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MULTIPLE SYMMETRIC POSITIVE SOLUTIONS TO FOUR-POINT BOUNDARY-VALUE PROBLEMS OF DIFFERENTIAL SYSTEMS WITH P-LAPLACIAN

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Abstract. In this article, we study the four-point boundary-value problem with the one-dimensional p-Laplacian

$$(\phi_{p_i}(u_i'))' + q_i(t)f_i(t,u_1,u_2) = 0, \quad t \in (0,1), \quad i = 1,2;$$
$$u_i(0) - g_i(u_i'(\xi)) = 0, \quad u_i(1) + g_i(u_i'(\eta)) = 0, \quad i = 1,2.$$

We obtain sufficient conditions such that by means of a fixed point theorem on a cone, there exist multiple symmetric positive solutions to the above boundary-value problem. As an application, we give an example that we illustrates our results.

1. Introduction

In this article, we discuss the existence of multiple symmetric positive solutions to the four-point boundary-value problem (BVP) for a differential system with the one-dimensional p-Laplacian,

$$(\phi_{p_i}(u_i'))' + q_i(t)f_i(t,u_1,u_2) = 0, \quad t \in (0,1),\ i = 1,2; \tag{1.1}$$
$$u_i(0) - g_i(u_i'(\xi)) = 0, \quad u_i(1) + g_i(u_i'(\eta)) = 0, \quad i = 1,2, \tag{1.2}$$

where $\phi_{p_i}(s) = |s|^{p_i-2}s$, $p_i > 1$, $0 < \xi < 1/2$, $\xi + \eta = 1$, and the functions q_i, f_i, g_i, $i = 1,2$ satisfy the following conditions:

- (H1) $q_i \in L^1[0,1]$ is nonnegative symmetric on $[0,1]$ (i.e., $q_i(t) = q_i(1-t)$, $t \in [0,1]$) and $q_i(t) \not\equiv 0$ on any subinterval of $[0,1]$;
- (H2) $f_i \in C([0,1] \times [0,+\infty) \times [0,+\infty), (0,+\infty))$ is symmetric on $[0,1]$ (i.e., $f_i(t,u_1,u_2) = f_i(1-t,u_1,u_2)$, for $t \in [0,1]$);
- (H3) $g_i \in C((-\infty,+\infty),(-\infty,+\infty))$ is strictly increasing, odd and satisfies the condition that there exists $m_i > 0$ such that $0 \le g_i(s) \le m_i s$ for all $s \ge 0$.

Multipoint boundary-value problems for ordinary differential equations and systems arise in a variety of areas of applied mathematics and physics. The study of

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multipoint BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [13]. Since then, many authors studied more general nonlinear multi-point boundary-value problems by Leray-Schauder continuation theorem, coincidence degree theory, the method of lower and upper solutions, monotone iterative technique, fixed point theorem in cones and so on. We refer readers to [3, 6, 11, 12, 21, 25, 27] and the references cited therein. On the other hand, the existence of symmetric positive solutions of second or higher order boundary-value problems have received more and more attention in the recent literature. The results of existence of symmetric positive solutions were obtained by some authors, see [2, 4, 5, 8, 9, 15, 16, 23, 24, 29].

In recent years, there were many works done for a variety of nonlinear second order ordinary differential systems with different boundary conditions, see [1, 7, 20, 22, 26, 28]. However, to the best of our knowledge, there were only a few results on the existence of multiple positive solutions to boundary-value problems for differential systems with the one-dimensional p-Laplacian. Especially, there were few papers on the existence of symmetric positive solutions.

Recently, Liu [18] studied the existence of positive solutions of singular boundary value systems with p-Laplacian,

$$
\begin{gathered}
(\phi_p(x'))' + a_1(t)f(t,x(t),y(t)) = 0, \quad t \in (0,1),\\
(\phi_p(y'))' + a_2(t)g(t,x(t),y(t)) = 0, \quad t \in (0,1),\\
x(0) - \beta_1 x'(0) = 0, \quad x(1) + \delta_1 x'(1) = 0,\\
y(0) - \beta_2 y'(0) = 0, \quad y(1) + \delta_2 y'(1) = 0.
\end{gathered}
$$

By using fixed-point index theory, the existence of one and multiple positive solutions for the boundary value systems under some conditions were established.

Liu and zhang [19] considered the existence of positive solutions for the nonlinear system

$$
\begin{gathered}
(\varphi_1(x'))' + a(t)f(x,y) = 0, \ (\varphi_2(y'))' + b(t)g(x,y) = 0, \quad t \in (0,1),\\
\alpha\varphi_1(x(0)) - \beta\varphi_1(x'(0)) = 0, \quad \alpha\varphi_2(y(0)) - \beta\varphi_2(y'(0)) = 0,\\
\alpha\varphi_1(x(1)) - \beta\varphi_1(x'(1)) = 0, \quad \alpha\varphi_2(y(1)) - \beta\varphi_2(y'(1)) = 0,
\end{gathered}
$$

where φ_1, φ_2 are the increasing homeomorphism and positive homomorphism and $\varphi_1(0) = 0, \varphi_2(0) = 0$. They showed the sufficient conditions for the existence of positive solutions by means of the norm type cone expansion-expression fixed point theorem.

Recently, Ji, Feng and Ge [14] discussed the existence of symmetric positive solutions for the boundary-value system with p-Laplacian,

$$
\begin{gathered}
(\phi_{p_1}(u'))' + a_1(t)f(u,v) = 0, \quad t \in (0,1),\\
(\phi_{p_2}(v'))' + a_2(t)g(t,u,v) = 0, \quad t \in (0,1),\\
u(0) - \alpha u'(\xi) = 0, \quad u(1) + \alpha u'(\eta) = 0,\\
v(0) - \alpha u'(\xi) = 0, \quad v(1) + \alpha v'(\eta) = 0,
\end{gathered}
$$

Motivated by the above works, our purpose in this paper is to give some conditions that guarantee the existence of multiple symmetric positive solutions for boundary value systems (1.1), (1.2).

The main tool of this article is the fixed point index theorem in cones.

Lemma 1.1 ([10, 17]). *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with $D_k = D \cap K \neq \phi$ and $\overline{D}_k \neq K$. Assume that $A : \overline{D}_k \to K$ is a compact map such that $x \neq Ax$ for $x \in \partial D_k$. Then the following results hold:*

(1) *If $\|Ax\| \leq \|x\|$, $x \in \partial D_k$, then $i_k(A, D_k) = 1$;*
(2) *If there exists $e \in K\backslash\{0\}$ such that $x \neq Ax + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(A, D_k) = 0$;*
(3) *Let U be open in X such that $\overline{U} \subset D_k$. If $i_k(A, D_k) = 1$ and $i_k(A, U_k) = 0$, then A has a fixed point in $D_k\backslash\overline{U}_k$. The same results holds if $i_k(A, D_k) = 0$ and $i_k(A, U_k) = 1$.*

2. Preliminaries

Let $E = C[0,1] \times C[0,1]$, then E is a Banach space with the norm $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \max_{t\in[0,1]} | u(t) |$, $\|v\| = \max_{t\in[0,1]} | v(t) |$.

Definition 2.1. We define a partial ordering in E. For $(u_1,u_2),(v_1,v_2) \in E$: $(u_1,u_2) \leq (v_1,v_2)$ if and only if $u_i(t) \leq v_i(t)$, $t \in [0,1]$, $i = 1,2$.

Definition 2.2. $(u_1,u_2) \in E$ is concave and symmetric on $[0,1]$ if and only if $u_i(t)$, $i = 1,2$, is concave and symmetric on $[0,1]$.

So, define a cone $K \subset E \times E$ by

$K = \{(u_1,u_2) \in E \times E : (u_1,u_2)$ is nonnegative, concave and symmetric on $[0,1]\}$.

For $h_i \in L^1[0,1]$, let (u_1,u_2) be a solution of the BVP

$$(\phi_{p_i}(u_i'))' + h_i(t) = 0, \quad t \in (0,1),\ i = 1,2, \tag{2.1}$$

$$u_i(0) - g_i(u_i'(\xi)) = 0, \quad u_i(1) + g_i(u_i'(\eta)) = 0, \quad i = 1,2. \tag{2.2}$$

By integrating (2.1), it follows that

$$\begin{gathered}
u_i'(t) = \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^t h_i(\tau)d\tau\Big), \\
u_i(t) = u_i(0) + \int_0^t \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^s h_i(\tau)d\tau\Big)ds, \\
u_i(t) = u_i(1) - \int_t^1 \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^s h_i(\tau)d\tau\Big)ds.
\end{gathered}$$

Using the boundary condition (2.2), we can easily obtain

$$u_i(t) = g_i \circ \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^\xi h_i(\tau)d\tau\Big) + \int_0^t \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^s h_i(\tau)d\tau\Big)ds$$

or

$$u_i(t) = -g_i \circ \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^\eta h_i(\tau)d\tau\Big) - \int_t^1 \phi_{p_i}^{-1}\Big(A_{h_i} - \int_0^s h_i(\tau)d\tau\Big)ds,$$

where A_{h_i} satisfies (2.3).

$$\begin{aligned}
&g_i \circ \phi_{p_i}^{-1}(A_{h_i} - \int_0^\xi h_i(\tau)d\tau) + g_i \circ \phi_{p_i}^{-1}(A_{h_i} - \int_0^\eta h_i(\tau)d\tau) \\
&+ \int_0^1 \phi_{p_i}^{-1}(A_{h_i} - \int_0^s h_i(\tau)d\tau)ds = 0, \quad i = 1,2.
\end{aligned} \tag{2.3}$$

Lemma 2.3. *If $h_i \in L^1[0,1]$ is nonnegative on $[0,1]$ and $h_i(t) \not\equiv 0$ on any subinterval of $[0,1]$, then there exists a unique $A_{h_i} \in (-\infty,+\infty)$ satisfying (2.3). Moreover, there is a unique $\sigma_{h_i} \in (0,1)$ such that $A_{h_i} = \int_0^{\sigma_{h_i}} h_i(\tau)d\tau$ for $i=1,2$.*

Proof. For any $h_i(t)$ in Lemma 2.3, define

$$\begin{aligned} H_{h_i}(c) &= g_i \circ \phi_{p_i}^{-1}\Big(c - \int_0^{\xi} h_i(\tau)d\tau\Big) + g_i \circ \phi_{p_i}^{-1}\Big(c - \int_0^{\eta} h_i(\tau)d\tau\Big) \\ &\quad + \int_0^1 \phi_{p_i}^{-1}(c - \int_0^s h_i(\tau)d\tau)ds, \quad i=1,2. \end{aligned}$$

So, $H_{h_i} : R \to R$ is continuous and strictly increasing since g_i is strictly increasing. It is easy to see that $H_{h_i}(0) < 0$, $H_{h_i}(\int_0^1 h_i(\tau)d\tau) > 0$. Therefore, there exists a unique $A_{h_i} \in (0, \int_0^1 h_i(\tau)d\tau) \subset (-\infty,+\infty)$ satisfying (2.3). Furthermore, if $F_i(t) = \int_0^t h_i(\tau)d\tau$, then $F_i(t)$ is continuous and strictly increasing on $[0,1]$, $F_i(0)=0$, and $F_i(1) = \int_0^1 h_i(\tau)d\tau$. Thus,

$$0 = F_i(0) < A_{h_i} < F_i(1) = \int_0^1 h_i(\tau)d\tau.$$

Therefore, the intermediate value theorem guarantees that there exists a unique $\sigma_{h_i} \in (0,1)$ such that $A_{h_i} = \int_0^{\sigma_{h_i}} h_i(\tau)d\tau$, $i=1,2$. □

Remark 2.4. By Lemma 2.3, if (u_1,u_2) is the unique solution of (2.1)-(2.2), then u_i $(i=1,2)$ can be rewritten as

$$u_i(t) = \begin{cases} g_i \circ \phi_{p_i}^{-1}\Big(\int_{\xi}^{\sigma_{h_i}} h_i(\tau)d\tau\Big) + \int_0^t \phi_{p_i}^{-1}\Big(\int_s^{\sigma_{h_i}} h_i(\tau)d\tau\Big)ds, & 0 \le t \le \sigma_{h_i}, \\ g_i \circ \phi_{p_i}^{-1}\Big(\int_{\sigma_{h_i}}^{\eta} h_i(\tau)d\tau\Big) + \int_t^1 \phi_{p_i}^{-1}\Big(\int_{\sigma_{h_i}}^{s} h_i(\tau)d\tau\Big)ds, & \sigma_{h_i} \le t \le 1, \end{cases} \tag{2.4}$$

for $i=1,2$.

Lemma 2.5. *If $h_i \in L^1[0,1]$, $i=1,2$, is nonnegative symmetric on $[0,1]$ and $h_i(t) \not\equiv 0$ on any subinterval of $[0,1]$, then the unique solution (u_1,u_2) of (2.1)-(2.2) is nonnegative, concave and symmetric.*

Proof. Suppose that (u_1,u_2) is the solution of (2.1)-(2.2). From the fact that

$$(\phi_{p_i}(u_i'))'(t) = -h_i(t) \le 0, \quad i=1,2,$$

we know that $\phi_{p_i}(u_i'(t))$ is non-increasing. It follows that $u_i'(t)$ is also non-increasing. Thus, we know that the graph of u_i is concave down on $[0,1]$.

It is easy to know that $H_{h_i}(\int_0^{\frac{1}{2}} h_i(\tau)d\tau) = 0$; i.e., $\sigma_{h_i} = 1/2$ from the symmetry of $h_i(t)$. So, from (2.4) and for $t \in [0,1/2]$, by the transformation $\tau = 1-\widehat{\tau}$, we have

$$\begin{aligned} u_i(t) &= g_i \circ \phi_{p_i}^{-1}\Big(\int_{\xi}^{1/2} h_i(\tau)d\tau\Big) + \int_0^t \phi_{p_i}^{-1}\Big(\int_s^{1/2} h_i(\tau)d\tau\Big)ds \\ &= -g_i \circ \phi_{p_i}^{-1}\Big(\int_{1-\xi}^{1/2} h_i(\widehat{\tau})d\widehat{\tau}\Big) - \int_0^t \phi_{p_i}^{-1}\Big(\int_{1-s}^{1/2} h_i(\widehat{\tau})d\widehat{\tau}\Big)ds. \end{aligned}$$

Again, let $s = 1-\widehat{s}$; then

$$u_i(t) = -g_i \circ \phi_{p_i}^{-1}\Big(\int_{\eta}^{1/2} h_i(\widehat{\tau})d\widehat{\tau}\Big) + \int_1^{1-t} \phi_{p_i}^{-1}\Big(\int_{\widehat{s}}^{1/2} h_i(\widehat{\tau})d\widehat{\tau}\Big)d\widehat{s}$$

$$
\begin{aligned}
&= g_i \circ \phi_{p_i}^{-1}\Big(\int_{1/2}^{\eta} h_i(\tau)d\tau\Big) + \int_{1-t}^{1} \phi_{p_i}^{-1}\Big(\int_{1/2}^{s} h_i(\tau)d\tau\Big)ds \\
&= u_i(1-t), \quad i=1,2.
\end{aligned}
$$

So, u_i is symmetric on $[0,1]$.

Combining the concavity and symmetry of u_i, we have $u_i(1/2) = \max_{0\le t\le 1} u_i$ and $u_i'(1/2) = 0$. So, $u_i'(t) \ge 0$ for $t \in (0,1/2)$ and $u_i'(t) \le 0$ for $t \in (1/2,1)$. It follows that $u_i(0) = g(u_i'(\xi)) = -g(u_i'(\eta)) = u_i(1) \ge 0$. Therefore, $u_i \ge 0$ on $[0,1]$, $i = 1,2$. $\square$

Lemma 2.6. *Let $(u_1,u_2) \in K$, and $\delta \in (0,\frac{1}{2})$, then $\min_{\delta\le t\le 1-\delta}(u_1(t)+u_2(t)) \ge \delta(\|u_1\|+\|u_2\|)$, $t \in [\delta, 1-\delta]$.*

The proof of the above lemma uses standard arguments only; we omit it here.

Lemma 2.7. *Assume that (H1)–(H3) hold. For any $(u_1,u_2) \in K$, define an operator*

$$
T(u_1,u_2)(t) = (T_1(u_1,u_2), T_2(u_1,u_2))(t),
$$

where

$$
T_i(u_1,u_2)(t) = \begin{cases}
g_i \circ \phi_{p_i}^{-1}\Big(\int_{\xi}^{1/2} q_i(\tau) f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big) \\
+ \int_0^t \phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau) f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big)ds, \quad 0 \le t \le 1/2, \\
g_i \circ \phi_{p_i}^{-1}\Big(\int_{1/2}^{\eta} q_i(\tau) f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big) \\
+ \int_t^1 \phi_{p_i}^{-1}\Big(\int_{1/2}^{s} q_i(\tau) f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big)ds, \quad 1/2 \le t \le 1,
\end{cases}
$$

for $i = 1,2$. Then $T : K \to K$ is completely continuous.

Proof. We first verify that $T : K \to K$. To do so, let $(u_1,u_2) \in K$. According to the definition of T and Lemma 2.5, it follows $(\phi_{p_i}(T_i(u_1,u_2))')'(t) = -q_i(t) f_i(t,u_1(t),u_2(t)) \le 0$ this implies $T_i(u_1,u_2)$ is concave on $[0,1]$. Again, from the definition of symmetry of f_i and q_i, it is easy to know that $T_i(u_1,u_2)(t) = T_i(u_1,u_2)(1-t)$ for $t \in [0,1/2]$, that is $T_i(u_1,u_2)$ is symmetric on $[0,1]$. So, indeed $TK \subset K$ from definition 2.2.

Next we show that $T : K \to K$ is completely continuous.

(1) We prove T is compact. Let $U \subset K$ is a bounded subset, then there exists a constant $D > 0$, such that $\|(u_1,u_2)\| \le D$ for any $(u_1,u_2) \in U$. By the discussion about T_i $(i = 1,2)$ and condition (H3), for any $(u_1,u_2) \in U$ and $0 \le t \le 1/2$ (the case $1/2 \le t \le 1$ can be proved similarly), we have

$$
\begin{aligned}
\|T_i(u_1,u_2)\| &= T_i(u_1,u_2)(\frac{1}{2}) \\
&= g_i \circ \phi_{p_i}^{-1}\Big(\int_{\xi}^{1/2} q_i(\tau) f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big) \\
&\quad + \int_0^{\frac{1}{2}} \phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau) f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big)ds, \\
&\le \Big[m_i \phi_{p_i}^{-1}\Big(\int_{\xi}^{1/2} q_i(\tau)d\tau\Big) + \int_0^{\frac{1}{2}} \phi_{p_i}^{-1}(\int_s^{1/2} q_i(\tau)d\tau)ds\Big] \\
&\quad \times \phi_{p_i}^{-1}\Big(\sup\{f_i(t,u_1,u_2) : t \in [0,1],\ (u_1,u_2) \in U\}\Big).
\end{aligned}
$$

Thus $T_i(U)$ $(i=1,2)$ is bounded, this implies $T(U)$ is bounded.

Next, for any $(u_1,u_2)\in U$ and $0\le t\le 1/2$, we have

$$\begin{aligned}\|T_i'(u_1,u_2)\| &\le \phi_{p_i}^{-1}\Big(\int_0^{1/2} q_i(\tau)f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big)\\ &\le \phi_{p_i}^{-1}\Big(\int_0^{1/2} q_i(\tau)d\tau\Big)\phi_{p_i}^{-1}(\sup\{f_i(t,u_1,u_2): t\in[0,1],\ (u_1,u_2)\in U\}).\end{aligned}$$

Then $T(U)$ is equicontinuous; that is, $T(U)$ is a relatively compact set according to the Ascoli-Arzela theorem.

(2) We show that T is continuous. Let $(u_1^{(n)},u_2^{(n)})\in U$ and converge uniformly to $(u_1^{(0)},u_2^{(0)})$, then

$$\begin{aligned}&T_i(u_1^{(n)},u_2^{(n)})(t)\\ &\le T_i(u_1^{(n)},u_2^{(n)})(\frac{1}{2})\\ &=\begin{cases} g_i\circ\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)f_i(\tau,u_1^{(n)}(\tau),u_2^{(n)}(\tau))d\tau\Big)\\ +\int_0^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)f_i(\tau,u_1^{(n)}(\tau),u_2^{(n)}(\tau))d\tau\Big)ds, & 0\le t\le 1/2,\\ g_i\circ\phi_{p_i}^{-1}\Big(\int_{1/2}^{\eta} q_i(\tau)f_i(\tau,u_1^{(n)}(\tau),u_2^{(n)}(\tau))d\tau\Big)\\ +\int_{1/2}^{1}\phi_{p_i}^{-1}\Big(\int_{1/2}^{s} q_i(\tau)f_i(\tau,u_1^{(n)}(\tau),u_2^{(n)}(\tau))d\tau\Big)ds, & 1/2\le t\le 1,\end{cases}\\ &\le\begin{cases} \Big[m_i\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)d\tau\Big)+\int_0^{\frac12}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\Big)ds\Big]\\ \times\phi_{p_i}^{-1}(\sup\{f_i(t,u_1,u_2): t\in[0,1],\ (u_1,u_2)\in U\}), & 0\le t\le 1/2,\\ \Big[m_i\phi_{p_i}^{-1}\Big(\int_{1/2}^{\eta} q_i(\tau)d\tau\Big)+\int_{\frac12}^{1}\phi_{p_i}^{-1}\Big(\int_{1/2}^{s} q_i(\tau)d\tau\Big)ds\Big]\\ \times\phi_{p_i}^{-1}(\sup\{f_i(t,u_1,u_2): t\in[0,1],\ (u_1,u_2)\in U\}) & 1/2\le t\le 1.\end{cases}\end{aligned}$$

Thus, by the dominated convergence theorem, we can get the limit

$$\begin{aligned}&\lim_{n\to\infty} T_i(u_1^{(n)},u_2^{(n)})(t)\\ &=\begin{cases} g_i\circ\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)f_i(\tau,u_1^{(0)}(\tau),u_2^{(0)}(\tau))d\tau\Big)\\ +\int_0^{t}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)f_i(\tau,u_1^{(0)}(\tau),u_2^{(0)}(\tau))d\tau\Big)ds, & 0\le t\le 1/2,\\ g_i\circ\phi_{p_i}^{-1}\Big(\int_{1/2}^{\eta} q_i(\tau)f_i(\tau,u_1^{(0)}(\tau),u_2^{(0)}(\tau))d\tau\Big)\\ +\int_{t}^{1}\phi_{p_i}^{-1}\Big(\int_{1/2}^{s} q_i(\tau)f_i(\tau,u_1^{(0)}(\tau),u_2^{(0)}(\tau))d\tau\Big)ds, & 1/2\le t\le 1;\end{cases}\end{aligned}$$

i.e., $\lim_{n\to\infty} T_i(u_n,u_n)(t)=T_i(u_0,u_0)(t)$. So T_i $(i=1,2)$ is continuous on U. It follows that $T(U)$ is continuous on U. Hence we complete the proof of Lemma 2.7. □

3. Existence of multiple symmetric positive solutions to (1.1)-(1.2)

Now for convenience we use the following notation. Let

$$\overline{\gamma}_i=\frac{\delta\int_\delta^{1/2}\phi_{p_i}^{-1}\big(\int_s^{1/2} q_i(\tau)d\tau\big)ds}{m_i\phi_{p_i}^{-1}\big(\int_\xi^{1/2} q_i(\tau)d\tau\big)+\int_0^{1/2}\phi_{p_i}^{-1}\big(\int_s^{1/2} q_i(\tau)d\tau\big)ds},\quad i=1,2,$$

$\gamma_i = \delta\overline{\gamma}_i$, $\gamma = \min\{\gamma_1, \gamma_2\}$, $K_\rho = \{(u_1, u_2) \in K : \|(u_1, u_2)\| < \rho\}$,

$$\begin{aligned}\Omega_\rho &= \{(u_1, u_2) \in K : \min_{\delta \le t \le 1-\delta}(u_1(t) + u_2(t)) < \gamma\rho\},\\ &= \{(u_1, u_2) \in K,\ \gamma\|(u_1, u_2)\| \le \min_{\delta \le t \le 1-\delta}(u_1(t) + u_2(t)) < \gamma\rho\},\end{aligned}$$

$$\begin{gathered}f_{i[\gamma\rho,\rho]} = \min\{\min_{t\in[\delta,1-\delta]} \frac{f_i(t, u_1, u_2)}{\phi_{p_i}(\rho)} : u_1 + u_2 \in [\gamma\rho, \rho]\},\\ f_i^{[0,\rho]} = \max\{\max_{t\in[0,1]} \frac{f_i(t, u_1, u_2)}{\phi_{p_i}(\rho)} : u_1 + u_2 \in [0, \rho]\},\\ f_{i\alpha} = \liminf_{(u_1,u_2)\to\alpha} \min_{t\in[\delta,1-\delta]} \frac{f_i(t, u_1, u_2)}{\phi_{p_i}(u_1 + u_2)},\\ f_i^{\alpha} = \limsup_{(u_1,u_2)\to\alpha} \max_{t\in[0,1]} \frac{f_i(t, u_1, u_2)}{\phi_{p_i}(u_1 + u_2)}, \quad (\alpha := \infty,\ \text{or } 0),\\ \frac{1}{N_i} = 2\Big[m_i\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)d\tau\Big) + \int_0^{1/2} \phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\Big)ds\Big],\\ \frac{1}{M_i} = 2\delta\int_\delta^{1/2} \phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\Big)ds,\end{gathered}$$

where $i = 1, 2$ and $(u_1, u_2) \to \alpha$ if and only if $\|u_1\| + \|u_2\| \to \alpha$

Remark 3.1. By (H1) it is to show that $0 < N_i, M_i < \infty$ and $M_i\gamma \le M_i\gamma_i = M_i\delta\overline{\gamma}_i = \delta N_i < N_i$, $i = 1, 2$.

Lemma 3.2 ([17]). *The set Ω_ρ defined above has the following properties:*

(a) *Ω_ρ is open relative to K;*
(b) *$K_{\gamma\rho} \subset \Omega_\rho \subset K_\rho$;*
(c) *$(u_1, u_2) \in \partial\Omega_\rho$ if and only if $\min_{\delta\le t\le 1-\delta}(u_1(t) + u_2(t)) = \gamma\rho$;*
(d) *If $(u_1, u_2) \in \partial\Omega_\rho$, then $\gamma\rho \le u_1 + u_2 \le \rho$ for $t \in [\delta, 1-\delta]$.*

We are now ready to apply Lemma 1.1 to the operator T to give sufficient conditions for the existence of multiple symmetric positive solutions to (1.1)-(1.2).

Theorem 3.3. *Assume that* (H1)–(H3) *hold, and suppose that f_i satisfies the following conditions:*

(H4) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \gamma\rho_2 < \rho_2 < \rho_3$ such that*
$$f_i^{[0,\rho_1]} < \phi_{p_i}(N_i), \quad f_{i[\gamma\rho_2,\rho_2]} > \phi_{p_i}(\gamma M_i), \quad f_i^{[0,\rho_3]} \le \phi_{p_i}(N_i), \quad i = 1, 2.$$

Then (1.1)-(1.2) *has three symmetric positive solutions in K.*

Proof. Recall that (1.1)-(1.2) has a solution (u_1, u_2) if and only if the operator T has a fixed point. Thus we set out to verify that the operator T satisfies Lemma 1.1 which will prove the existence of three fixed points of T which satisfies the conclusion of the theorem.

Firstly, we show that $i_k(T, K_{\rho_1}) = 1$. In fact, by the definition of T and $f_i^{[0,\rho_1]} < \phi_{p_i}(N_i)$, for $(u_1, u_2) \in \partial K_{\rho_1}$, we have

$$\begin{aligned}\|T_i(u_1, u_2)\| &= \max_{0\le t\le 1} |T_i(u_1, u_2)(t)| = T_i(u_1, u_2)(1/2)\\ &= g_i \circ \phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)f_i(\tau, u_1(\tau), u_2(\tau))d\tau\Big)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)f_i(\tau,u_1(\tau),u_2(\tau))d\tau\Big)ds\\
&< m_i\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)d\tau\phi_{p_i}(\rho_1N_i)\Big)\\
&\quad+\int_0^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\phi_{p_i}(\rho_1N_i)\Big)ds\\
&\le \rho_1N_i\Big[m_i\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)d\tau\Big)+\int_0^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\Big)ds\Big]\\
&=\frac{\rho_1}{2}=\frac{\|(u_1,u_2)\|}{2},\quad i=1,2.
\end{aligned}
$$

Thus,

$$
\|T(u_1u_2)\|=\|(T_1(u_1u_2),T_2(u_1u_2))\|=\|(T_1(u_1u_2)\|+\|T_2(u_1u_2))\|<\|(u_1,u_2)\|,
$$

by Lemma1.1 (1), one has $i_k(T,K_{\rho_1})=1$.

Secondly, we show that $i_k(T,\Omega_{\rho_2})=0$. Let $(e_1(t),e_2(t))\equiv(\frac12,\frac12)$ for $t\in[0,1]$, then $(e_1(t),e_2(t))\in\partial K_1$. We claim that

$$
\begin{aligned}
(u_1,u_2)\neq T(u_1,u_2)+\lambda(e_1,e_2)&=(T_1(u_1,u_2),T_2(u_1,u_2))+\lambda(e_1,e_2)\\
&=(T_1(u_1,u_2)+\lambda e_1,T_2(u_1,u_2)+\lambda e_2);
\end{aligned}
$$

that is, $u_i\neq T_i(u_1,u_2)+\lambda e_i$, for $(u_1,u_2)\in\partial\Omega_{\rho_2}$, $\lambda>0$, $i=1,2$.

In fact, if not, there exist $(u_1^0,u_2^0)\in\partial\Omega_{\rho_2}$, $\lambda_0>0$ such that $(u_1^0,u_2^0)=T(u_1^0,u_2^0)+\lambda_0(e_1,e_2)$. From Lemma 2.6 and $f_{i[\gamma\rho_2,\rho_2]}>\phi_{p_i}(\gamma M_i)$, we have

$$
\begin{aligned}
u_i^0(t)&=T_i(u_1^0,u_2^0)(t)+\lambda_0e_i(t)\\
&\ge\delta\|T_i(u_1^0,u_2^0)\|+\frac{\lambda_0}{2}=\delta T_i(u_1^0,u_2^0)(\frac12)+\frac{\lambda_0}{2}\\
&=\delta\Big[g_i\circ\phi_{p_i}^{-1}\Big(\int_\xi^{1/2} q_i(\tau)f_i(\tau,u_1^0(\tau),u_2^0(\tau))d\tau\Big)\\
&\quad+\int_0^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)f_i(\tau,u_1^0(\tau),u_2^0(\tau))d\tau\Big)ds\Big]+\frac{\lambda_0}{2}\\
&\ge\delta\int_\delta^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)f_i(\tau,u_1^0(\tau),u_2^0(\tau))d\tau\Big)ds+\frac{\lambda_0}{2}\\
&>\delta\int_\delta^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\phi_{p_i}(\gamma\rho_2M_i)\Big)ds+\frac{\lambda_0}{2}\\
&=\delta\gamma\rho_2M_i\int_\delta^{1/2}\phi_{p_i}^{-1}\Big(\int_s^{1/2} q_i(\tau)d\tau\Big)ds+\frac{\lambda_0}{2}\\
&=\frac{\gamma\rho_2+\lambda_0}{2},\ i=1,2.
\end{aligned}
$$

Thus $u_1^0(t)+u_2^0(t)>\gamma\rho_2+\lambda_0$. This implies that $\gamma\rho_2>\gamma\rho_2+\lambda_0$, which is a contradiction. Hence by Lemma 1.1 (2), it follows that $i_k(T,\Omega_{\rho_2})=0$.

Finally, similar to the proof of $i_k(T,K_{\rho_1})=1$, we can obtain that $i_k(T,K_{\rho_3})=1$. Therefore, it follows from Lemma 1.1 that T has three fixed points $u_1\in K_{\rho_1}$, $u_2\in\Omega_{\rho_2}\backslash\overline{K}_{\rho_1}$ and $u_3\in K_{\rho_3}\backslash\overline{\Omega}_{\rho_2}$. □

Theorem 3.4. *Assume that* (H1)–(H3) *hold, and suppose that* f_i *satisfies the conditions:*

(H5) *There exist* $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, *with* $\rho_1 < \rho_2 < \gamma\rho_3$ *such that*
$$f_{i[\gamma\rho_1,\rho_1]} > \phi_{p_i}(\gamma M_i), \quad f_i^{[0,\rho_2]} < \phi_{p_i}(N_i), \quad f_{i[\gamma\rho_3,\rho_3]} \geq \phi_{p_i}(\gamma M_i), \quad i = 1, 2.$$

Then (1.1)-(1.2) *has two symmetric positive solutions in* K.

The proof of the above theorem is similar to that of Theorem 3.3; we omit it here. As a special case of Theorem 3.3, we obtain the following result.

Corollary 3.5. *Assume that* (H1)–(H3) *hold. In addition, if there exists* $\rho \in (0, \infty)$ *such that*

(H6) $0 \leq f_i^0 < \phi_{p_i}(N_i)$, $f_{i[\gamma\rho,\rho]} > \phi_{p_i}(\gamma M_i)$, $0 \leq f_i^\infty < \phi_{p_i}(N_i)$, $i = 1, 2$.

Then (1.1)-(1.2) *has three symmetric positive solutions in* K.

Proof. We show that (H6) implies (H4). It is easy to verify that $0 \leq f_i^0 < \phi_{p_i}(N_i)$ implies that there exists $\rho_1 \in (0, \gamma\rho)$ such that $f_i^{[0,\rho_1]} < \phi_{p_i}(N_i)$. Let $k_i \in (f_i^\infty, \phi_{p_i}(N_i))$, then there exists $r > \rho$ such that $\max_{0\leq t\leq 1} f_i(t, u_1, u_2) \leq k_i\phi_{p_i}(u_1 + u_2)$ for $u_1 + u_2 \in [r, \infty)$ since $0 \leq f_i^\infty < \phi_{p_i}(N_i)$. Let
$$\beta_i = \max\big\{\max_{0\leq t\leq 1} f_i(t, u_1, u_2) : 0 \leq u_1 + u_2 \leq r\big\},$$
$$\rho_3 > \max\big\{\phi_{p_1}^{-1}(\frac{\beta_1}{\phi_{p_1}(N_1) - k_1}), \phi_{p_2}^{-1}(\frac{\beta_2}{\phi_{p_2}(N_2) - k_2}), \rho\big\}.$$

Then
$$\max_{0\leq t\leq 1} f_i(t, u_1, u_2) \leq k_i\phi_{p_i}(u_1 + u_2) + \beta_i \leq k_i\phi_{p_i}(\rho_3) + \beta_i < \phi_{p_i}(N_i)\phi_{p_i}(\rho_3)$$
for $u_1 + u_2 \in [0, \rho_3]$. This implies $f_i^{[0,\rho_3]} \leq \phi_{p_i}(N_i)$ and (H4) holds. □

Similarly, as a special case of Theorem 3.4, we obtain the following result.

Corollary 3.6. *Assume that* (H1)–(H3) *hold. In addition, if there exists* $\rho \in (0, \infty)$ *such that the following conditions hold*

(H7) $\phi_{p_i}(M_i) < f_{i0} \leq \infty$, $f_i^{[0,\rho]} < \phi_{p_i}(N_i)$, $\phi_{p_i}(M_i) < f_{i\infty} \leq \infty$, $i = 1, 2$.

Then (1.1)-(1.2) *has two symmetric positive solutions in* K.

By an argument similar to that of Theorem 3.3, we can obtain the following results.

Theorem 3.7. *Assume that* (H1)–(H3) *hold. In addition, one of the following two condition holds*

(H8) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \gamma\rho_2$ *such that*
$$f_i^{[0,\rho_1]} \leq \phi_{p_i}(N_i), \quad f_{i[\gamma\rho_2,\rho_2]} \geq \phi_{p_i}(\gamma M_i) \quad i = 1, 2;$$

(H9) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that*
$$f_{i[\gamma\rho_1,\rho_1]} \geq \phi_{p_i}(\gamma M_i), \quad f_i^{[0,\rho_2]} \leq \phi_{p_i}(N_i) \quad i = 1, 2.$$

Then (1.1)-(1.2) *has one symmetric positive solution in* K.

Corollary 3.8. *Assume that* (H1)–(H3) *hold. In addition, one of the following two conditions holds*

(H10) $0 \leq f_i^0 < \phi_{p_i}(N_i)$, $\phi_{p_i}(M_i) < f_{i\infty} \leq \infty$, $i = 1, 2$.

(H11) $0 \le f_i^{\infty} < \phi_{p_i}(N_i)$, $\phi_{p_i}(M_i) < f_{i0} \le \infty$, $i = 1, 2$.

Then (1.1)-(1.2) *has one symmetric positive solution in* K.

4. Example

Let $p_i = 3$, $q_i(t) = 2$, $i = 1, 2$, in (1.1) and $\xi = 1/3$, $\eta = 2/3$, and g_i satisfies (H3) with $m_i = 1$, $i = 1, 2$, in (1.2). We consider the boundary-value problem

$$(|u_i'|u_i')'(t) + q_i(t)f_i(t, u_1, u_2) = 0, \quad t \in (0, 1), \quad i = 1, 2; \tag{4.1}$$

$$u_i(0) - g_i(u_i'(\frac{1}{3})) = 0, \quad u_i(1) + g_i(u_i'(\frac{2}{3})) = 0, \quad i = 1, 2, \tag{4.2}$$

where

$$f_1(t, u_1, u_2) = \begin{cases} t(1-t)(u_1 + u_2)^{14} + \frac{1}{1000}, & 0 \le u + v \le 3, \\ t(1-t) \cdot 3^{14} + \frac{1}{1000}, & u + v > 3, \end{cases}$$

$$f_2(t, u_1, u_2) = \begin{cases} \sqrt{t(1-t)}(u_1 + u_2)^{13} + \frac{1}{1000}, & 0 \le u + v \le 3, \\ \sqrt{t(1-t)} \cdot 3^{13} + \frac{1}{1000}, & u + v > 3, \end{cases}$$

Choose $\rho_1 = 1$, $\rho_2 = 64(\sqrt{6} + \sqrt{2})$, $\rho_3 = 1500$, $\delta = \frac{1}{4}$. we note that

$$M_i = 24, \quad N_i = \frac{3(\sqrt{6} - \sqrt{2})}{4}, \quad \gamma = \frac{\sqrt{6} - \sqrt{2}}{128}.$$

Consequently, $f_i(t, u_1, u_2)$, $i = 1, 2$, satisfies

$$f_1^{[0,\rho_1]} = 0.25 < \phi_{p_1}(N_1) = 0.60, \quad f_2^{[0,\rho_1]} = 0.5 < \phi_{p_1}(N_1) = 0.60,$$
$$f_{1[\gamma\rho_2,\rho_2]} = 0.05 > \phi_{p_1}(\gamma M_1) = 0.04, \quad f_{2[\gamma\rho_2,\rho_2]} = 0.06 > \phi_{p_1}(\gamma M_1) = 0.04,$$
$$f_1^{[0,\rho_3]} = 0.53 < \phi_{p_1}(N_1) = 0.60, \quad f_2^{[0,\rho_3]} = 0.35 < \phi_{p_1}(N_1) = 0.60.$$

Then all the conditions for Theorem 3.3 hold. Thus, (1.1)-(1.2) has three symmetric positive solutions in K.

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References

[1] R. P. Agarwal, D. ORegan; *A coupled system of boundary-value problems*, Appl. Anal. 69 (1998) 381-385.

[2] R. I. Avery, A. C. Henderson; *Three symmetric positive solutions for a second-order boundary-value problem*, Appl. Math. Lett. 13 (2000) 1-7.

[3] Z. Bai; *Positive solutions of some nonlocal fourth-order boundary-value problem*, Appl. Math. Comput. 215 (2010) 4191-4197.

[4] H. Feng, W. Ge; *Triple symmetric positive solutions for multipoint boundary-value problem with one-dimensional p-Laplacian*, Math. Comput. Model. 47 (2008) 186-195.

[5] H. Feng, H. Pang, W. Ge; *Multiplicity of symmetric positive solutions for a multipoint boundary-value problem with a one-dimensional p-Laplacian*, Nonlinear Anal. 69 (2008) 3050-3059.

[6] W. Feng, J. R. L. Webb; *Solvability of a m-point boundary-value problem with nonlinear growth*, J. Math. Anal. Appl. 212 (1997) 467-480.

[7] A. M. Fink, J. A. Gatica; *Positive solutions of second order systems of boundary-value problems*, J. Math. Anal. Appl. 180 (1993) 93-108.

[8] J. R. Graef, L. Kong, Q. Kong; *Symmetric positive solutions of nonlinear boundary-value problems*, J. Math. Anal. Appl. 326 (2007) 1310-1327.

[9] J. R. Graef, L. Kong; *Necessary and sufficient conditions for the existence of symmetric positive solutions of singular boundary-value problems*, J. Math. Anal. Appl. 331 (2007) 1467-1484.

[10] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[11] C. P. Gupta; *Solvability of a three-point nonlinear boundary-value problem under for a second order ordinary differential equation*, J. Math. Anal. Appl. 168 (1992) 540-551.

[12] C. P. Gupta; *A generalized multi-point boundary-value problem for second order ordinary differential equation*, Appl. Math. Comput. 89 (1998) 133-146.

[13] V. A. Il'in, E. I. Moiseev; *nonlocal boundary-value problem of the second kind for a Sturm-Liouville operator*, Diff. Eqs. 23 (1987) 979-987.

[14] D. Ji, H. Feng, W. Ge; *The existence of symmetric positive solutions for some nonlinear equation systems*, Appl. Math. Comput. 197 (2008) 51-59.

[15] N. Kosmatov; *Symmetric solutions of a muti-point boundary-value problem*, J. Math. Anal. Appl. 309 (2005) 25-36.

[16] N. Kosmatov; *A symmetric solution of a multipoint boundary-value problem at resonance*, Abstr. Appl. Anal. 2006 (2006), Article ID 54121, 1-11.

[17] K. Lan, Multiple positive solutions of semilinear differential equations with singularities, J. London Math. Soc. 63 (2001) 690-704.

[18] B. Liu; *The existence of positive solutions of singular boundary value systems with p-Laplacian*, Acta Math. Sin. 1 (2005) 35-50.

[19] B. Liu, J. Zhang; *The existence of positive solutions for some nonlinear equation systems*, J. Math. Anal. Appl. 324 (2006) 970-981.

[20] L. Liu, L. Hu, Y. Wu; *Positive solutions of two-point boundary-value problems for systems of nonlinear second-order singular and impulsive differential*, Nonlinear Anal. 69 (2008) 3774-3789.

[21] R. Ma; *Existence theorems for a second order three-point boundary-value problem*, J. Math. Anal. Appl. 212 (1997) 430-442.

[22] R. Ma; *Multiple nonnegative solutions of second-order systems of boundary-value problems*, Nonlinear Anal. 42 (2000) 1003-1010.

[23] Y. Sun; *Existence and multiplicity of symmetric positive solutions for three-point boundary-value problem*, J. Math. Anal. Appl. 329 (2007) 998-1009.

[24] Y. Sun; *Optimal existence criteria for symmetric positive solutions to a three-point boundary-value problem*, Nonlinear Anal. 66 (2007) 1051-1063.

[25] P. Wang; *Iterative methods for the boundary-value problem of a fourth-order differential-difference equation*, Appl. Math. Comput., 73 (1995) 257-270.

[26] Z. Wei; *Positive solution of singular Dirichlet boundary-value problems for second order differential equation system*, J. Math. Anal. Appl. 328 (2007) 1255-1267.

[27] F. Wong; *Existence of positive solutions for m-Laplacian boundary-value problems*, Appl. Math. Lett. 12 (1999) 11-17.

[28] J. Yang, Z. Wei; *On existence of positive solutions of Sturm-Liouville boundary-value problems for a nonlinear singular differential system*, Appl. Math. Comput. 217 (2011) 6097-6104.

[29] Q. Yao; *Existence and Iteration of n symmetric positive solutions for a singular two-point boundary-value problem*, Comput. Math. Appl. 47 (2004) 1195-1200.

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