta| = (\eta\hat{u}^{\frac{\beta-1}{2}}|\nabla u|)(2\hat{u}^{\frac{\beta+1}{2}}|\nabla\eta|).$$

From the interpolation inequality $ab \le \epsilon a^2 + \frac{1}{\epsilon}b^2$ valid for non-negative numbers a, b, ϵ, we find

$$2\eta\hat{u}^\beta|\nabla u||\nabla\eta| \le (\epsilon/2)(\eta\hat{u}^{\frac{\beta-1}{2}}|\nabla u|)^2 + \frac{1}{(\epsilon/2)}(2\hat{u}^{\frac{\beta+1}{2}}|\nabla\eta|)^2. \tag{5.7}$$

Similarly,

$$\begin{aligned} |G(\nabla u)\eta^2\hat{u}^\beta| &\le \mathcal{C}|\nabla u|\eta^2\hat{u}^\beta + \mathcal{C}_0\eta^2\hat{u}^\beta \\ &= (\eta\hat{u}^{\frac{\beta-1}{2}}|\nabla u|)(\mathcal{C}\eta\hat{u}^{\frac{\beta+1}{2}}) + \mathcal{C}_0\eta^2\hat{u}^\beta \\ &\le (\epsilon/2)(\eta\hat{u}^{\frac{\beta-1}{2}}|\nabla u|)^2 + \frac{1}{(\epsilon/2)}(\mathcal{C}\eta\hat{u}^{\frac{\beta+1}{2}})^2 + \mathcal{C}_0\eta^2\hat{u}^\beta. \end{aligned} \tag{5.8}$$

From $\hat{u}^\beta \le \hat{u}^{\beta+1}$, (5.6), (5.7) and (5.8),

$$\begin{aligned} &|\beta|\int|\nabla u|^2\eta^2\hat{u}^{\beta-1}dx \\ &\le \int|g|\eta^2\hat{u}^\beta dx + \int|G(\nabla u)\eta^2\hat{u}^\beta|dx + 2\int\eta\hat{u}^\beta|\nabla u||\nabla\eta|dx + \int|c|\eta^2\hat{u}^{\beta+1}dx \\ &\le \int|g|\eta^2\hat{u}^{\beta+1}dx + \epsilon\int\eta^2\hat{u}^{\beta-1}|\nabla u|^2dx + \frac{8}{\epsilon}\int\hat{u}^{\beta+1}|\nabla\eta|^2dx + \int|c|\eta^2\hat{u}^{\beta+1}dx \end{aligned}$$

$$+\frac{2\mathcal{C}^2}{\epsilon}\int \eta^2\hat{u}^{\beta+1}dx+\mathcal{C}_0\int \eta^2\hat{u}^{\beta+1}.dx$$

Then we obtain

$$(|\beta|-\epsilon)\int |\nabla u|^2\eta^2\hat{u}^{\beta-1}dx \le \int \big\{(|g|+|c|+\frac{2\mathcal{C}^2}{\epsilon}+\mathcal{C}_0)\eta^2+\frac{8}{\epsilon}|\nabla\eta|^2\big\}\hat{u}^{\beta+1}dx. \quad (5.9)$$

Now, if we set $w=\hat{u}^{\frac{\beta+1}{2}}$, we obtain

$$\nabla w=\frac{\beta+1}{2}\hat{u}^{\frac{\beta-1}{2}}\nabla u,$$
$$|\eta\nabla w|^2=\frac{(\beta+1)^2}{4}\hat{u}^{\beta-1}|\nabla u|^2\eta^2.$$

It follows from (5.9) that

$$\int |\eta\nabla w|^2dx \le \frac{(\beta+1)^2}{4(|\beta|-\epsilon)}\int \big\{(|g|+|c|+\frac{2\mathcal{C}^2}{\epsilon}+\mathcal{C}_0)\eta^2+\frac{8}{\epsilon}|\nabla\eta|^2\big\}w^2dx. \quad (5.10)$$

If we set

$$\hat{b}\equiv |g|+|c|+\frac{2\mathcal{C}^2}{\epsilon}+\mathcal{C}_0,$$

applying the Hölder inequality,

$$\int \hat{b}(\eta w)^2 \le \|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}\|(\eta w)^2\|_{\frac{q}{q-2}}=\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}\|\eta w\|^2_{\frac{2q}{q-2}}.$$

We arrive at

$$\int |\eta\nabla w|^2dx \le \frac{(\beta+1)^2}{4(|\beta|-\epsilon)}\big\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}\|\eta w\|^2_{\frac{2q}{q-2}}+\frac{8}{\epsilon}\int |\nabla\eta|^2\big\}w^2dx.$$

At this point we use the interpolation inequality for L^p norms: If $p\le s\le r$ then $\|u\|_s\le \epsilon_1\|u\|_r+\epsilon_1^{-\mu}\|u\|_p$, where

$$\mu=\frac{\frac{1}{p}-\frac{1}{s}}{\frac{1}{s}-\frac{1}{r}}.$$

The condition $q>N$ implies $2<\frac{2q}{q-2}<\frac{2N}{N-2}$, therefore

$$\|\eta w\|_{\frac{2q}{q-2}}\le \epsilon_1\|\eta w\|_{\frac{2N}{N-2}}+\epsilon_1^{\frac{N}{N-q}}\|\eta w\|_2.$$

So, we have

$$\begin{aligned}&\int |\eta\nabla w|^2dx\\ &\le \frac{(\beta+1)^2}{4(|\beta|-\epsilon)}\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}(\epsilon_1\|\eta w\|_{\frac{2N}{N-2}}+\epsilon_1^{\frac{N}{N-q}}\|\eta w\|_2)^2+\frac{8}{\epsilon}\int |\nabla\eta|^2w^2dx\}.\end{aligned} \quad (5.11)$$

By the Sobolev inequality,

$$C(N)\|\eta w\|^2_{\frac{2N}{N-2}}\le \int |\eta\nabla w|^2dx+\int |w\nabla\eta|^2dx.$$

We deduce that

$$\begin{aligned} C(N)\|\eta w\|^2_{\frac{2N}{N-2}} &\le \frac{(\beta+1)^2}{4(|\beta|-\epsilon)}\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}(\epsilon_1\|\eta w\|_{\frac{2N}{N-2}} \\ &\quad + \epsilon_1^{\frac{N}{N-q}}\|\eta w\|_2)^2 + \frac{8}{\epsilon}\int|\nabla\eta|^2w^2dx\} + \int|w\nabla\eta|^2 \\ &\le \max\big\{\frac{(\beta+1)^2}{4(|\beta|-\epsilon)}\max\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})},\frac{8}{\epsilon}\},1\big\} \\ &\quad \times\big\{(\epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \epsilon_1^{\frac{N}{N-q}}\|\eta w\|_2)^2 + 2\int|w\nabla\eta|^2\big\}. \end{aligned} \tag{5.12}$$

Using the triangle inequality for the Euclidean norm

$$\begin{aligned} &\Big((\epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \epsilon_1^{\frac{N}{N-q}}\|\eta w\|_2)^2 + 2\int|w\nabla\eta|^2dx\Big)^{1/2} \\ &\le \epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \Big(\epsilon_1^{\frac{2N}{N-q}}\|\eta w\|_2^2 + 2\int|w\nabla\eta|^2dx\Big)^{1/2} \\ &\le \epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \sqrt{\max(\epsilon_1^{\frac{2N}{N-q}},2)}\Big(\|\eta w\|_2^2 + \int|w\nabla\eta|^2dx\Big)^{1/2} \\ &= \epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \max(\epsilon_1^{\frac{N}{N-q}},\sqrt{2})\Big(\|\eta w\|_2^2 + \int|w\nabla\eta|^2dx\Big)^{1/2}. \end{aligned}$$

We are able to compute

$$\begin{aligned} \sqrt{C(N)}\|\eta w\|_{\frac{2N}{N-2}} &\le \sqrt{\max\big\{\frac{(\beta+1)^2}{4(|\beta|-\epsilon)}\max\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})},\frac{8}{\epsilon}\},1\big\}} \\ &\quad\times\Big(\epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \max\{\epsilon_1^{\frac{N}{N-q}},\sqrt{2}\}\big\{\|\eta w\|_2^2 + \int|w\nabla\eta|^2\big\}^{1/2}\Big). \end{aligned}$$

Therefore, if we choose $\epsilon = |\beta|/2$ then

$$\begin{aligned} \sqrt{C(N)}\|\eta w\|_{\frac{2N}{N-2}} &\le \sqrt{\max\big\{\frac{(\beta+1)^2}{2(|\beta|)}\max\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})},\frac{16}{|\beta|}\},1\big\}} \\ &\quad\times\Big(\epsilon_1\|\eta w\|_{\frac{2N}{N-2}} + \max\{\epsilon_1^{\frac{N}{N-q}},\sqrt{2}\}\big\{\|\eta w\|_2^2 + \int|w\nabla\eta|^2\big\}^{1/2}\Big). \end{aligned}$$

If

$$\epsilon_1 = \frac{\frac{1}{2}\sqrt{C(N)}}{\sqrt{\max\{\frac{(\beta+1)^2}{2(|\beta|)}\max\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}\frac{16)}{|\beta|}\},1\}}},$$

then

$$\begin{aligned} \frac{1}{2}\sqrt{C(N)}\|\eta w\|_{\frac{2N}{N-2}} &\le \sqrt{\max\big\{\frac{(\beta+1)^2}{2(|\beta|)}\max\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})},\frac{16}{|\beta|}\},1\big\}} \\ &\quad\times\max\{\epsilon_1^{\frac{N}{N-q}},\sqrt{2}\}\big\{\int w^2(\eta^2 + |\nabla\eta|^2dx)\big\}^{1/2}. \end{aligned}$$

We are led to

$$\|w\|_{L^{\frac{2N}{N-2}}(B_{R_1})} \le \frac{1}{\epsilon_1}\max\{\epsilon_1^{\frac{N}{N-q}},\sqrt{2}\}\big(1 + \frac{1}{R_2-R_1}\big)\|w\|_{L^2(B_{R_2})}.$$

Finally, using

$$\begin{aligned}\|w\|_{L^{\frac{2N}{N-2}}(B_{R_1})} &= \Big(\int_{B_{R_1}} \hat{u}^{\frac{\beta+1}{2}\frac{2N}{N-2}}dx\Big)^{(N-2)/(2N)}\\ &= \Big\{\Big(\int_{B_{R_1}} \hat{u}^{\frac{(\beta+1)N}{N-2}}dx\Big)^{\frac{N-2}{N(\beta+1)}}\Big\}^{\frac{\beta+1}{2}} = \|\hat{u}\|^{\frac{\beta+1}{2}}_{L^{\frac{N(\beta+1)}{N-2}}(B_{R_1})},\end{aligned}$$

and

$$\|w\|_{L^2(B_{R_2})} = \|\hat{u}\|^{\frac{\beta+1}{2}}_{L^{\beta+1}(B_{R_2})}.$$

This completes the proof. □

6. Proofs of main results

Proof of Theorem 3.17. Like in classical statements, we introduce the quantities

$$\Phi(p,R) = \Big(\int_{B_R} |u+1|^p dx\Big)^{1/p}.$$

From (5.4) and (5.5),

$$\Phi(\frac{N}{N-2}\beta, R_1)^{\beta/2} \le \Big\{C(\beta, \|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})})(1+\frac{1}{R_2-R_1})\Big\}\Phi(\beta,R_2)^{\beta/2},$$

where $\beta\in\mathbb{R}$ and $\hat{b} = |c|+|g|+(4\mathcal{C}^2/|\beta|)+\mathcal{C}_0$. We define

$$C(\beta,R_2) = C(\beta,\|\hat{b}\|_{L^{\frac{q}{2}}(B_{R_2})}),\quad \chi = \frac{N}{N-2}.$$

Therefore,

$$\Phi(\chi\beta,R_1)^{\beta/2} \le \big\{C(\beta,R_2)(1+\frac{1}{R_2-R_1})\big\}\Phi(\beta,R_2)^{\beta/2}. \tag{6.1}$$

We consider $R>0$ such that $B_{2R}\subset\Omega$ and the sequence $R<(1+2^{-m})R<(1+2^{-m+1})R<(1+2^{-m+2})R\dots<(1+2^{-1})R$. We deduce

$$\begin{aligned}1+\frac{1}{(1+2^{-m+j+1})R-2^{-m+j}R} &= \frac{1+(1+2^{-m+j+1})R-(1+2^{-m+j})R}{(1+2^{-m+j+1})R-(1+2^{-m+j})R}\\ &= \frac{1+2^{-m+j}R}{2^{-m+j}R}\\ &< \frac{1+2^{-1}R}{2^{-m+j}R}.\end{aligned}$$

In this framework, we set

$$C(\chi^{m-j}p) = C(\chi^{m-j}p,\|\hat{b}\|_{L^{\frac{q}{2}}(B_{(1+2^{-m+j-1})R})}).$$

Using (6.1),

$$\begin{aligned}\Phi(\chi^m p,R) &\le \Big(\frac{C(\chi^{m-1}p)(1+2^{-1}R)}{(1+2^{-m})R-R}\Big)^{\frac{1}{\chi^{m-1}p}}\Phi(\chi^{m-1}p,(1+2^{-m})R)\\ &= \Big(\frac{C(\chi^{m-1}p)(1+2^{-1}R)}{2^{-m}R}\Big)^{\frac{1}{\chi^{m-1}p}}\Phi(\chi^{m-1}p,(1+2^{-m})R)\\ &\le \Big(\frac{C(\chi^{m-1}p)(1+2^{-1}R)}{2^{-m}R}\Big)^{\frac{1}{\chi^{m-1}p}}\big(\frac{C(\chi^{m-2}p)(1+2^{-1}R)}{(1+2^{-m+1})R-(1+2^{-m})R}\big)^{\frac{1}{\chi^{m-2}p}}\\ &\quad\times\Phi(\chi^{m-2}p,(1+2^{-m+1})R)\end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{C(\chi^{m-1}p)(1+2^{-1}R)}{2^{-m}R}\Big)^{\frac{1}{\chi^{m-1}p}} \big(\frac{C(\chi^{m-2}p)(1+2^{-1}R)}{2^{-m}R}\big)^{\frac{1}{\chi^{m-2}p}}, \\
&\quad \times \Phi(\chi^{m-2}p, (1+2^{-m+1})R).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Phi(\chi^m p, R) \\
&\le \Big\{(1+2^{-1}R)^{\sum_{j=1}^{m}\frac{1}{\chi^{m-j}p}} \Big(\frac{C(\chi^{m-1}p)}{2^{-m}R}\Big)^{\frac{1}{\chi^{m-1}p}} \prod_{j=2}^{m} \Big(\frac{C(\chi^{m-j}p)}{2^{j-2-m}R}\Big)^{\frac{1}{\chi^{m-j}p}}\Big\}\Phi(p, \frac{3R}{2}).
\end{aligned}
$$

Setting $s = m - j$ it follows that

$$
\begin{aligned}
\Phi(\chi^m p, R) &\le \Big\{(1+2^{-1}R)^{\sum_{j=0}^{m-1}\frac{1}{\chi^{j}p}} \Big(\frac{C(\chi^{m-1}p)}{2^{-m}R}\Big)^{\frac{1}{\chi^{m-1}p}}\Big\} \\
&\quad \times \Big\{\prod_{s=0}^{m-2}\Big(\frac{2^{s+2}C(\chi^s p)}{R}\Big)^{\frac{1}{\chi^s p}}\Big\}\Phi(p, \frac{3R}{2}) \\
&= \big\{2^{\frac{m}{\chi^{m-1}p}} 2^{\frac{1}{p}\sum_{s=0}^{m-2}\frac{s+2}{\chi^s}} R^{\frac{-1}{p}\sum_{s=0}^{m-1}\frac{1}{\chi^s}} (1+2^{-1}R)^{\sum_{j=0}^{m-1}\frac{1}{\chi^j p}}\big\} \\
&\quad \times \Big\{\prod_{s=0}^{m-1}(C(\chi^s p))^{\frac{1}{\chi^s p}}\Big\}\Phi(p, \frac{3R}{2}).
\end{aligned}
$$

Now

$$
\prod_{s=0}^{m-1}(C(\chi^s p))^{\frac{1}{\chi^s p}} = \exp\Big\{\sum_{s=0}^{m-1}\frac{\log C(\chi^s p))}{\chi^s p}\Big\}.
$$

Therefore, we study the convergence of

$$
\begin{aligned}
\sum_{s=0}^{\infty}\frac{\log C(\chi^s p))}{\chi^s p} &= \sum_{s=0}^{\infty}\frac{\log\big(\frac{1}{\epsilon_1(\chi^s p)}\max\{\sqrt{2}, (\epsilon_1(\chi^s p))^{\frac{N}{N-q}}\}\big)}{\chi^s p} \\
&= \sum_{s=0}^{\infty}\frac{\log\big(\max\{\sqrt{2}(\epsilon_1(\chi^s p))^{-1}, ((\epsilon_1(\chi^s p))^{-1})^{\frac{q}{q-N}}\}\big)}{\chi^s p},
\end{aligned} \tag{6.2}
$$

where

$$
\epsilon_1(\chi^s p) = \frac{\frac{1}{2}\sqrt{C(N)}}{\sqrt{\max\{\frac{(\chi^s p)^2}{2(\chi^s p-1)}\max\{\|\hat b\|_{L^{\frac{q}{2}}(B_{(1+2^{-s-1})R})}, \frac{16}{\chi^s p-1}\}, 1\}}}.
$$

Now because the function $s \mapsto 16/(\chi^s p - 1)$ is non-increasing, the function $s \mapsto 2(\chi^s p-1)/(\chi^s p)^2$ is bounded and the inequality $\frac{\chi^s p}{\sqrt{2(\chi^s p-1)}} \le C(p,\chi)\sqrt{\chi^s p}$ holds. Then we have

$$
\begin{aligned}
(\epsilon_1(\chi^s p))^{-1} &\le \frac{\sqrt{\max\{\frac{(\chi^s p)^2}{2(\chi^s p-1)}\max\{\|\hat b\|_{L^{\frac{q}{2}}(B_{\frac{3R}{2}})}, \frac{16}{p-1}\}, 1\}}}{\frac{1}{2}\sqrt{C(N)}} \\
&\le \frac{\chi^s p}{\sqrt{2(\chi^s p-1)}}\frac{\sqrt{\max\{\max\{\|\hat b\|_{L^{\frac{q}{2}}(B_{\frac{3R}{2}})}, \frac{16}{p-1}\}, \frac{2(\chi^s p-1)}{(\chi^s p)^2}\}}}{\frac{1}{2}\sqrt{C(N)}}
\end{aligned}
$$

$$\leq C(p,\chi)\sqrt{\chi^s p}\frac{\sqrt{\max\{\max\{\|\hat{b}\|_{L^{\frac{q}{2}}(B_{\frac{3R}{2}})}, \frac{16}{p-1}\}, C(p,\chi)\}}}{\frac{1}{2}\sqrt{C(N)}}.$$

From this inequality, we deduce, using the integral test, the convergence of the series (6.2). Finally,

$$\begin{aligned}
&\sup_{B_R}\{u+1\}\\
&= \lim_{m\to\infty}\Phi(\chi^m p, R)\\
&\leq \Phi(p,\frac{3R}{2})\lim_{m\to\infty}\Big\{2^{\frac{m}{\chi^{m-1}p}}2^{\frac{1}{p}\sum_{s=0}^{m-2}\frac{s+2}{\chi^s}}R^{\frac{-1}{p}\sum_{s=0}^{m-1}\frac{1}{\chi^s}}(1+2^{-1}R)^{\sum_{j=0}^{m-1}\frac{1}{\chi^j p}}\Big\}\\
&\quad\times\prod_{s=0}^{m-1}(C(\chi^s p))^{\frac{1}{\chi^s p}}\\
&= \Phi(p,\frac{3R}{2})CR^{-N/(2p)}(1+\frac{R}{2})^{N/(2p)},
\end{aligned}$$

where

$$C = C(|c|,|g|,\mathcal{C},\mathcal{C}_0,N,p,q) = 2^{\frac{1}{p}\sum_{s=0}^{\infty}\frac{s+2}{\chi^s}}\prod_{s=0}^{\infty}(C(\chi^s p))^{1/\chi^s p},$$

In a similar manner, we can prove (3.29). $\square$

Proof of Theorem 3.5. If a non-negative function $u \in W^{1,2}(\Omega)$ solves the equation

$$-\Delta u = du + f(u) + G(\nabla u) + g \quad \text{in } \Omega \subset \mathbb{R}^N,$$

with the conditions in Theorem 3.5, then $-\Delta u \geq C_1 u^r$. We set $c = d + f(u)/u$. Now, we apply Theorem 3.17. From (3.28) and (3.3), we deduce

$$\begin{aligned}
\sup_{B_R(y)}\{u+1\} &\leq \Big(\int_{B_{\frac{3R}{2}}(y)}|u+1|^p dx\Big)^{1/p}CR^{-N/(2p)}(1+\frac{R}{2})^{N/(2p)}\\
&\leq \Big(\|u\|_{L^p(B_{\frac{3R}{2}}(y))}+\omega_N^{1/p}(\frac{3R}{2})^{N/p}\Big)CR^{-N/(2p)}(1+\frac{R}{2})^{N/(2p)}\\
&\leq \Big(C(N,p,r,C_1)R^{\frac{N}{p}-\frac{2}{r-1}}+\omega_N^{1/p}(\frac{3R}{2})^{N/p}\Big)CR^{-N/(2p)}(1+\frac{R}{2})^{N/(2p)},
\end{aligned}$$

where $\max\{r-1,1\} < p < r$ and $C = C(|c|,|g|,\mathcal{C},\mathcal{C}_0,N,p,q)$. Condition (iii) in Theorem 3.5 implies

$$\begin{aligned}
\|\frac{f(u)}{u}\|_{L^{q/2}(B_{(1+2^{-s-1})R})} &\leq C_2\|u^{r-1}\|_{L^{q/2}(B_{(1+2^{-s-1})R})}\\
&= C_2\|u\|^{\frac{1}{r-1}}_{L^{\frac{(r-1)q}{2}}(B_{(1+2^{-s-1})R})}.
\end{aligned} \tag{6.3}$$

To use Lemma 3.4 and Theorem 3.17, we need to satisfy the conditions $r-1 < (r-1)q/2 < r$, $N/2 \leq q/2$. Moreover, to get simple statements, we set an additional condition $1 < (r-1)q/2$. For $1 < r < N/(N-2)$ there exists $q(r)$ satisfying all the required restrictions. We deduce

$$\|u\|^{\frac{1}{r-1}}_{L^{\frac{(r-1)q(r)}{2}}(B_{(1+2^{-s-1})R})} \leq C(r,N,C_1)(R^{\frac{N}{(r-1)\frac{q(r)}{2}}-\frac{2}{r-1}})^{\frac{1}{r-1}}. \tag{6.4}$$

Therefore, the constant $C = C(|c|, |g|, \mathcal{C}, \mathcal{C}_0, N, p, q)$ is bounded above without dependence on u. Finally (3.6) is a consequence of [13, Lemma 1.1]. This completes the proof. $\square$

Proof of Theorem 3.11. Because $-\Delta\{u+1\} \geq 0$, there exists $p_0 < 0$ such that
$$\int_{B_R(y)} \{u+1\}^{p_0} dx \int_{B_R(y)} \{u+1\}^{-p_0} dx \leq C(R).$$
For the demonstration of this affirmation, see for example the proof of [5, Lemma 1.36]. Thus
$$\begin{aligned}
\Big(\int_{B_R(y)} \{u+1\}^{p_0} dx\Big)^{1/p_0} &= \Big(\int_{B_R(y)} \{u+1\}^{p_0} dx \int_{B_R(y)} \{u+1\}^{-p_0} dx\Big)^{1/p_0} \\
&\quad \times \Big(\int_{B_R(y)} \{u+1\}^{-p_0} dx\Big)^{-1/p_0} \\
&\geq C(R, p_0)\Big(\int_{B_R(y)} \{u+1\}^{-p_0} dx\Big)^{-1/p_0}.
\end{aligned}$$
Collecting this inequality with (3.29) and (3.28) in Theorem 3.17, we derive the conclusion of Theorem 3.11. $\square$

Proof of Theorem 3.12. From equations (3.14) (3.15), we compute
$$\begin{aligned}
-\Delta(u+v) &\leq (d_1+d_2)(u+v) + \Big(\sum_{i=1}^{2} \frac{f_{i,1}(u)}{u} + \frac{f_{i,2}(v)}{v}\Big)(u+v) + (g_1+g_2) \\
&\leq (d_1+d_2)(u+v) + \mathcal{C}(u^{r-1} + v^{r-1})(u+v) + (g_1+g_2).
\end{aligned}$$
From $-\Delta u \geq C_{0,1,1} u^r$, $-\Delta v \geq C_{0,2,2} v^r$ using Lemma 3.4 and with the same method of proof of Theorem 3.5, we obtain the desired result. $\square$

Proof of Theorem 3.15. By Theorem 3.12, if the non-negative pair (u, v) solves equations (3.21) and (3.22), then $(u, v) \in L^\infty_{\rm loc}(\Omega)$, where there are not dependence in the local bound on (u, v). Therefore the result follows from Theorem 3.11. $\square$

Acknowledgments. The author would like to express his gratitude with the Embassy of France at Ottawa and Revenu Quebec for their kind support. The author would like to thank the anonymous referees for their useful suggestions.

References

[1] M. F. Bidaut Veron, S. Pohozaev; *Nonexistence results and estimates for some nonlinear elliptic problems.* J. Anal. Math., 84 (2001), 1-49.

[2] O. Bendinelli, C. Cacciari, S. Djorgorski, L. Federici, .. Ferraro, Fi Fusi Pecci, G. Parmeggiani, N. Weir, E Zaratti; *The first detection of a collapsed core globular cluster in M31.* Astrophy. I., 409 (1993), LI7-LI9.

[3] D. Blackwell; *Conditional expectation and unbiased sequential estimation.* Ann. Math. Statist., 1947 18, 105-110.

[4] J. Busca, B. Sirakov; *Harnack type estimates for nonlinear elliptic systems and applications.* Ann. Inst. H. Poincar Anal. Non Linaire 21 (2004), no. 5, 543–590.

[5] Y. Z. Chen, L. C. Wu; *Second order elliptic equations.* Translations of mathematical monographs, vol. 174 AMS.

[6] E. N. Dancer; *Superlinear problems on domains with holes of asymptotic shape and exterior problems.* Math. Z., 229 (1998), 475-491.

[7] L. D. Landau, E. M. Lifshitz; *Quantum mechanics non-relativistic theory.* Volume 3 of Course of Theoretical Physics, third edition Pergamon Press.

[8] R. A. Fisher; *On the mathematical fundations of theoretical statistics.* Phil. Trans. Roy. Soc. A., 1922, 222, 309-368.
[9] Bernhard Ruf; *Superlinear Elliptic Equations and Systems.* Dipartimento de Matematica, Universita degli Studi di Milano, May 23, 2007.
[10] B. Gidas, J. Spruck; *Global and local behavior of positive solutions of nonlinear elliptic equations.* Comm. Pure Appl. Math., 34 (1981), 525-598.
[11] D. Gilbarg, N. Trudinger; *Elliptic partial differential equations of second order.* Second edition 1983-Springer-Verlag.
[12] M. Hénon; *Numerical experiments on the stability of spherical stellar systems.* Astronomy and Astrophysics 24 (1973), 229-238.
[13] O. Ladyzhenskaya, N. Ural'tseva; *Linear and quasilinear elliptic equations.* 1968 Academic Press.
[14] E. L. Lehmann, H. Scheffe; *Completeness, similar regions and unbiased estimation.* Pt. I. Sankhya, 1950, 10 1950, 10, 305-340.
[15] P.I.E. Peebles; *Black holes are where you find them.* Gen. Rel. Grav., 3 (1972), 63-82.
[16] P.I.E. Peebles; *Star distribution near a collapsed object.* Ast. I., 178 (1972), 371-375.
[17] T. Matukuma; *Sur la dynamique des amas globulaires stellaires.* Proc. Imp. Acad. 6 (1930), 133-136.
[18] C. B. Morrey; *Second order elliptic equations in several variables and Hölder continuity.* Math. Z. 72, 146-164 (1959).
[19] J. Moser; *On Harnack Theorem for elliptic differential equations.* Comm. Pure Appl. Math. 14 577-591 (1961).
[20] J. Neyman; *Su un teorema concernente le cossidette statistiche sufficienti.* Inst. Ital. Atti. Giorn. (1935), 6 320-334.
[21] J. Rao; *Sufficient statistics and minimum variances estimates.* Proc. Cambr. Phill. Soc. 1949, 45, 213-218.
[22] J. Serrin, H. Zou; *Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities.* Acta Math.,189 (2002),79-142.
[23] R. Schoen; *Courses at Stanford University*, 1988, and New York University, 1989.
[24] G. Stampacchia; *Problemi al contorno ellitici, con dati discontinui, dotati di soluzioni hölderiane.* Ann. Mat. Pura Appl. (4) 51, 1-37 (1960).
[25] S. Zacks; *The theory of statistical inference.* Wiley, N. Y., 1971.

Carlos Cesar Aranda
Blue Angel Navire research laboratory, Rue Eddy 113 Gatineau, QC, Canada
E-mail address: carloscesar.aranda@gmail.com