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EXISTENCE OF PERIODIC SOLUTIONS FOR RAYLEIGH EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract. We establish sufficient conditions for the existence of periodic solutions for a Rayleigh-type equation with state-dependent delay. Our approach is based on the continuation theorem in degree theory, and some analysis techniques. An example illustrates that our approach to this problem is new.

1. Introduction

Lord Rayleigh (John William Strutt: 1842–1919) [18] introduced the equation

$$x''(t) + f(x'(t)) + ax(t) = 0 \tag{1.1}$$

to model the oscillations of a clarinet reed. This equation is used for studying problems arising in acoustics, and is referred in the literature as Rayleigh equation. Later on, the Rayleigh equation of the form

$$x''(t) + f(x'(t)) + g(t, x(t)) = 0 \tag{1.2}$$

was studied in the monographs [1, 3, 6]. In many circumstances, however, it is known that the forces intervening in the system depend depend not only at the current time considered, but also on previous times. Thus, the forced Rayleigh equation with delay

$$x''(t) + f(t, x'(t)) + g(t, x(t - \tau)) = p(t) \tag{1.3}$$

has been taken into consideration, see [19, 20, 21]. Recently, it has been recognized that (1.3) has widespread applications in many applied sciences such as physics, mechanics and engineering techniques fields. In such applications, it is crucial to know the periodic behavior of solutions for Rayleigh equation. This justifies the intensive interest among researchers in investigating the existence of periodic solutions for this equation in the last decade. Publications [8, 5, 9, 10, 11, 13, 14, 15, 17, 22, 23, 24, 25, 26] are devoted to various generalizations of equation (1.3). Nevertheless, one can realize that all the results obtained in the above mentioned papers have been proved under the assumptions that τ is a constant, g is bounded and $\int_0^{2\pi} p(t)\,\mathrm{d}t = 0$. However, it is known that the delay may not be only related to time t but also it relates to the current state x. Thus, it is worth while to consider

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a type of Rayleigh equation with state-dependent delay. In this paper, particularly, we consider Rayleigh equation of the form

$$x''(t) + f(t, x(t)) + g(x(t - \tau(t, x(t)))) = p(t). \tag{1.4}$$

We shall utilize the continuation theorem of degree theory to obtain sufficient conditions for the existence of periodic solutions of (1.4). The main result is proved by bypassing the boundedness of g and the integral condition on p. To the best of authors' observations, there exists no paper establishing sufficient conditions for the existence of periodic solutions for (1.4). Thus, our result presents a new approach.

2. Preliminaries

Let

$$C_{2\pi} = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t + 2\pi) \equiv x(t),\ \forall\, t \in \mathbb{R}\}$$

with the norm $\|x\|_0 = \max_{t\in[0,2\pi]} |x(t)|$, for $x \in C_{2\pi}$ and

$$C^1_{2\pi} = \{x : x \in C^1(\mathbb{R}, \mathbb{R}), x(t + 2\pi) \equiv x(t),\ \forall\, t \in \mathbb{R}\}$$

with the norm $\|x\|_1 = \max_{t\in[0,2\pi]}\{\|x(t)\|_0, \|x'(t)\|_0\}$, for $x \in C^1_{2\pi}$. We shall consider (1.4) under the assumptions that $f \in C_{2\pi}(\mathbb{R}^2, \mathbb{R})$ with $f(\cdot, 0) = 0$, $g \in C_{2\pi}(\mathbb{R}, \mathbb{R})$, $\tau \in C_{2\pi}(\mathbb{R}^2, \mathbb{R}^+)$ and $p \in C_{2\pi}(\mathbb{R}, \mathbb{R})$.

Let $0 \le \tau(t) \in C_{2\pi}$, then there must exist two integers $k \ge 0$ and $m \ge 1$ such that

$$\tau(t) \in [2\pi k, 2\pi(k + m)], \quad \tau(t) \notin (0, 2\pi k) \cup (2\pi(k + m), \infty). \tag{2.1}$$

Denote $\Delta_i = \{t : t \in [0, 2\pi],\ \tau(t) \in [2\pi(k+i), 2\pi(k+i+1)]\}$, $i = 0, 1, 2, \dots, m-1$,

$$\tau_0(t) = \begin{cases} \tau(t), & t \in \Delta_0, \\ 2\pi(k + 1), & t \in [0, 2\pi]\backslash\Delta_0, \end{cases}$$

and

$$\tau_j(t) = \begin{cases} \tau(t), & t \in \Delta_j, \\ 2\pi(k + j), & t \in [0, 2\pi]\backslash\Delta_j. \end{cases}$$

Then, it is clear that $\cup_{i=0}^{m-1}\Delta_i = [0, 2\pi]$; $2\pi(k+1) - \tau_0(t) \in [0, 2\pi]$ and $\tau_j(t) - 2\pi(k+j) \in [0, 2\pi]$ for all $t \in [0, 2\pi]$, $j = 1, 2, \dots, m - 1$.

Let $\delta_0 = \sup_{t\in[0,2\pi]}[2\pi(k + 1) - \tau_0(t)]$, $\delta_j = \sup_{t\in[0,2\pi]}[\tau_j(t) - 2\pi(k + j)]$, then we have $\delta_0, \delta_{m-1} \in [0, 2\pi]$, $\delta_j = 2\pi,\ j = 1, 2, \dots, m - 2$.

The following lemma plays a key role in proving the main result.

Lemma 2.1 ([2]). *Let $\tau(t, x(t)) \in C_{2\pi}$ satisfying (2.1) and $x \in C^1_{2\pi}$, then*

$$\int_0^{2\pi} |x(t-\tau(t, x(t)))-x(t)|^2\, \mathrm{d}t \le \Big(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\Big) \int_0^{2\pi} |x'(t)|^2\, \mathrm{d}t,\ 2 < m < \infty$$

and

$$\int_0^{2\pi} |x(t - \tau(t, x(t))) - x(t)|^2\, \mathrm{d}t \le \big(\beta_0 + \beta_{m-1}\big) \int_0^{2\pi} |x'(t)|^2\, \mathrm{d}t,\ 1 \le m \le 2,$$

where

$$\beta_0 = \max_{\sigma\in[0,2\pi-\delta_0]} \int_\sigma^{\sigma+\delta_0} \tau(t, x(t))\, \mathrm{d}t,\ \beta_{m-1} = \max_{\sigma\in[0,2\pi-\delta_{m-1}]} \int_\sigma^{\sigma+\delta_{m-1}} \tau(t, x(t))\, \mathrm{d}t$$

and

$$\beta_j = \int_0^{2\pi} \tau(t, x(t))\,\mathrm{d}t, \quad j = 1, 2, \dots, m-2.$$

Lemma 2.2 ([12]). *Let $x \in C^1_{2\pi}$ and there exists a constant $\xi \in \mathbb{R}$ such that $x(\xi) = 0$. Then we have*

$$\int_0^{2\pi} |x(t)|^2\,\mathrm{d}t \le 4 \int_0^{2\pi} |x'(t)|^2\,\mathrm{d}t.$$

Degree theory has been used to prove the existence of solutions of a wide variety of differential, integral, functional and difference equations. Of particular interest is its use in the investigation of periodic solutions. We shall use a result by Mawhin [16] to prove the existence of a 2π-periodic solution of equation (1.4). We refer the reader to [4] for more information. Here are some basic concepts in the framework of this theory.

Define a linear operator

$$L : D(L) \subset C^1_{2\pi} \to C_{2\pi}, \quad Lx = x'',$$

where $D(L) = \{x : x \in C^2(\mathbb{R}, \mathbb{R}), x(t + 2\pi) \equiv x(t)\}$ and a nonlinear operator

$$N : C^1_{2\pi} \to C_{2\pi}, \quad Nx = -f(t, x(t)) - g(x(t - \tau(t, x(t)))) + p(t).$$

It is easy to see that

$$\ker(L) = \{a,\ a \in \mathbb{R}\}, \quad \operatorname{Im} L = \Big\{y : y \in C_{2\pi},\ \int_0^{2\pi} y(s)\,\mathrm{d}s = 0\Big\}.$$

Therefore, $\operatorname{Im} L$ is closed in $C_{2\pi}$ and $\dim\ker L = \operatorname{codim}\operatorname{Im} L = 1$. It follows that the operator L is a Fredholm operator with index zero.

Define the continuous projectors

$$P : C_{2\pi} \to \ker L, \quad Px = x(0),$$

$$Q : C_{2\pi} \to C_{2\pi}/\operatorname{Im} L, \quad Qy = \frac{1}{2\pi}\int_0^{2\pi} y(s)\,\mathrm{d}s.$$

It is easy to see that $\operatorname{Im} P = \ker L$ and $\ker Q = \operatorname{Im} L$. Set the operators

$$L_p = L\big|_{D(L)\cap\ker P} : D(L) \cap \ker P \to \operatorname{Im} L.$$

Then, L_p has a unique continuous inverse operator L_p^{-1} on $\operatorname{Im} L$ defined by

$$(L_p^{-1}y)(t) = \int_0^{2\pi} G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} \frac{s(2\pi - t)}{2\pi}, & 0 \le s < t \\ \frac{t(2\pi - s)}{2\pi}, & t \le s \le 2\pi. \end{cases}$$

Lemma 2.3 ([3]). *Let X and Y be two Banach spaces. Suppose that $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero and $N : X \to Y$ is L-compact on $\overline{\Omega}$ where $\Omega \subset X$ is an open bounded set. If the following conditions hold:*

(i) *$Lx \ne \lambda Nx$, for all $x \in \partial\Omega \cap D(L)$, for all $\lambda \in (0, 1)$;*
(ii) *$Nx \notin \operatorname{Im} L$, for all $x \in \partial\Omega \cap \ker L$;*
(iii) *The Brouwer degree $\deg\{QN, \Omega \cap \ker L, 0\} \ne 0$.*

Then equation $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

3. Existence result

Theorem 3.1. *Assume that there exist constants $K > 0$, $d > 0$ and $L \geq 0$ such that*

(C1) $|f(t,x)| \leq K$ *for all* $(t,x) \in \mathbb{R}^2$*;*
(C2) $xg(x) < 0$ *and* $|g(x)| \leq \|p\|_0$ *implies* $|x| \leq d$*;*
(C3) $|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$, *for all* $x_1, x_2 \in \mathbb{R}$.

If

$$2L\Big(\beta_0 + \beta_{m-1} + \sum_{j=1}^{m-2} \beta_j\Big)^{1/2} < 1. \tag{3.1}$$

Then (1.4) *has at least one* 2π*-periodic solution.*

Proof. Consider the auxiliary equation

$$x''(t) + \lambda f(t,x(t)) + \lambda g(x(t - \tau(t,x(t)))) = \lambda p(t), \quad \lambda \in (0,1). \tag{3.2}$$

To complete the proof of this theorem, one can see that it suffices to show that all possible 2π-periodic solutions of (3.2) are bounded. In other words, we shall prove that there exist positive constants M_2 and M_4 independent of λ and x such that if $x(t)$ is a 2π-periodic solution of equation (3.2) then $\|x\|_0 < M_2$ and $\|x'\|_0 < M_4$.

Let $x(t)$ be any 2π-periodic solution of (3.2). Then there exist $t_1, t_2 \in [0, 2\pi]$ such that

$$x(t_1) = \min_{t\in[0,2\pi]} x(t), \quad x(t_2) = \max_{t\in[0,2\pi]} x(t). \tag{3.3}$$

We claim that there exists $t^* \in [0, 2\pi]$ such that

$$|x(t^*)| \leq d. \tag{3.4}$$

From (3.3), it follows that $x'(t_1) = 0$ and thus $x''(t_1) \geq 0$. Therefore, we have

$$g(x(t_1 - \tau(t_1, x(t_1)))) \leq p(t_1) - f(t_1, x(t_1)). \tag{3.5}$$

In a similar manner, we deduce that $x'(t_2) = 0$ and thus $x''(t_2) \leq 0$. Hence,

$$g(x(t_2 - \tau(t_2, x(t_2)))) \geq p(t_2) - f(t_2, x(t_2)). \tag{3.6}$$

In view of (3.5) and (3.6), we may write

$$g(x(t_1 - \tau(t_1, x(t_1)))) \leq p(t_1) \leq \|p\|_0, \tag{3.7}$$

$$g(x(t_2 - \tau(t_2, x(t_2)))) \geq p(t_2) - K \geq -\|p\|_0. \tag{3.8}$$

Combining (3.7) and (3.8), we can find a point $\xi \in [0, 2\pi]$ such that

$$|g(x(\xi - \tau(\xi, x(\xi))))| \leq \|p\|_0.$$

By (C2), the above inequality implies

$$\big|x(\xi - \tau(\xi, x(\xi)))\big| \leq d.$$

Since $x(t)$ is a 2π-periodic function then there exists $t^* \in [0, 2\pi]$ such that $\xi - \tau(\xi, x(\xi)) = 2\pi k + t^*$. Therefore, one can see that (3.4) holds. It follows that

$$\|x\|_0 \leq |x(t^*)| + \int_0^{2\pi} |x'(s)|\,\mathrm{d}s \leq d + \int_0^{2\pi} |x'(s)|\,\mathrm{d}s. \tag{3.9}$$

Let

$$E_1 = \{t \in [0, 2\pi] : \ |x(t)| > d\}, \quad E_2 = \{t \in [0, 2\pi] : \ |x(t)| \leq d\}.$$

Multiplying both sides of (3.2) by $x(t)$ and integrating over $[0,2\pi]$, we have

$$\begin{aligned}-\int_0^{2\pi}\Big(x'(t)\Big)^2\,\mathrm{d}t &= -\lambda\int_0^{2\pi} g\Big(x(t-\tau(t,x(t)))\Big)x(t)\,\mathrm{d}t-\lambda\int_0^{2\pi} f(t,x(t))x(t)\,\mathrm{d}t\\ &\quad+\lambda\int_0^{2\pi} p(t)x(t)\,\mathrm{d}t\end{aligned}$$

or

$$\begin{aligned}\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t &= \lambda\int_0^{2\pi} g\Big(x(t-\tau(t,x(t)))\Big)x(t)\,\mathrm{d}t+\lambda\int_0^{2\pi} f(t,x(t))x(t)\,\mathrm{d}t\\ &\quad-\lambda\int_0^{2\pi} p(t)x(t)\,\mathrm{d}t.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\\ &=\lambda\int_0^{2\pi}\Big[g(x(t-\tau(t,x(t))))-g(x(t))\Big]x(t)\,\mathrm{d}t+\lambda\int_0^{2\pi} g(x(t))x(t)\,\mathrm{d}t\\ &\quad+\lambda\int_0^{2\pi} f(t,x(t))x(t)\,\mathrm{d}t-\lambda\int_0^{2\pi} p(t)x(t)\,\mathrm{d}t\end{aligned}$$

or

$$\begin{aligned}&\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\\ &=\lambda\int_0^{2\pi}\Big[g(x(t-\tau(t,x(t))))-g(x(t))\Big]x(t)\,\mathrm{d}t+\lambda\int_{E_1} g(x(t))x(t)\,\mathrm{d}t\\ &\quad+\lambda\int_{E_2} g(x(t))x(t)\,\mathrm{d}t+\lambda\int_0^{2\pi} f(t,x(t))x(t)\,\mathrm{d}t-\lambda\int_0^{2\pi} p(t)x(t)\,\mathrm{d}t.\end{aligned}$$

This implies

$$\begin{aligned}\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t &\le \int_0^{2\pi}\Big|g(x(t-\tau(t,x(t))))-g(x(t))\Big||x(t)|\,\mathrm{d}t+g_d\int_0^{2\pi}|x(t)|\,\mathrm{d}t\\ &\quad+K\int_0^{2\pi}|x(t)|\,\mathrm{d}t+\int_0^{2\pi}|p(t)||x(t)|\,\mathrm{d}t,\end{aligned}$$

where $g_d=\max_{t\in E_2}|g(x(t))|$. Furthermore,

$$\begin{aligned}&\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\\ &\le L\int_0^{2\pi}\Big|x(t-\tau(t,x(t)))-x(t)\Big||x(t)|\,\mathrm{d}t+g_d(2\pi)^{1/2}\|x\|_2+K\|x\|_2+\|p\|_2\|x\|_2,\end{aligned}$$

where $\|x\|_2=\Big(\int_0^{2\pi}|x(s)|^2\,\mathrm{d}s\Big)^{1/2}$. It follows that

$$\begin{aligned}\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t &\le L\Big(\int_0^{2\pi}\Big|x(t-\tau(t,x(t)))-x(t)\Big|^2\,\mathrm{d}t\Big)^{1/2}\|x\|_2+g_d(2\pi)^{1/2}\|x\|_2\\ &\quad+K\|x\|_2+\|p\|_2\|x\|_2.\end{aligned}$$

By the consequence of Lemma 2.1, we obtain

$$\begin{aligned}\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t &\le L\Big(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\Big)^{1/2}\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}\|x\|_2\\ &\quad+g_d(2\pi)^{1/2}\|x\|_2+K\|x\|_2+\|p\|_2\|x\|_2,\end{aligned}\tag{3.10}$$

Denote $u(t)=x(t)-x(t^*)$ where t^* is defined as in (3.9). Then we have

$$|x(t)|\le|x(t^*)|+|x(t)-x(t^*)|\le d+|u(t)|.$$

Using the Minkowski inequality [7], we obtain

$$\|x\|_2=\Big(\int_0^{2\pi}|x(t)|^2\,\mathrm{d}t\Big)^{1/2}\le(2\pi)^{1/2}d+\Big(\int_0^{2\pi}|u(t)|^2\,\mathrm{d}t\Big)^{1/2}.\tag{3.11}$$

However, since $u(t^*)=0,\ u(t+2\pi)=u(t)$ and $u'(t)=x'(t)$ then by the consequence of Lemma 2.2, we have

$$\Big(\int_0^{2\pi}|u(t)|^2\,\mathrm{d}t\Big)^{1/2}\le2\Big(\int_0^{2\pi}|u'(t)|^2\,\mathrm{d}t\Big)^{1/2}=2\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}.$$

Substituting back in (3.11), we obtain

$$\Big(\int_0^{2\pi}|x(t)|^2\,\mathrm{d}t\Big)^{1/2}\le(2\pi)^{1/2}d+2\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}.\tag{3.12}$$

It follows from (3.10) and (3.12) that

$$\begin{aligned}\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t &\le L\Big(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\Big)^{1/2}\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}\\ &\quad\times\Big((2\pi)^{1/2}d+2\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big)\\ &\quad+\Big(g_d(2\pi)^{1/2}+K+\|p\|_2\Big)\Big((2\pi)^{1/2}d+2\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}\Big)\\ &=2L\Big(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\Big)^{1/2}\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\\ &\quad+(2\pi)^{1/2}dL\Big(\beta_0+\beta_{m-1}+\sum_{j=1}^{m-2}\beta_j\Big)^{1/2}\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}\\ &\quad+2\Big(g_d(2\pi)^{1/2}+K+\|p\|_2\Big)\Big(\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\Big)^{1/2}\\ &\quad+(2\pi)^{1/2}d\Big(g_d(2\pi)^{1/2}+K+\|p\|_2\Big).\end{aligned}$$

By (3.1), we deduce that there exists a constant $M_1>0$ such that

$$\int_0^{2\pi}|x'(t)|^2\,\mathrm{d}t\le M_1.$$

From (3.9), we end up with

$$\|x\|_0\le d+(2\pi)^{1/2}M_1^{1/2}:=M_2.$$

In view of equation (1.4), one can obtain

$$\|x''(t)\| \le g_{M_2} + K + \|p\|_0 := M_3, \tag{3.13}$$

where $g_{M_2} = \max_{|x|\le M_2} |g(x)|$. However, since $x(0) = x(2\pi)$ then there exists a constant $\eta \in [0, 2\pi]$ such that $x'(\eta) = 0$. Therefore, by (3.13) we have

$$\|x'\|_0 \le |x'(\eta)| + \int_0^{2\pi} |x''(s)|\,\mathrm{d}s \le 2\pi M_3 := M_4.$$

Clearly, M_2 and M_4 are independent of λ and x. Take $\Omega = \{x : x \in X,\ \|x\|_0 < M_2,\ \|x'\|_0 < M_4\}$ and $\Omega_1 = \{x : x \in \ker L,\ Nx \in \operatorname{Im} L\}$. Clearly for all $x \in \Omega_1$, $x \equiv c$ is a constant and $f(t, c) + g(c) = p(t)$ thus by assumption (C2) we have $|c| \le d$ and hence $\Omega_1 \subset \Omega$. This tells that conditions (i)–(ii) of Lemma 2.3 are satisfied. Let

$$H(x, \mu) = \mu x - \frac{1-\mu}{2\pi}\int_0^{2\pi} \Big(f(t, x) + g(x) - p(t)\Big)\,\mathrm{d}t.$$

Then, one can easily realize that $H(x, \mu) \ne 0$ for all $(x, \mu) \in (\partial\Omega \cap \ker L) \times [0, 1]$. Hence

$$\begin{aligned}\deg\{QN, \Omega \cap \ker L, 0\} &= \deg\{H(x, 0), \Omega \cap \ker L, 0\} = \deg\{H(x, 1), \Omega \cap \ker L, 0\}\\ &= \deg\{1, \Omega \cap \ker L, 0\} \ne 0.\end{aligned}$$

Therefore, condition (iii) of Lemma 2.3 holds. This shows that equation (1.4) has at least one 2π-periodic solution. The proof is complete. □

Example 3.2. Consider the equation

$$x''(t) + \frac{1}{2}\sin x(t) - \frac{x^3\Big(t - \frac{1}{40}\tau(t, x(t))\Big)}{1 + x^2\Big(t - \frac{1}{20}\tau(t, x(t))\Big)} = e^{\cos^2 t}, \tag{3.14}$$

where $f(t, x) = \frac{1}{2}\sin x(t)$, $g(x) = -x^3/(1 + x^2)$, $\tau(t, x(t)) = \frac{1}{80}|\cos(t + 10x(t))|$ and $p(t) = e^{\cos^2 t}$. Clearly, $K = 1/2$, $L = 1$ and $\tau(t, x(t)) \in [0, 4\pi]$ for $t \in [0, 2\pi]$. Therefore, $k = 0$, $m = 2$, $\delta_0 = \delta_1 = 2\pi$, $\beta_0 = \beta_1 = \pi/40$. Thus, it is straightforward to realize that conditions (C1)–(C3) and (3.1) hold. By Theorem 3.1, equation (3.14) has at least one 2π-periodic solution.

We remark that the results obtained in [2, 8, 5, 9, 10, 11, 13, 14, 15, 17, 22, 23, 24, 25, 26] can not be applied to (3.14). This tells that the result in this paper is essentially new.

References

[1] K. Deimling; *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, New York, 1985.

[2] B. Du, C. Bai, X. Zhao; *Problems of periodic solutions for a type of Duffing equation with state–dependent delay*, J. Comput. Appl. Math. 233 (11) (2010), 2807–2813 .

[3] R. E. Gaines, J. L. Mawhin; *Coincidence Degree and Nonlinear Differential Equations*, Springer, Berlin, 1977.

[4] R. E. Gaines; *Existence of periodic solutions to second-Order nonlinear ordinary differential equations*, J. Differential Equations 16 (1074), 186–199.

[5] H. Gao, B. Liu; *Existence and uniqueness of periodic solutions for forced Rayleigh-type equations*, Appl. Math. Comput. 211 (1) (2009), 148-1-54.

[6] I. Grasman; *Asymptotic Methods for Relaxations Oscillations and Applications*, Springer, New York, 1987.

[7] G. Hardy, J. E. Littlewood, G. Polya; *Inequalities*, Cambridge University Press, London, 1952.
[8] C. Huang, Y. He, L. Huang, W. Tan; *New results on the periodic solutions for a kind of Rayleigh equation with two deviating arguments*, Math. Comput. Modelling 46 (5–6) (2007), 604–611.
[9] Y. Li, L. Huang; *New results of periodic solutions for forced Rayleigh-type equations*, J. Comput.Appl. Math. 221 (1) (2008), 98–105.
[10] B. Liu; *Existence and uniqueness of periodic solutions for a kind of Rayleigh equation with two deviating arguments*, Computers Math. Appl. 55 (9) (2008), 2108–2117.
[11] B. Liu, L. Huang; *Periodic solutions for a kind of Rayleigh equation with a deviating argument*, J. Math. Anal. Appl. 321 (2) (2006), 491–500.
[12] X. Liu, M. Jia, R. Ren; *On the existence and uniqueness of periodic solutions to a type of Duffing equation with complex deviating argument*, Acta Math. Sci. 27 (2007), 037–044. In Chinese.
[13] S. Lu, W. Ge, Z. Zheng; *A new result on the existence of periodic solutions for a kind of Rayleigh equation with a deviating argument*, Acta Math. Sinica 47 (2004), 299–304. In Chinese.
[14] S. P. Lu, W. G. Ge, Z. X. Zheng; *Periodic solutions for a kind of Rayleigh equation with a deviating argument*, Appl. Math. Lett. 17 (2004), 443–449.
[15] S. Lu, G. Weigao; *Some new results on the existence of periodic solutions to a kind of Rayleigh equation with a deviating argument*, Nonlinear Anal. 56 (2004), 501–514.
[16] J. Mawhin; *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector spaces*, J. Differential Equations 12 (1972), 610–636.
[17] L. Peng, B. Liu, Q. Zhou, L. Huang; *Periodic solutions for a kind of Rayleigh equation with two deviating arguments*, J. Franklin Inst. 343 (7) (2006), 676–687.
[18] J. W. Strutt (Lord Rayleigh); *Theory of Sound*, vol. 1. New York: Dover Publications; 1877. re-issued 1945.
[19] G. Wang, S. Cheng; *A priori bounds for periodic solutions of a delay Rayleigh equation*, Appl. Math. Lett. 12 (1999), 41–44.
[20] G. Q. Wang, J. R. Yan; *On existence of periodic solutions of the Rayleigh equation of retarded type*, Internat. J. Math. Math. Sci. 23 (1) (2000), 65–68.
[21] G. Q. Wang, J. R. Yan; *Existence theorem of periodic positive solutions for the Rayleigh equation of retarded type*, Portugal. Math. 57 (2) (2000), 153–160.
[22] L. Wang, J. Shao; *New results of periodic solutions for a kind of forced Rayleigh-type equations*, Nonlinear Anal.: Real World Appl. 11 (1) (2010), 99–105.
[23] Y. Wang, L. Zhang; *Existence of asymptotically stable periodic solutions of a Rayleigh type equation*, Nonlinear Anal. 71 (5–6) (2009), 1728–1735.
[24] Y. Zhou, X. Tang; *On existence of periodic solutions of Rayleigh equation of retarded type*, J. Comput.Appl. Math. 203(1) (2007), 1–5.
[25] Y. Zhou, X. Tang; *Periodic solutions for a kind of Rayleigh equation with a deviating argument*, Computers Math. Appl. 53 (5) (2007), 825–830.
[26] Y. Zhou, X. Tang; *On existence of periodic solutions of a kind of Rayleigh equation with a deviating argument*, Nonlinear Anal. 69 (8) (2008), 2355–2361.

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