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SOLVABILITY OF A SECOND-ORDER SINGULAR BOUNDARY-VALUE PROBLEM

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Dedicated to Professor Stepan Tersian on his 60th birthday

Abstract. Using the barrier strips technique, we study the existence of solutions to the boundary-value problem

$$x'' = f(t, x, x'), \quad t \in (0, 1),$$
$$x'(0) = A, \quad x(1) = Bx'(1) + C,$$

where the scalar function f may be singular at $t = 0$.

1. Introduction

In this article we give sufficient conditions that guarantee the solvability of the boundary-value problem (BVP)

$$\begin{gathered} x'' = f(t, x, x'), \quad t \in (0, 1), \\ x'(0) = A, \quad x(1) = Bx'(1) + C, \end{gathered} \tag{1.1}$$

where $f(t, x, p)$ is a scalar function defined for $(t, x, p) \in (0, 1] \times D_x \times D_p$, with $D_x, D_p \subseteq \mathbb{R}$, and f may be unbounded at $t = 0$.

Our work is motivated by Ferguson and Finlagson [4], Klokov [8] and Vasil'ev and Klokov [11]. The work [4] is devoted to the solvability of the BVP

$$\begin{gathered} x'' = -\frac{k}{t}x' + g(t, x), \quad t \in (0, 1),\ k = 0, 1, 2, \\ x'(0) = 0, \quad x(1) = Bx'(1) + C, \end{gathered} \tag{1.2}$$

which arises as a model for processes in chemical reactors. The more general problem

$$x'' = -\frac{k}{t}x' + g(t, x, x'), \quad t \in (0, 1),$$
$$x'(0) = 0, \quad x'(1) = C - Bx(1),$$

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where $k \geq 0$ and $g \in C([0,1] \times R^2)$ is considered in [8]. In [11], existence and uniqueness results are given for the problem

$$\begin{gathered} x'' = -a(t)x' + g(t, x, x'), \quad t \in (0,1), \\ x'(0) = 0, \quad x(1) = Bx'(1) + C, \end{gathered} \tag{1.3}$$

where $g \in C([0,1] \times R^2)$, and $a \in C((0,1], [0, +\infty))$ is such that $\lim_{t \to 0^+} a(t) = +\infty$. The results obtained in [8] and [11] rely on the assumption that the considered problem has lower and upper solutions.

The literature devoted to the solvability of BVPs for second order ordinary differential equations with various singularities is too vast. We quote here only Agarwal and O'Regan [1], Cabada and Pouso [2], De Coster and Habets [3], Heikkilä and Lakshmikantham [6], Kiguradze [7], O'Regan [9], Rachůnková et al [10], Vasil'ev and Klokov [11] for results, methods and references.

In this paper, we establish firstly an existence result for nonsingular problem (1.1). It is proved by a combination of the barrier strips technique with a global existence theorem which is due to Granas et al [5]. Next, we apply the obtained existence result to construct a sequence $\{x_n\}$ of $C^2[n^{-1}, 1]$ solutions to the nonsingular problems

$$\begin{gathered} x'' = f(t, x, x'), \quad t \in (0,1), \\ x'(n^{-1}) = A, \quad x(1) = Bx'(1) + C, \quad n \in N \setminus \{1\}. \end{gathered}$$

Finally, using the Arzela-Ascoli theorem, we obtain a $C^1[0,1] \cap C^2(0,1]$ solution to singular problem (1.1) as the limit of an uniformly convergent subsequence of $\{x_n\}$.

2. Existence of a global solution

Let $\Lambda x = x'' + p(t)x' + q(t)x$, where the functions p and q are continuous on the interval $[a, b]$, and

$$V_1(x) = a_1 x(a) + b_1 x'(a), \quad V_2(x) = a_2 x(b) + b_2 x'(b),$$

where the constants are such that $a_i^2 + b_i^2 > 0$ for $i = 1, 2$. Let B_0 be the set of functions satisfying the homogeneous boundary conditions $V_i(x) = 0$, $i = 1, 2$, and $C^2_{B_0}[a, b] = C^2[a, b] \cap B_0$.

In this setting, we consider the boundary-value problem

$$\begin{gathered} \Lambda x = f(t, x, x'), \quad t \in (a, b), \\ V_1(x) = r_1, \quad V_2(x) = r_2, \end{gathered} \tag{2.1}$$

where $f : [a, b] \times D_x \times D_p \to R$, with $D_x, D_p \subseteq \mathbb{R}$, and $r_i \in \mathbb{R}$, $i = 1, 2$.

The proof of our theorem guaranteeing the existence of nonsingular problems is based on the following global existence theorem which is a slight modification of a well-known result.

Theorem 2.1 ([5, Theorem 5.1]). *Assume that:*

(i) *The map* $\Lambda : C^2_{B_0}[a, b] \to C[a, b]$ *is one-to-one.*
(ii) *Each solution* $x \in C^2[a, b]$ *to the family of problems*

$$\begin{gathered} \Lambda x = \lambda f(t, x, x'), \quad t \in (a, b), \\ V_1(x) = r_1, \ V_2(x) = r_2, \end{gathered}$$

with $\lambda \in [0,1]$*, satisfies the bounds*

$$m_i \le x^{(i)}(t) \le M_i, \quad i = 0,1,2, \text{ for } t \in [a,b],$$

where the constants $-\infty < m_i, M_i < \infty$*,* $i = 0,1,2$*, are independent of* λ *and* x*.*

(iii) *There is a sufficiently small* $\tau > 0$ *such that*

$$[m_0 - \tau, M_0 + \tau] \subseteq D_x, \quad [m_1 - \tau, M_1 + \tau] \subseteq D_p$$

and $f(t,x,p)$ *is continuous for* $(t,x,p) \in [a,b] \times [m_0 - \tau, M_0 + \tau] \times [m_1 - \tau, M_1 + \tau]$*.*

Then (2.1) *has at least one solution in* $C^2[a,b]$*.*

3. Nonsingular problem

Consider the nonsingular problem

$$\begin{gathered} x'' = f(t,x,x'), \quad t \in (a,b), \\ x'(a) = A, \; x(b) = Bx'(b) + C, \end{gathered} \tag{3.1}$$

where $f : [a,b] \times D_x \times D_p \to \mathbb{R}$, $D_x, D_p \subseteq R$.

In the next lemma, we use the assumption

(R1) There are constants $L_i, F_i, i = 1,2$, and a sufficiently small $\tau > 0$ such that

$$\begin{gathered} L_2 - \tau \ge L_1 \ge A \ge F_1 \ge F_2 + \tau, \\ [m_0 - \tau, M_0 + \tau] \subseteq D_x, \quad [F_2, L_2] \subseteq D_p, \end{gathered}$$

where

$$\begin{gathered} m_0 = -\max\{|L_1|, |F_1|\}(|B| + b - a) + C, \\ M_0 = \max\{|L_1|, |F_1|\}(|B| + b - a) + C, \\ f(t,x,p) \in C\big([a,b] \times [m_0 - \tau, M_0 + \tau] \times [F_1 - \tau, L_1 + \tau]\big), \\ f(t,x,p) \le 0 \quad \text{for } (t,x,p) \in [a,b] \times D_x \times [L_1, L_2], \\ f(t,x,p) \ge 0 \quad \text{for } (t,x,p) \in [a,b] \times D_x \times [F_2, F_1]. \end{gathered}$$

Our first result shows that the strips $[a,b] \times [L_1, L_2]$ and $[a,b] \times [F_2, F_1]$ are barriers to the values of all $C^2[a,b]$ solutions to the family

$$\begin{gathered} x'' = \lambda f(t,x,x'), \quad t \in (a,b), \\ x'(a) = A, \; x(b) = Bx'(b) + C, \end{gathered} \tag{3.2}$$

where $\lambda \in [0,1]$.

Lemma 3.1. *Let* (R1) *hold and let* $x \in C^2[a,b]$ *be a solution to family* (3.2) *with* $\lambda \in [0,1]$*. Then*

$$m_0 \le x(t) \le M_0, \quad F_1 \le x'(t) \le L_1, \quad m_2 \le x''(t) \le M_2$$

for $t \in [a,b]$*, where* $m_2 = \min f(t,x,p)$ *and* $M_2 = \max f(t,x,p)$ *on* $[a,b] \times [m_0, M_0] \times [F_1, L_1]$*.*

Proof. Suppose that the set

$$S_- = \{t \in [a,b] : L_1 < x'(t) \le L_2\}$$

is not empty. Then $x'(a) = A \le L_1$ and $x' \in C[a,b]$ imply that there exists a $\gamma \in S_-$ such that $x''(\gamma) > 0$. Since $x(t)$ is a solution of the differential equation,

$(t, x(t), x'(t)) \in (a, b) \times D_x \times D_p$. In particular, we have $(\gamma, x(\gamma), x'(\gamma)) \in S_- \times D_x \times (L_1, L_2]$. So, we can use (R1) to obtain

$$x''(\gamma) = \lambda f(\gamma, x(\gamma), x'(\gamma)) \le 0,$$

a contradiction. Thus, S_- is empty which means

$$x'(t) \le L_1 \quad textfor t \in [a, b].$$

Further, assuming that the set

$$S_+ = \{t \in [a, b] : F_2 \le x'(t) < F_1\}$$

is not empty and arguing as in the above part of the proof, we obtain

$$F_1 \le x'(t) \quad \text{for } t \in [a, b].$$

Now, by the mean value theorem, for each $t \in [a, b)$ there exists $\xi \in (t, b)$ such that $x(b) - x(t) = x'(\xi)(b - t)$, from where, using the proved bounds for $x'(t)$, we obtain

$$m_0 \le x(t) \le M_0 \quad \text{for } t \in [a, b].$$

Finally, the bounds for $x''(t)$ are an elementary consequence of the continuity of $f(t, x, p)$ on the compact set $[a, b] \times [m_0, M_0] \times [F_1, L_1]$. □

We are ready to formulate an existence result.

Theorem 3.2. *Let* (R1) *hold. Then nonsingular problem* (3.1) *has at least one solution in* $C^2[a, b]$.

Proof. A combination of the a priori bounds of Lemma 3.1 with Theorem 2.1 gives the assertion at once. Notice only that (i) of Theorem 2.1 follows from the fact that for each $y \in C[a, b]$ the homogeneous BVP

$$x'' = y, \quad x'(a) = 0, \quad x(b) - Bx'(b) = 0$$

has a unique solution. □

4. Singular problem

Now, we turn our attention to problem (1.1) by considering the case

$$\begin{gathered} f(t, x, p) \text{ is defined for } (t, x, p) \in ([0, 1] \times D_x \times D_p) \setminus S, \text{ where} \\ D_x, D_p \subseteq \mathbb{R} \text{ and } S = \{0\} \times X \times P \text{ for some sets } X \subseteq D_x \text{ and } P \subseteq D_p \end{gathered} \tag{4.1}$$

which allows $f(t, x, p)$ to be unbounded at $t = 0$ if $(x, p) \in X \times P$.

Now, assume that

(S1) There are constants L_i, F_i, $i = 1, 2$, and a sufficiently small $\tau > 0$ such that

$$L_2 - \tau \ge L_1 \ge A \ge F_1 \ge F_2 + \tau,$$
$$[m_0 - \tau, M_0 + \tau] \subseteq D_x, \quad [F_2, L_2] \subseteq D_p,$$

where

$$m_0 = -K(|B| + 1) + C, \; M_0 = K(|B| + 1) + C,$$
$$K = \max\{|L_1|, |F_1|\},$$
$$f(t, x, p) \in C\big((0, 1] \times [m_0 - \tau, M_0 + \tau] \times [F_1 - \tau, L_1 + \tau]\big),$$
$$f(t, x, p) \le 0 \quad \text{for } (t, x, p) \in (0, 1] \times D_x \times [L_1, L_2],$$
$$f(t, x, p) \ge 0 \quad \text{for } (t, x, p) \in (0, 1] \times D_x \times [F_2, F_1].$$

(S2) The functions $f_t(t,x,p)$, $f_x(t,x,p)$ and $f_p(t,x,p)$ are continuous for (t,x,p) in $(0,1] \times [m_0, M_0] \times [F_1, L_1]$, where m_0, M_0, F_1 and L_1 are as in (S1).

We are now in position to state the main existence theorem of this paper.

Theorem 4.1. *Assume* (4.1) *is satisfied. Assume also* (S1) *and* (S2) *hold. Then singular BVP* (1.1) *has at least one* $C^1[0,1] \cap C^2(0,1]$ *solution.*

Proof. It is easy to see that for each fixed $n \in N \setminus \{1\}$, (R1) holds for the corresponding nonsingular BVP

$$\begin{gathered} x'' = f(t,x,x'), \\ x'(n^{-1}) = A, \quad x(1) = Bx'(1) + C. \end{gathered} \tag{4.2}$$

Consequently, for each $n \in N \setminus \{1\}$ we can apply Theorem 3.2 to construct a sequence $\{x_n\}$ of $C^2[n^{-1},1]$ solutions to family (4.2).

Further, we introduce a numerical sequence $\{\theta_i\}$, $i \in N$, such that $\theta_i \in (0,1)$, $\theta_{i+1} < \theta_i$ for $i \in N$ and $\lim_{i\to\infty} \theta_i = 0$.

In view of Lemma 3.1, for each $n \in N_1$ we have the bounds

$$m_0 \le x_n(t) \le M_0 \quad \text{for } t \in [\theta_1, 1], \tag{4.3}$$

$$F_1 \le x_n'(t) \le L_1 \quad \text{for } t \in [\theta_1, 1], \tag{4.4}$$

independent of n. Also, the continuity of $f(t,x,p)$ on $[\theta_1,1] \times [m_0,M_0] \times [F_1,L_1]$ yields the bound

$$m_{\theta_1} \le x_n''(t) \le M_{\theta_1} \quad \text{for } t \in [\theta_1, 1], \tag{4.5}$$

independent of n. Thus, we can use (S2) to conclude that $x_n'''(t)$ exists, $x_n''' \in C[\theta_1,1]$ and

$$x_n'''(t) = f_t(t,x_n(t),x_n'(t)) + f_x(t,x_n(t),x_n'(t))x_n'(t) + f_p(t,x_n(t),x_n'(t))x_n''(t),$$

from where it follows that there is a constant $\overline{M}_{\theta_1}$, independent of n, such that

$$|x_n'''(t)| \le \overline{M}_{\theta_1} \quad \text{for } t \in [\theta_1, 1] \text{ and all } n \in N_1. \tag{4.6}$$

Bounds (4.3)-(4.6) allow us to apply the Arzela-Ascoli theorem on the sequence $\{x_n\}$ to conclude that there are a subsequence $\{x_{1,n_k}\}$, $k \in N$, $n_k \in N_1$, and a function $x_{\theta_1} \in C^2[\theta_1,1]$ such that $\{x_{1,n_k}\}$, $\{x'_{1,n_k}\}$ and $\{x''_{1,n_k}\}$ converge uniformly on $[\theta_1,1]$ to x_{θ_1}, x'_{θ_1} and x''_{θ_1}, respectively.

As a solution of (4.2), each function x_{1,n_k}, $n_k \in N_1$, is such that

$$\begin{gathered} x''_{1,n_k}(t) = f(t, x_{1,n_k}(t), x'_{1,n_k}(t)) \quad \text{for } t \in [\theta_1, 1), \\ x_{1,n_k}(1) = Bx'_{1,n_k}(1) + C, \end{gathered}$$

from where, keeping in mind (4.3), (4.4) and the continuity of f on the compact set $[\theta_1,1] \times [m_0,M_0] \times [F_1,L_1]$, we obtain

$$\begin{gathered} x''_{\theta_1}(t) = f(t, x_{\theta_1}(t), x'_{\theta_1}(t)) \quad \text{for } t \in [\theta_1, 1), \\ x_{\theta_1}(1) = Bx'_{\theta_1}(1) + C. \end{gathered}$$

Now consider the sequence $\{x_{1,n_k}\}$ on the interval $[\theta_2,1]$. Arguing as above, we extract a subsequence $\{x_{2,n_k}\}$, $n_k \in N_2 = \{n \in N : n^{-1} < \theta_2\}$, of $\{x_{1,n_k}\}$ converging uniformly on the new interval $[\theta_2,1]$ to a new function $x_{\theta_2}(t)$, which is a $C^2[\theta_2,1]$ solution to the differential equation $x'' = f(t,x,x')$ on the interval $[\theta_2,1)$ with the property $x_{\theta_2}(1) = Bx'_{\theta_2}(1) + C$ and

$$x_{\theta_2}(t) = x_{\theta_1}(t) \quad \text{for } t \in [\theta_1, 1].$$

Continuing this process, we establish that for each $i \in N$ there is a function $x_{\theta_i}(t)$ which is a $C^2[\theta_i, 1]$ solution of $x'' = f(t, x, x')$ on the interval $[\theta_i, 1)$, $x_{\theta_i}(1) = Bx'_{\theta_i}(1) + C$ and

$$x_{\theta_{i+1}}(t) = x_{\theta_i}(t) \quad \text{for } t \in [\theta_i, 1].$$

Moreover, for each $i \in N$ there is a subsequence $\{x_{i,n_k}\}$, $n_k \in N_i = \{n \in N : n^{-1} < \theta_i\}$, such that

$$\|x_{i,n_k} - x_{\theta_i}\|_2 \to 0 \quad \text{on the interval } [\theta_i, 1], \tag{4.7}$$

where

$$\|x\|_2 = \max\big\{ \max_{t\in[\theta_i,1]} |x(t)|, \max_{t\in[\theta_i,1]} |x'(t)|, \max_{t\in[\theta_i,1]} |x''(t)|\big\}$$

is the norm in the Banach space $C^2[\theta_i, 1]$.

The existence of the sequence $\{x_{\theta_i}\}$ allows us to conclude that there is a function $x_0(t)$, which is a $C^2(0, 1]$ solution of $x'' = f(t, x, x')$ on the interval $(0, 1)$, $x_0(1) = Bx'_0(1) + C$,

$$x_0(t) = x_{\theta_i}(t) \quad \text{for } t \in [\theta_i, 1]. \tag{4.8}$$

In what follows we will show in addition that

$$\lim_{t\to 0^+} x'_0(t) = A. \tag{4.9}$$

Reasoning by contradiction, assume that there are sufficiently small $\varepsilon > 0$ and $\delta_0 > 0$ such that

$$x'_0(t) \notin (A - \varepsilon, A + \varepsilon) \quad \text{for } t \in (0, \delta_0). \tag{4.10}$$

Now, from $x'_n \in C[n^{-1}, 1]$ and $x'_n(n^{-1}) = A$ it follows that for each sufficiently large n and the chosen ε there exists a sufficiently small $\delta_n > 0$, depending on n and ε, such that $(n^{-1}, \delta_n) \subset (0, \delta_0)$ and

$$x'_n(t) \in (A - \varepsilon/2, A + \varepsilon/2) \quad \text{for } t \in (n^{-1}, \delta_n).$$

Besides, for each sufficiently large n there exists any $i \in N$ such that $\theta_i > n^{-1}$ and

$$[\theta_i, \theta_{i-1}] \subset (n^{-1}, \delta_n) \subset (0, \delta_0);$$

the assumption that the interval $[\theta_i, \theta_{i-1}]$ does not exist contradicts to the fact that $t = 0$ is an accumulation point of the sequence $\{\theta_i\}$. In summary, for each sufficiently large n there exists $i \in N$ such that

$$x'_n(t) \in (A - \varepsilon/2, A + \varepsilon/2) \quad \text{for } t \in [\theta_i, \theta_{i-1}] \subset (0, \delta_0). \tag{4.11}$$

But, for each sufficiently large n and its i from (4.7) ($\theta_i > n^{-1}$ means that $n \equiv n_k \in N_i$ for any $k \in N$), (4.8) and (4.10) we obtain

$$x'_n(t) \notin (A - \varepsilon, A + \varepsilon) \quad \text{for } t \in [\theta_i, \theta_{i-1}],$$

which contradicts (4.11). Thus, (4.9) is true.

Now, we introduce a function $x(t)$ such that

$$x(t) = x_0(t) \quad \text{for } t \in (0, 1]$$

and

$$x'(t) = \begin{cases} x'_0(t) & \text{for } t \in (0, 1] \\ A & \text{for } t = 0\,. \end{cases}$$

Because of (4.9), $x'(t)$ is continuous on $[0, 1]$, which means $x(t)$ is also continuous on $[0, 1]$, from where it follows that $x(t)$ is a $C^1[0, 1] \cap C^2(0, 1]$ solution to singular BVP (1.1). □

As an application of the above theorem, we will establish existence results for singular problems (1.2) and (1.3). The first result concerns (1.2).

Corollary 4.2. *Assume that:*

(i) *$g(t,x)$ is bounded; i.e., there is a constant $M > 0$ such that*
$$|g(t,x)| \le M \quad \text{for } (t,x) \in [0,1] \times D_x.$$
(ii) *There is a $\tau > 0$ such that $[m_0 - \tau, M_0 + \tau] \subseteq D_x$ and*
$$f(t,x) \in C([0,1] \times [m_0 - \tau, M_0 + \tau]),$$
where $m_0 = -M(|B|+1)/k + C$ and $M_0 = M(|B|+1)/k + C$.

(iii) *g_t, g_x are continuous for $(t,x) \in (0,1] \times [-M/k - \tau, M/k + \tau]$.*

Then (1.2) with $k = 1,2$ has at least one solution in $C^1[0,1] \cap C^2(0,1]$.

Proof. It is easy to check that (S1) and (S2) hold for $F_1 = -M/k$, $F_2 = -M/k - 1$, $L_1 = M/k$ and $L_2 = M/k + 1$ and $\tau = 0.5$. So we can apply Theorem 4.1 to conclude that the assertion is true. □

The following two results concern problem (1.3).

Corollary 4.3. *Assume $g(t,x,p)$ satisfies (S1) and (S2). Then (1.3) has at least one solution in $C^1[0,1] \cap C^2(0,1]$.*

Proof. Since $a(t) \ge 0$ for $t \in (0,1]$, the function $f(t,x,p) = -a(t)p + g(t,x,p)$ satisfies (S1) and (S2) for the same constants $L_i, F_i, i = 1,2$, for which g satisfies them. Thus, the assertion follows from Theorem 4.1. □

Corollary 4.4. *Assume that*

(i) *$g(t,x,p)$ is bounded; i.e., there is a constant $M > 0$ such that*
$$|g(t,x,p)| \le M \quad \text{for } (t,x,p) \in [0,1] \times D_x \times D_p$$
and $a(t) \ge h > 0$ for $t \in (0,1]$.

(ii) *There is a $\tau > 0$ such that*
$$[m_0 - \tau, M_0 + \tau] \subseteq D_x, \quad [-M/h - \tau, M/h + \tau] \subseteq D_p,$$
$$g(t,x,p) \in C([0,1] \times [m_0 - \tau, M_0 + \tau] \times [-M/h - \tau, M/h + \tau]),$$
where $m_0 = -M(|B|+1)/h + C$ and $M_0 = M(|B|+1)/h + C$.

(iii) *$g_t(t,x,p)$, $g_x(t,x,p)$ and $g_p(t,x,p)$ are continuous for $(t,x,p) \in [0,1] \times [m_0 - \tau, M_0 + \tau] \times [-M/h - \tau, M/h + \tau]$.*

Then (1.3) has at least one solution in $C^1[0,1] \cap C^2(0,1]$.

Proof. We can choose, for example, $F_1 = -M/h$, $F_2 = -M/h - \tau$, $L_1 = M/h$ and $L_2 = M/h + \tau$ to see that (S1) and (S2) hold and so the assertion is true by Theorem 4.1. □

5. Examples

As a first example, we have the boundary-value problem
$$x'' = x' ln(t) + \frac{(x'+3)(x'-4)}{\sqrt{100 - x^2}}, \quad t \in (0,1),$$
$$x'(0) = 2, \quad x(1) = x'(1),$$

which is solvable in $C^1[0,1] \cap C^2(0,1]$, by Theorem 4.1, since (S1) and (S2) hold for $F_2 = -4$, $F_1 = -3$, $L_1 = 3$, $L_2 = 4$ and $\tau = 0.1$.

As a second example, we have the boundary-value problem

$$x'' = -2t^{-1}x' + \sqrt{25 - x^2} + 1, \quad t \in (0,1),$$
$$x'(0) = 0, \quad x(1) = 0.5x'(1) - 1,$$

which has a $C^1[0,1] \cap C^2(0,1]$ solution by Corollary 4.2.

As a third example, we have boundary-value problem

$$x'' = -15t^{-1}x' + \sqrt{25 - x^2}\sqrt{9 - x'^2} + \cos 5t, \quad t \in (0,1),$$
$$x'(0) = 0,\ x(1) = -2x'(1) + 1,$$

which has a $C^1[0,1] \cap C^2(0,1]$ solution by Corollary 4.4.

References

[1] R. P. Agarwal, D. O'Regan; *Singular Differential and Integral Equations with Applications*, Kluwer Academic, 2003.

[2] A. Cabada, R. L. Pouso; *Extremal solutions of strongly nonlinear discontinuous second-order equations with nonlinear functional boundary conditions*, Nonl. Anal. 42 (2000), 1377-1396.

[3] C. De Coster, P. Habets; *Two Points Boundary Value Problems: Lower and Upper Solutions*, Elsevier, Amsterdam et al., 2006.

[4] B. Ferguson, A. Finlagson; *Error bounds for approximate solutions to nonlinear ordinary differential equations*, AIChE J. 8 (1972), 1053-1059.

[5] A. Granas, R. B. Guenther, J. W. Lee; *Nonlinear Boundary Value Problems for Ordinary Differential Equations*, Dissnes. Math. 244 (1985), 1-128.

[6] S. Heikkilä, V. Lakshmikantham; *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, New York, 1994.

[7] I. Kiguradze; *Some Singular Boundary Value Problems for Ordinary Differential Equations*, Tbilisi, 1975.(Russian)

[8] Yu. Klokov; *New solvability conditions for the Neumann problem for ordinary singular differential equations*, Differential Equations 36 (2000), 1316-1418.

[9] D. O'Regan; *Existence Theory for Nonlinear Ordinari Differential equations*, Kluwer Acaemic, Dordrecht, 1997.

[10] I. Rachunková, S. Staněk, M. Tvrdý; *Solvability of Nonlinear Singular Problems for Ordinary Differential Equations*, Hindawi Publishing Corporation, New York, 2009.

[11] N. Vasil'ev, Yu. Klokov; *Foundations of the Theory of Boundary Value Problems for Ordinary Differential Equations*, Zinatne, Riga, 1974. (Russian)

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