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UNIQUENESS OF POSITIVE SOLUTIONS FOR A FRACTIONAL DIFFERENTIAL EQUATION VIA A FIXED POINT THEOREM OF A SUM OPERATOR

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Abstract. In this work, we study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary-value problems. Our analysis relies on a fixed point theorem of a sum operator. Our results guarantee the existence of a unique positive solution, and can be applied for constructing an iterative scheme for obtaining the solution.

1. Introduction

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering and biological sciences,etc; see [3, 4, 5, 9, 13, 14, 15, 16, 17, 18] for example. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Miller and Ross [15], Podlubny [17], Kilbas et al [9], and the articles [1, 2, 7, 8, 10, 11, 12, 21, 22, 20, 24, 25, 26, 27] and the references therein. In these papers, many authors have investigated the existence of positive solutions for nonlinear fractional differential equation boundary value problems. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by some authors, see [22, 25, 26, 27] for example.

By means of a fixed point theorem for mixed monotone operators, Xu, Jiang and Yuan [21] considered the existence and the uniqueness of positive solutions for the following

$$\begin{gathered} D_{0+}^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 3 < \alpha \le 4, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{1.1}$$

where $f(t,u) = q(t)[g(u) + h(u)]$, $g : [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing, $h : (0,+\infty) \to (0,+\infty)$ is continuous and nonincreasing, and $q \in C((0,1),(0,+\infty))$ satisfies

$$\int_0^1 s^{2-\eta(2-\alpha)}(1-s)^{\alpha-2-2\eta}q(s)ds < +\infty, \quad \eta \in (0,1).$$

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By a similar method, Zhang [25] studied a unique positive solution for the singular boundary value problem

$$\begin{gathered} D_{0+}^{\alpha}u(t)+q(t)f(t,u,u',\dots,u^{(n-2)})=0, \quad 0<t<1,\ n-1<\alpha\le n,\ n\ge 2, \\ u(0)=u'(0)=\dots=u^{(n-2)}(0)=u^{(n-2)}(1)=0, \end{gathered} \tag{1.2}$$

where $f=g+h$ and g,h have different monotone properties.

By means of a fixed point theorem for u_0 concave operators, Yang and Chen [22] investigated the existence and uniqueness of positive solutions for the following boundary value problem

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u,u',\dots,u^{(n-2)})=0, \quad 0<t<1,\ n-1<\alpha\le n,\ n\ge 2, \\ u(0)=u'(0)=\dots=u^{(n-2)}(0)=u^{(n-2)}(1)=0, \end{gathered} \tag{1.3}$$

where $f\in C([0,1]\times[0,+\infty)\times R^{n-2}\to[0,+\infty))$, $f(t,y_1,y_2,\dots,y_{n-1})$ is increasing for $y_i\ge 0, i=1,2,\dots n-1$, and $f\not\equiv 0$.

Different from the works mentioned above, we will use a fixed point theorem for a sum operator to show the existence and uniqueness of positive solutions for the following fractional equation boundary value problem

$$\begin{gathered} D_{0+}^{\alpha}u(t)+f(t,u(t))+g(t,u(t))=0, \quad 0<t<1,\ 1<\alpha\le 2, \\ u(0)=u(1)=0. \end{gathered} \tag{1.4}$$

Moreover, we can construct a sequence for approximating the unique solution. It must be pointed out that the method used in this article can be applied to (1.1)-(1.3).

2. Preliminaries

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proof of our theorem.

Definition 2.1 ([19, Definiton 2.1])**.** The integral

$$I_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad x>0$$

is called the Riemann-Liouville fractional integral of order α, where $\alpha>0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2.2 ([19, page 36-37])**.** For a function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\big(\frac{d}{dx}\big)^n\int_0^x\frac{f(t)}{(x-t)^{\alpha-n+1}}dt,$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Lemma 2.3 ([1])**.** *Given $y\in C[0,1]$ and $1<\alpha\le 2$, the boundary-value problem*

$$\begin{gathered} D_{0+}^{\alpha}u(t)+y(t)=0, \quad 0<t<1, \\ u(0)=u(1)=0, \end{gathered} \tag{2.1}$$

has a unique solution

$$u(t)=\int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ [t(1-s)]^{\alpha-1}, & 0\le t\le s\le 1,\end{cases}$$

which is the Green function for this boundary-value problem.

In [7], the authors obtained the following result.

Lemma 2.4. *Let $1<\alpha\le 2$. Then the Green function $G(t,s)$ in Lemma 2.3 satisfies*

$$\frac{\alpha-1}{\Gamma(\alpha)}h(t)(1-s)^{\alpha-1}s\le G(t,s)\le\frac{1}{\Gamma(\alpha)}h(t)(1-s)^{\alpha-2}\quad \text{for } t,s\in[0,1],$$

where $h(t)=t^{\alpha-1}(1-t)$, $t\in[0,1]$.

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed-point theorem which we will be used later. For convenience of readers, we suggest that one refer to [6, 23] for details.

Suppose that $(E,\|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P\subset E$; i.e., $x\le y$ if and only if $y-x\in P$. If $x\le y$ and $x\neq y$, then we denote $x<y$ or $y>x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P\subset E$ is a cone if it satisfies (i) $x\in P,\lambda\ge 0$ implies $\lambda x\in P$; (ii) $x\in P$ and $-x\in P$ imply $x=\theta$.

Putting $\mathring{P}=\{x\in P: x \text{ is an interior point of } P\}$, a cone P is said to be solid if $\mathring{P}$ is non-empty. Moreover, P is called normal if there exists a constant $N>0$ such that, for all $x,y\in E$, $\theta\le x\le y$ implies $\|x\|\le N\|y\|$; in this case N is called the normality constant of P. We say that an operator $A:E\to E$ is increasing if $x\le y$ implies $Ax\le Ay$.

For all $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\le y\le\mu x$. Clearly, $\sim$ is an equivalence relation. Given $h>\theta$ (i.e., $h\ge\theta$ and $h\neq\theta$), we denote by P_h the set $P_h=\{x\in E|\ x\sim h\}$. It is easy to see that $P_h\subset P$.

Definition 2.5. Let $D=P$ or $D=\mathring{P}$ and β be a real number with $0\le\beta<1$. An operator $A:P\to P$ is said to be β-concave if it satisfies

$$A(tx)\ge t^{\beta}Ax,\forall\ t\in(0,1),\ x\in D.$$

Notice that the definition of a β-concave operator mentioned above is different from that in [6], because we need not require the cone to be solid in general.

Definition 2.6. An operator $A:E\to E$ is said to be homogeneous if it satisfies

$$A(\lambda x)=\lambda Ax,\quad \forall\lambda>0,\ x\in E.$$

An operator $A:P\to P$ is said to be sub-homogeneous if it satisfies

$$A(tx)\ge tAx,\forall\ t\in(0,1),\ x\in P.$$

In a recent paper Zhai and Anderson [23] considered the sum operator equation

$$Ax+Bx+Cx=x,$$

where A is an increasing β-concave operator, B is an increasing sub-homogeneous operator and C is a homogeneous operator. They established the existence and uniqueness of positive solutions for the above equation, and when C is a null operator, they present the following interesting result.

Lemma 2.7. *Let P be a normal cone in a real Banach space E, $A : P \to P$ be an increasing β-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \geq \delta_0 Bx$, for all $x \in P$.*

Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}, n = 1, 2, \dots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

3. Existence and uniqueness of positive solutions for (1.4)

In this section, we apply Lemma 2.7 to study problem (1.4) and we obtain a new result on the existence and uniqueness of positive solutions. The method used here is new to the literature and so is the existence and uniqueness result to the fractional differential equations.

In our considerations we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbf{R}$ is continuous$\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0,1], x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for } t \in [0,1].$$

Set $P = \{x \in C[0,1] | x(t) \geq 0,\ t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in $C[0,1]$ and the normality constant is 1. Our main result is summarized in the following theorem using the following assumptions:

(H1) $f, g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous and increasing respect to the second argument, $g(t,0) \not\equiv 0$;
(H2) $g(t, \lambda x) \geq \lambda g(t,x)$ for $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,+\infty)$, and there exists a constant $\beta \in (0,1)$ such that $f(t, \lambda x) \geq \lambda^\beta f(t,x)$, for all $t \in [0,1]$, $\lambda \in (0,1)$, $x \in [0,+\infty)$;
(H3) there exists a constant $\delta_0 > 0$ such that $f(t,x) \geq \delta_0 g(t,x)$, $t \in [0,1]$, $x \geq 0$.

Theorem 3.1. *Under assumptions* (H1)–(H3)*, problem* (1.4) *has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}(1-t)$, $t \in [0,1]$. Moreover, for any initial value $u_0 \in P_h$, the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s) f(s, u_n(s)) ds + \int_0^1 G(t,s) g(s, u_n(s)) ds, \quad n = 0, 1, 2, \dots$$

satisfies $u_n(t) \to u^*(t)$ as $n \to \infty$.

Proof. From [1], problem (1.4) has the integral formulation

$$u(t) = \int_0^1 G(t,s)[f(s,u(s)) + g(s,u(s))] ds,$$

where $G(t,s)$ is given as in Lemma 2.3. Define two operators $A : P \to E$ and $B : P \to E$ by

$$Au(t) = \int_0^1 G(t,s) f(s, u(s))) ds, \quad Bu(t) = \int_0^1 G(t,s) g(s, u(s)) ds.$$

It is easy to prove that u is the solution of (1.4) if and only if $u = Au + Bu$. From (H1) and Lemma 2.4, we know that $A : P \to P$ and $B : P \to P$. In the sequel we check that A, B satisfy all assumptions of Lemma 2.7.

Firstly, we prove that A, B are two increasing operators. In fact, by (H1) and Lemma 2.4, for $u, v \in P$ with $u \geq v$, we know that $u(t) \geq v(t)$, $t \in [0,1]$ and obtain

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \int_0^1 G(t,s)f(s,v(s))ds = Av(t).$$

That is, $Au \geq Av$. Similarly, $Bu \geq Bv$. Next we show that A is a β-concave operator and B is a sub-homogeneous operator. In fact, for any $\lambda \in (0,1)$ and $u \in P$, by (H_2) we obtain

$$A(\lambda u)(t) = \int_0^1 G(t,s)f(s,\lambda u(s))ds \geq \lambda^\beta \int_0^1 G(t,s)f(s,u(s))ds = \lambda^\beta Au(t).$$

That is, $A(\lambda u) \geq \lambda^\beta Au$ for $\lambda \in (0,1)$, $u \in P$. So the operator A is a β-concave operator. Also, for any $\lambda \in (0,1)$ and $u \in P$, from (H2) we know that

$$B(\lambda u)(t) = \int_0^1 G(t,s)g(s,\lambda u(s))ds \geq \lambda \int_0^1 G(t,s)g(s,u(s))ds = \lambda Bu(t);$$

that is, $B(\lambda u) \geq \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. So the operator B is sub-homogeneous.

Now we show that $Ah \in P_h$ and $Bh \in P_h$. Let $h_{\max} = \max\{h(t) = t^{\alpha-1}(1-t) : t \in [0,1]\}$. Then $h_{\max} > 0$. From (H1) and Lemma 2.4,

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s)f(s,h(s))ds \leq \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-2} f(s,h_{\max})ds, \\ Ah(t) &= \int_0^1 G(t,s)f(s,h(s))ds \geq \frac{\alpha-1}{\Gamma(\alpha)} h(t) \int_0^1 s(1-s)^{\alpha-1} f(s,0)ds. \end{aligned}$$

From (H1) and (H3), we have

$$f(s,h_{\max}) \geq f(s,0) \geq \delta_0 g(s,0) \geq 0.$$

Since $g(t,0) \not\equiv 0$, we can obtain

$$\int_0^1 f(s,h_{\max})ds \geq \int_0^1 f(s,0)ds \geq \delta_0 \int_0^1 g(s,0)ds > 0,$$

and in consequence,

$$\begin{aligned} l_1 &:= \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f(s,0)ds > 0, \\ l_2 &:= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} f(s,h_{\max})ds > 0. \end{aligned}$$

So $l_1 h(t) \leq Ah(t) \leq l_2 h(t)$, $t \in [0,1]$; and hence we have $Ah \in P_h$.

Similarly,

$$\frac{\alpha-1}{\Gamma(\alpha)} h(t) \int_0^1 s(1-s)^{\alpha-1} g(s,0)ds \leq Bh(t) \leq \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-2} g(s,h_{\max})ds,$$

from $g(t,0) \not\equiv 0$, we easily prove $Bh \in P_h$. Hence the condition (i) of Lemma 2.7 is satisfied.

In the following we show the condition (ii) of Lemma 2.7 is satisfied. For $u \in P$, from (H3),

$$Au(t) = \int_0^1 G(t,s)f(s,u(s))ds \geq \delta_0 \int_0^1 G(t,s)g(s,u(s))ds = \delta_0 Bu(t).$$

Then we get $Au \geq \delta_0 Bu, u \in P$. Finally, an application of Lemma 2.7 implies: the operator equation $Ax + Bx = x$ has a unique solution u^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}, n = 1, 2, \dots$ for any initial value $y_0 \in P_h$, we have $y_n \to u^*$ as $n \to \infty$. That is, problem (1.4) has a unique positive solution u^* in P_h. Moreover, for any initial value $u_0 \in P_h$, constructing successively the sequence

$$u_{n+1}(t) = \int_0^1 G(t,s)f(s,u_n(s))ds + \int_0^1 G(t,s)g(s,u_n(s))ds, \quad n = 0, 1, 2, \dots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$. □

Remark 3.2. A simple example that illustrates Theorem 3.1 is as follows: let $f(t,x) \equiv 2\delta$, $g(t,x) \equiv 1, \delta > 0$. Then the conditions (H1)–(H3) are satisfied and (1.4) has a unique solution $u(t) = (2\delta+1)\int_0^1 G(t,s)ds$, $t \in [0,1]$. Evidently,

$$\begin{gathered}
u(t) \geq \frac{(2\delta+1)(\alpha-1)}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} ds \cdot h(t) = \frac{(2\delta+1)(\alpha-1)}{\alpha(\alpha+1)\Gamma(\alpha)} \cdot h(t), \\
u(t) \leq \frac{2\delta+1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-2} ds \cdot h(t) = \frac{2\delta+1}{(\alpha-1)\Gamma(\alpha)} \cdot h(t), \quad t \in [0,1].
\end{gathered}$$

So the unique solution u is a positive solution and satisfies $u \in P_h = P_{t^{\alpha-1}(1-t)}$.

Example 3.3.

$$\begin{gathered}
D_{0+}^{\frac{3}{2}} u(t) + u^{1/2}(t) + \frac{u(t)}{1+u(t)} q(t) + t^2 + a = 0, \quad 0 < t < 1, \\
u(0) = u(1) = 0,
\end{gathered} \tag{3.1}$$

where $a > 0$ is a constant, $q : [0,1] \to [0,+\infty)$ is continuous with $q \not\equiv 0$.

In this example, we have $\alpha = 3/2$. Take $0 < b < a$ and let

$$\begin{gathered}
f(t,x) = x^{1/2} + t^2 + b, \quad g(t,x) = \frac{x}{1+x} q(t) + a - b, \\
\beta = \frac{1}{2}, \quad q_{\max} = \max\{q(t) : t \in [0,1]\}.
\end{gathered}$$

Obviously, $q_{\max} > 0$; $f, g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous and increasing respect to the second argument, $g(t,0) = a - b > 0$. Besides, for $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,+\infty)$, we have

$$\begin{gathered}
g(t,\lambda x) = \frac{\lambda x}{1+\lambda x} q(t) + a - b \geq \frac{\lambda x}{1+x} q(t) + \lambda(a-b) = \lambda g(t,x), \\
f(t,\lambda x) = \lambda^{1/2} x^{1/2} + t^2 + b \geq \lambda^{1/2}(x^{1/2} + t^2 + b) = \lambda^\beta f(t,x).
\end{gathered}$$

Moreover, if we take $\delta_0 \in (0, \frac{b}{q_{\max}+a-b}]$, then we obtain

$$\begin{aligned}
f(t,x) = x^{1/2} + t^2 + b \geq b &= \frac{b}{q_{\max}+a-b}(q_{\max} + a - b) \\
&\geq \delta_0 [\frac{x}{1+x} q(t) + a - b] = \delta_0 g(t,x).
\end{aligned}$$

Hence all the conditions of Theorem 3.1 are satisfied. This implies that (3.1) has a unique positive solution in P_h, where $h(t) = t^{\alpha-1}(1-t)$, $t \in [0,1]$.

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