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EXISTENCE OF INFINITELY MANY HOMOCLINIC ORBITS FOR SECOND-ORDER SYSTEMS INVOLVING HAMILTONIAN-TYPE EQUATIONS

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Abstract. We study the second-order differential system

$$\ddot{u} + A\dot{u} - L(t)u + \nabla V(t,u) = 0,$$

where A is an antisymmetric constant matrix and $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$. We establish the existence of infinitely many homoclinic solutions if W is of subquadratic growth as $|x| \to +\infty$ and L does not satisfy the global positive definiteness assumption. In the particular case where $A = 0$, earlier results in the literature are generalized.

1. Introduction

Let us consider the second-order differential system

$$\ddot{u} + A\dot{u} - L(t)u + \nabla V(t,u) = 0, \tag{1.1}$$

where A is an antisymmetric constant matrix with small size in $\mathbb{R}^{2N}$ (see the estimation (2.2)), $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function and $V \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is of class C^1 in the second variable. We will say that a solution u of (1.1) is *homoclinic* (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

For the particular case $A = 0$, (1.1) is just the Hamiltonian system

$$\ddot{u} - L(t)u + \nabla V(t,u) = 0. \tag{1.2}$$

In recent years, existence and multiplicity of homoclinic solutions for the second order Hamiltonian system (1.2) have been investigated by many authors via the critical point theory, see [1]–[13], [15]–[21] and references therein. Most of them treat the superquadratic case under the so-called global Ambrosetti-Rabinowitz condition; that is, there exists $\mu > 2$ such that

$$0 < \mu V(t,x) \le (\nabla V(t,x), x), \quad \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}^N \backslash \{0\}.$$

Exceptionally, in [5], the author considered, in part of the paper, the case where the potential is of subquadratic growth as $|x| \to +\infty$. Moreover, contrary to the previous works, he removed the global positive definiteness of the matrix $L(t)$ by assuming

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(L1) for the smallest eigenvalue of $L(t)$, i.e., $l(t) = \inf_{|x|=1}(L(t)x, x)$, there exists a constant $\alpha < 1$ such that

$$l(t)|t|^{\alpha-2} \to \infty \quad \text{as } |t| \to \infty,$$

(L2) for some positive constants a, r, one of the following is true:

(i) $L \in C^1(\mathbb{R}, \mathbb{R}^{N^2})$ and $|L'(t)x| \le a|L(t)x|$ for all $|t| > r$ and all $x \in \mathbb{R}^N$ with $|x| = 1$, or

(ii) $L \in C^2(\mathbb{R}, \mathbb{R}^{N^2})$ and $(aL(t)x - L''(t)x, x) \ge 0$ for all $|t| > r$ and all $x \in \mathbb{R}^N$ with $|x| = 1$,

where $L'(t) = (d/dt)L(t), L''(t) = (d^2/dt^2)L(t)$.

Under other suitable conditions he established the existence and multiplicity of homoclinic solutions for (1.2). Later, his results were partially improved in [17, 18].

Recently, the authors in [19, 20], treated the special case where $V(t, x) = a(t)|x|^\mu$ with $1 < \mu < 2$ and $L(t)$ is a positive definite matrix for all $t \in \mathbb{R}$. They proved the existence of a nontrivial homoclinic solution for (1.2) and (1.1) respectively; where the system (1.1) was considered for the first time. Later, multiplicity of homoclinics for (1.2) was studied in [15] for the same class of Hamiltonians. However, in mathematical physics, it is of frequent occurrence in (1.2) that the global definiteness of $L(t)$ is not satisfied (see [5] for an example).

As far as the authors know, there is no research concerning the existence and multiplicity of homoclinic solutions for (1.1) apart from [20]. In this paper, motivated by [5, 20] mainly, we study the existence of infinitely many homoclinic solutions for (1.1) in the case where L does not satisfy the global positive definiteness assumption. Also, the potential V will be of subquadratic growth as $|x| \to +\infty$ and is not necessarily of the form $V(t, x) = a(t)|x|^\mu$. In the first result we assume that $V(t, x) = a(t)W(x)$ with $W \in C^1(\mathbb{R}^N, \mathbb{R})$, $a \in C(\mathbb{R}, \mathbb{R}) \cap L^2(\mathbb{R}, \mathbb{R})$ are nonnegative functions and $a \not\equiv 0$. The difficulty in studying this class of nonlinearities comes essentially from the fact that $\inf_{t\in\mathbb{R}} a(t) = 0$ and then there is no constant $b > 0$ such that $V(t, x) \ge b|x|^\gamma$ for all $t \in \mathbb{R}$, which essential in previous works. Moreover, in the case where $A \ne 0$, we are unable to verify the Palais-Smale condition. To overcome this obstacle, we use a variant fountain theorem established in [22]. For our first theorem use the following assumptions:

(L3) $0 \notin \sigma\big(-(d^2/dt^2) + L(t) - A(d/dt)\big)$,

(V1) $W(0) = 0$ and there exist positive constants a_1, a_2, r and $1 \le \gamma \le \mu < 2$ such that

$$a_1|x|^\gamma \le W(x) \le a_2|x|^\mu, \quad \text{for all } |x| \ge r,$$

(V2) there exist positive constants a_3, ω and $\nu \in [1, 2)$ such that

$$W(x) \ge a_3|x|^\nu, \quad \text{for all } |x| \le \omega,$$

(V3) there exist constants $a_4 > 0$ and $\beta \in [1, 2)$ such that

$$|\nabla W(x)| \le a_4(|x|^{\beta-1} + 1) \quad \text{for all } x \in \mathbb{R}^N,$$

(V4) W is even,

(V5) $a \in L^2(\mathbb{R}, \mathbb{R})$ and $\text{meas}\{t \in \mathbb{R} : a(t) = 0\} = 0$.

Theorem 1.1. *Assume that L satisfies* (L1)–(L3) *and V satisfies* (V1)–(V5). *Then system* (1.1) *has infinitely many homoclinic solutions.*

In the particular for the case $A = 0$, we have the following result.

Corollary 1.2. *Under the assumptions of Theorem 1.1, system* (1.2) *has infinitely many homoclinic solutions.*

Remark 1.3. Consider

$$L(t) = (t^2 - 1)I_N \quad \text{and} \quad V(t, x) = \frac{|\sin t|}{|t| + 1}|x|^{5/4} \log(1 + |x|^{1/2}).$$

A straightforward computation shows that L and V satisfy the conditions of Theorem 1.1 but since $\inf_{t\in\mathbb{R}} V(t, x) = 0$, the assumptions of [5, Theorem 1.2] and [18, Theorem 1.1] do not hold. So, in some sense, Corollary 1.2 completes the corresponding results in [5, 18] and the one in [21] for the case $\beta = 0$. Moreover, Theorem 1.1 generalizes the result of [20].

Our second main result concerns a class of nonlinearities with bounded gradient which cover the functions of the type $V(t, x) = Ln(1 + |x|^{3/2})$ and which not necessarily of the form $V(t, x) = a(t)W(x)$. Homoclinic solutions to (1.2) for this class of Hamiltonians was investigated in [5, Theorem 1.3] under the assumption of positive definiteness of $L(t)$. Here, we omit this condition mainly. Precisely we have the following assumptions:

(V1') $V(t, 0) \equiv 0$ and $V(t, x) \to \infty$ as $|x| \to \infty$ uniformly in $t \in \mathbb{R}$,

(V2') there exist constants $a_1, \omega > 0$ and $\nu \in [1, 2)$ such that
$$V(t, x) \geq a_1|x|^\nu, \quad \text{for all } |x| \leq \omega, t \in \mathbb{R},$$

(V3') there exists a constant $M > 0$ such that
$$|\nabla V(t, x)| \leq M, \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}^N,$$

(V4') there exist constants $a_2, r > 0$ and $\beta \in [1/2, 1)$ such that
$$|\nabla V(t, x)| \leq a_2|x|^\beta, \quad \text{for all } |x| \leq r,\ t \in \mathbb{R},$$

(V5') $V(t, -x) = V(t, x) \geq 0, \quad$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^N$.

Theorem 1.4. *Assume that L satisfies* (L1)–(L3) *and V satisfies* (V1')–(V5'). *Then system* (1.1) *has infinitely many homoclinic solutions.*

Corollary 1.5. *Under the assumptions of Theorem 1.4, system* (1.2) *has infinitely many homoclinic solutions.*

Remark 1.6. Consider the function

$$V(t, x) = \log(1 + |x|^{3/2}).$$

A straightforward computation shows that V satisfies the conditions of Theorem 1.4 but does not satisfy condition (W4) in [18, Theorem 1.1]. Moreover, since $L(t)$ is unnecessarily positive definite, Corollary 1.5 improves the corresponding results in [5, 18].

2. Preliminary results

We establish our results by using critical point theory, but we first give some preliminaries (for details see [5]). We denote by B the selfadjoint extension of the operator $-(d^2/dt^2) + L(t)$ with the domain $\mathcal{D}(B) \subset L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^N)$. Let $|B|$ be the absolute value of B and $|B|^{1/2}$ be the square of $|B|$. Let $E = \mathcal{D}(|B|^{1/2})$, the domain of $|B|^{1/2}$, and define on E the inner product

$$(u, v)_0 = (|B|^{1/2}u, |B|^{1/2}v)_{L^2} + (u, v)_{L^2}$$

and norm

$$\|u\|_0 = (u,u)_0^{1/2},$$

where $(.,.)_{L^2}$ denotes the inner product of L^2. Then E is a Hilbert space.

It is easy to prove that the spectrum $\sigma(B)$ consists of eigenvalues numbered in $\lambda_1 \le \lambda_2 \le \dots \nearrow \infty$ (counted with their multiplicities), and a corresponding system of eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$ of B forms an orthonormal basis in L^2. Define

$$n^- = \#\{i : \lambda_i < 0\}, \quad n^0 = \#\{i : \lambda_i = 0\}, \quad \bar{n} = n^- + n^0 \tag{2.1}$$

and

$$E^- = \operatorname{span}\{e_1, \dots, e_{n^-}\}, \quad E^0 = \operatorname{span}\{e_{n^-+1}, \dots, e_{\bar{n}}\} = \ker B,$$
$$E^+ = \overline{\operatorname{span}\{e_{\bar{n}+1}, \dots\}}.$$

Then one has the orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$ with respect to the inner product $(\cdot,\cdot)_0$. Now we introduce on E the following inner product and norm:

$$(u,v) = (|B|^{1/2}u, |B|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}$$

and

$$\|u\| = (u,u)^{1/2},$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E = E^- \oplus E^0 \oplus E^+$. Clearly the norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent (see [5]). Furthermore, the decomposition $E = E^- \oplus E^0 \oplus E^+$ is orthogonal with respect to the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_{L^2}$. For the rest of this article, $\|\cdot\|$ will be the norm used on E. The following fact on E will be needed.

Lemma 2.1 ([5])**.** *Suppose that $L(t)$ satisfies* (L1)*. Then E is continuously embedded in $W^{1,2}(\mathbb{R},\mathbb{R}^N)$, and consequently there exists $\delta > 0$ such that*

$$\|u\|_{W^{1,2}(\mathbb{R},\mathbb{R}^N)} \le \delta\|u\|, \quad \textit{for all } u \in E,$$

where $\|u\|_{W^{1,2}(\mathbb{R},\mathbb{R}^N)} = (\|u\|_{L^2}^2 + \|\dot{u}\|_{L^2}^2)^{1/2}$.

Now, we make the following estimation on the norm of the matrix A,

$$|A| < \frac{1}{\delta^2}, \tag{2.2}$$

where $|\cdot|$ is the standard norm of $\mathbb{R}^{N^2}$.

Moreover, using (V5), we note that a is bounded and can be seen as a weight function. So, for $p \ge 1$, the weighted norm $\|\cdot\|_{L^p(a)}$ will be defined on E by

$$\|u\|_{L^p(a)} = \Big[\int_{\mathbb{R}} a(t)|u(t)|^p dt\Big]^{1/p}.$$

From [5, Lemmas 2.2 and 2.3], we have the following two lemmas.

Lemma 2.2 ([5])**.** *Suppose that $L(t)$ satisfies* (L1)*. Then E is compactly embedded in L^p for any $1 \le p \le \infty$, which implies that there exists a constant $C_p > 0$ such that*

$$\|u\|_{L^p} \le C_p\|u\|, \quad \textit{for all } u \in E. \tag{2.3}$$

Lemma 2.3 ([5])**.** *Suppose that $L(t)$ satisfies* (L1), (L2)*. Then $\mathcal{D}(B)$ is continuously embedded in $W^{2,2}(\mathbb{R},\mathbb{R}^N)$, and consequently, we have*

$$|u(t)| \to 0 \quad \textit{and} \quad |\dot{u}(t)| \to 0 \quad \textit{as } |t| \to \infty,$$

for all $u \in \mathcal{D}(B)$.

Lemma 2.4. *Suppose assumption* (V5) *holds. If $q_k \rightharpoonup q$ (weakly) in E, then $\nabla V(t, q_k) \to \nabla V(t, q)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$.*

Proof. Assume that $q_k \rightharpoonup q$ in E. By the Banach-Steinhaus Theorem the sequence $(q_k)_{k\in\mathbb{N}}$ is bounded in E and by (2.3), there exists a constant $d_1 > 0$ such that

$$\sup_{k\in\mathbb{N}} \|q_k\|_{L^\infty} \le d_1, \quad \|q\|_{L^\infty} \le d_1. \tag{2.4}$$

Since ∇W is continuous, by (2.4) there exists a constant $d_2 > 0$ such that

$$|\nabla W(q_k(t))| \le d_2, \quad |\nabla W(q(t))| \le d_2,$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|\nabla V(t, q_k(t)) - \nabla V(t, q(t))| \le 2d_2 a(t).$$

On the other hand, by Lemma 2.2, $q_k \to q$ in L^2, passing to a subsequence if necessary, we obtain $q_k \to q$ for almost every $t \in \mathbb{R}$. Then, using (V5), the Lebesgue's Convergence Theorem gives the conclusion. □

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider the C^1-functional $\Phi_\lambda : E \to \mathbb{R}$ defined by

$$\Phi_\lambda(u) := \mathcal{A}(u) - \lambda\mathcal{B}(u), \quad \lambda \in [1, 2].$$

Theorem 2.5 ([22, Theorem 2.2])**.** *Assume that the functional Φ_λ defined above satisfies*

- (T1) *Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Moreover, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$,*
- (T2) *$\mathcal{B}(u) \ge 0$ for all $z \in E; \mathcal{B}(u) \to \infty$ as $|z| \to \infty$ on any finite dimensional subspace of E,*
- (T3) *there exist $\rho_k > r_k > 0$ such that*

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \ge 0 > b_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \Phi_\lambda(u),$$

for all $\lambda \in [1, 2]$, and

$$d_k(\lambda) := \inf_{u\in Z_k, \|u\|\le\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty, \text{ uniformly for } \lambda \in [1, 2].$$

Then there exist $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \to f_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty.$$

Particularly, if $\{u_{\lambda_n}\}$ has a convergent subsequence for every $k \in \mathbb{N}$, then Φ_1 has infinitely many nontrivial critical points $\{u_k\} \in E\backslash\{0\}$ satisfying $\Phi_1(u_k) \to 0^-$ as $k \to \infty$.

3. Proof of Theorem 1.1

Let Φ be the functional defined on E by

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} \Big[|\dot{u}(t)|^2 + (L(t)u(t), u(t))\Big]dt + \frac{1}{2}\int_{\mathbb{R}} (Au(t), \dot{u}(t))dt - \int_{\mathbb{R}} V(t, u(t))dt \\ &= \frac{1}{2}\Big(\|u^+\|^2 - \|u^-\|^2\Big) + \frac{1}{2}\int_{\mathbb{R}} (Au(t), \dot{u}(t))dt - \int_{\mathbb{R}} V(t, u(t))dt,\end{aligned} \tag{3.1}$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$.

Lemma 3.1. *Under the conditions of Theorem 1.1, $\Phi \in C^1(E,\mathbb{R})$ and*

$$\Phi'(u)v = \int_{\mathbb{R}} \Big[(\dot u(t), \dot v(t)) + (L(t)u(t), v(t))\Big]dt + \int_{\mathbb{R}} (Au(t), \dot v(t))dt \\ - \int_{\mathbb{R}} (\nabla V(t,u(t)), v(t))dt.$$

for all $u = u^- + u^0 + u^+$, $v = v^- + v^0 + v^+$ in $E = E^- \oplus E^0 \oplus E^+$. Moreover, any critical point of Φ on E is a homoclinic solution of (1.1).

Proof. Rewrite $\Phi = \Psi_1 + \Psi_2 - \Psi_3$ where

$$\Psi_1(u) := \frac{1}{2}\int_{\mathbb{R}} \Big[|\dot u(t)|^2 + (L(t)u(t), u(t))\Big]dt, \quad \Psi_2(u) := \frac{1}{2}\int_{\mathbb{R}} (Au(t), \dot u(t))dt,$$
$$\Psi_3(u) := \int_{\mathbb{R}} V(t,u(t))dt.$$

It is known [5] that $\Psi_1 \in C^1(E,\mathbb{R})$ and for all $u, v \in E$,

$$\Psi_1'(u)v = \int_{\mathbb{R}} \Big[(\dot u(t), \dot v(t)) + (L(t)u(t), v(t))\Big]dt.$$

Also, we have $\Psi_2 \in C^1(E,\mathbb{R})$, and for all $u, v \in E$,

$$\Psi_2'(u)v = \int_{\mathbb{R}} (Au(t), \dot v(t))dt.$$

Indeed, using Lemma 2.1, the quadratic form Ψ_2 is continuous and therefore it is of class C^1. Furthermore, by the use of the antisymmetric property of A, we obtain the result.

It remains to show that $\Psi_3 \in C^1(E,\mathbb{R})$ and for all $q, v \in E$,

$$\Psi_3'(q)v = \int_{\mathbb{R}} (\nabla V(t,q(t)), v(t))dt.$$

Fix $q \in E$, let $c_1 = \sup_{|x| \le \|q\|_{L^\infty}} |\nabla W(x)|$ and define $J(q) : E \to \mathbb{R}$ as follows

$$J(q)v = \int_{\mathbb{R}} (\nabla V(t,q(t)), v(t))dt, \quad \forall v \in E.$$

Then $J(q)$ is linear and bounded. Indeed,

$$|\nabla V(t,q(t))| = a(t)|\nabla W(q(t))| \le c_1 a(t), \quad \forall t \in \mathbb{R}$$

and by (2.3), we obtain

$$\begin{aligned} |J(q)v| &= |\int_{\mathbb{R}} (\nabla V(t,q(t)), v(t))dt| \\ &\le c_1 \int_{\mathbb{R}} a(t)|v(t)|dt \\ &\le c_1 \|a\|_2 \|v\|_2 \\ &\le c_1 C_2 \|a\|_2 \|v\|. \end{aligned} \tag{3.2}$$

Moreover, for $q, v \in E$, by the Mean Value Theorem, we have

$$\int_{\mathbb{R}} V(t, q(t) + v(t))dt - \int_{\mathbb{R}} V(t,q(t))dt = \int_{\mathbb{R}} (\nabla V(t, q(t) + h(t)v(t)), v(t))dt,$$

where $h(t) \in (0,1)$. Also, by Lemma 2.4 and the Hölder inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}}(\nabla V(t,q(t)+h(t)v(t)),v(t))dt-\int_{\mathbb{R}}(\nabla V(t,q(t)),v(t))dt\\&=\int_{\mathbb{R}}(\nabla V(t,q(t)+h(t)v(t))-\nabla V(t,q(t)),v(t))dt\to 0,\end{aligned}\tag{3.3}$$

as $v \to 0$ in E. Combining (3.2) and (3.3) we obtain the result.

Now, we prove that Ψ_3' is continuous. Suppose that $q \to q_0$ in E and note that

$$\Psi_3'(q)v-\Psi_3'(q_0)v=\int_{\mathbb{R}}(\nabla V(t,q(t))-\nabla V(t,q_0(t)),v(t))dt.$$

By Lemma 2.4 and the Hölder inequality, we obtain

$$\Psi_3'(q)v-\Psi_3'(q_0)v\to 0,\quad \text{as } q\to q_0.$$

Now, we check that critical points of Φ are homoclinic solutions for (1.1). In fact, if u is a critical point of Φ, by Lemma 3.1, we have $L(t)u(t)-\nabla V(t,u(t))$ is the weak derivative of $\dot u + Au$. Since $L \in C(\mathbb{R},\mathbb{R}^{N^2})$ and $V \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, we see that $\dot u + Au$ is continuous and consequently $\dot u$ is continuous which yields $u \in C^2(\mathbb{R},\mathbb{R}^N)$; i.e., u is a classical solution of (1.1).

Finally, to prove that $\dot u(t) \to 0$ as $|t| \to \infty$, note that by Lemma 2.3 it suffices to show that any critical point of Φ on E is an element of $\mathcal{D}(B)$. Indeed, by Lemma 2.1, we know that $u \in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ and hence $u(t) \to 0$ as $|t| \to \infty$. Moreover, since $W \in C^1(\mathbb{R}^N,\mathbb{R})$, there exists $d > 0$ such that

$$|\nabla W(u(t))|\le d,\quad \forall t\in\mathbb{R}.\tag{3.4}$$

From (1.1) and this inequality, we receive

$$\begin{aligned}\|Bu\|_{L^2}^2&=\|A\dot u+\nabla V(t,u)\|_{L^2}^2\\&\le 2\int_{\mathbb{R}}|A\dot u(t)|^2dt+2d^2\int_{\mathbb{R}}|a(t)|^2dt.\end{aligned}\tag{3.5}$$

By (3.5) and the fact $|\dot u|, a \in L^2(\mathbb{R},\mathbb{R})$ one sees that $\|Bu\|_{L^2} < \infty$; i.e., $u \in \mathcal{D}(B)$. $\square$

To apply Theorem 2.5 for proving Theorem 1.1, we define the functionals $\mathcal{A},\mathcal{B}$ and Φ_λ on the space E by

$$\begin{aligned}\mathcal{A}(u)&=\frac12\|u^+\|^2+\frac12\int_{\mathbb{R}}(Au(t),\dot u(t))dt,\quad \mathcal{B}(u)=\frac12\|u^-\|^2+\int_{\mathbb{R}}V(t,u(t))dt,\\ \Phi_\lambda(u)&:=\mathcal{A}(u)-\lambda\mathcal{B}(u)\\&=\frac12\|u^+\|^2+\frac12\int_{\mathbb{R}}(Au(t),\dot u(t))dt-\lambda\Big(\frac12\|u^-\|^2+\int_{\mathbb{R}}V(t,u(t))dt\Big)\end{aligned}$$

for all $u = u^- + u^0 + u^+$ in $E = E^- \oplus E^0 \oplus E^+$ and $\lambda \in [1,2]$. From Lemma 3.1, we know that $\Phi_\lambda \in C^1(E,\mathbb{R})$ for all $\lambda \in [1,2]$. Let $X_j = span\{e_j\}$ for all $j \in \mathbb{N}$, where $\{e_n; n \in \mathbb{N}\}$ is the system of eigenfunctions given below. Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (3.1).

Lemma 3.2. *Under the assumption* (V1), *we have* $\mathcal{B}(u) \ge 0$ *and* $\mathcal{B}(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* E.

Proof. Since a and W are nonnegative it is obvious, by the definition of $\mathcal{B}$, that $\mathcal{B}(u) \geq 0$. We claim that for any finite dimensional subspace $F \subset E$, there exists $\epsilon > 0$ such that

$$\operatorname{meas}\big(\{t \in \mathbb{R} : a(t)|u(t)|^\gamma \geq \epsilon \|u\|^\gamma\}\big) \geq \epsilon, \quad \forall u \in F\backslash\{0\}. \tag{3.6}$$

If not, for any $n \in \mathbb{N}$, there exists $u_n \in F\backslash\{0\}$ such that

$$\operatorname{meas}(\{t \in \mathbb{R} : a(t)|u_n(t)|^\gamma \geq \frac{1}{n}\|u_n\|^\gamma\}) < \frac{1}{n}.$$

Let $v_n := \frac{u_n}{\|u_n\|}$. Then $v_n \in F$, $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\operatorname{meas}(\{t \in \mathbb{R} : a(t)|v_n(t)|^\gamma \geq \frac{1}{n}\}) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in E for some $v_0 \in F$ since F is of finite dimension. Evidently, $\|v_0\| = 1$. By the equivalence of norms on F, we have $v_n \to v_0$ in $L^\gamma(a)$; i.e.,

$$\int_{\mathbb{R}} a(t)|v_n - v_0|^\gamma dt \to 0, \quad \text{as } n \to \infty. \tag{3.8}$$

Moreover, since $\|v_0\|_{L^\infty} > 0$, by (V5) and the definition of $\|\cdot\|_{L^\infty}$, it is easy to see that there exists a constant $\delta_0 > 0$ such that

$$\operatorname{meas}(\{t \in \mathbb{R}; a(t)|v_0(t)|^\gamma \geq \delta_0\}) \geq \delta_0. \tag{3.9}$$

For any $n \in \mathbb{N}$, let

$$\Lambda_n = \{t \in \mathbb{R} : a(t)|v_n(t)|^\gamma < \frac{1}{n}\}, \quad \Lambda_n^c = \mathbb{R}\backslash\Lambda_n = \{t \in \mathbb{R} : a(t)|v_n(t)|^\gamma \geq \frac{1}{n}\}.$$

Set $\Lambda_0 = \{t \in \mathbb{R} : a(t)|v_0(t)|^\gamma \geq \delta_0\}$. Then, for n large enough, by (3.7) and (3.9), we have

$$\operatorname{meas}(\Lambda_n \cap \Lambda_0) \geq \operatorname{meas}(\Lambda_0) - \operatorname{meas}(\Lambda_n^c) \geq \delta_0 - 1/n \geq \delta_0/2.$$

Consequently, for n large enough, there holds

$$\begin{aligned}
\int_{\mathbb{R}} a(t)|v_n - v_0|^\gamma dt &\geq \int_{\Lambda_n\cap\Lambda_0} a(t)|v_n - v_0|^\gamma dt \\
&\geq \frac{1}{2^{\gamma-1}}\Big(\int_{\Lambda_n\cap\Lambda_0} a(t)|v_0|^\gamma dt - \int_{\Lambda_n\cap\Lambda_0} a(t)|v_n|^\gamma dt\Big) \\
&\geq \frac{1}{2^{\gamma-1}}(\delta_0 - 1/n)\operatorname{meas}(\Lambda_n \cap \Lambda_0) \\
&\geq \frac{\delta_0^2}{2^{\gamma+1}} > 0.
\end{aligned}$$

This contradicts (3.8) and therefore (3.6) holds. For the ϵ given in (3.6). Let

$$\Lambda_u = \{t \in \mathbb{R} : a(t)|u(t)|^\gamma \geq \epsilon\|u\|^\gamma\}, \quad \forall u \in F\backslash\{0\}.$$

Then

$$\operatorname{meas}(\Lambda_u) \geq \epsilon, \quad \forall u \in F\backslash\{0\}. \tag{3.10}$$

Observing that for $u \in F$ with $\|u\| \geq r(\|a\|_{L^\infty}/\epsilon)^{1/\gamma}$, there holds

$$|u(t)| \geq r, \quad \forall t \in \Lambda_u. \tag{3.11}$$

Combining (3.10), (3.11) and (V1), for any $u \in F$ with $\|u\| \geq r(\|a\|_{L^\infty}/\epsilon)^{1/\gamma}$, we obtain

$$\begin{aligned}\mathcal{B}(u) &= \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} V(t,u(t))dt \\ &\geq \int_{\Lambda_u} V(t,u(t))dt \\ &\geq a_1 \int_{\Lambda_u} a(t)|u(t)|^\gamma dt \\ &\geq a_1\epsilon\|u\|^\gamma \operatorname{meas}(\Lambda_u) \geq a_1\epsilon^2\|u\|^\gamma,\end{aligned}$$

which implies that $\mathcal{B}(u) \to \infty$ as $\|u\| \to \infty$ on F. $\square$

Lemma 3.3. *Under the assumptions of Theorem 1.1, there exist a positive integer k_0 and a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that*

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_0,$$

and

$$d_k(\lambda) := \inf_{u\in Z_k,\|u\|\leq\rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k\to\infty, \text{ uniformly for } \lambda \in [1,2],$$

where $Z_k = \overline{\oplus_{j=k}^\infty X_j}$.

Proof. Note that $Z_k \subset E^+$ for all $k \geq \bar{n}+1$ where $\bar{n}$ is the integer defined in (2.1). So, for any $k \geq \bar{n}+1$ and $(\lambda,u) \in [1,2]\times Z_k$, we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}|A|\|u\|_{L^2}\|\dot{u}\|_{L^2} - 2\int_{\mathbb{R}} V(t,u(t))dt \\ &\geq \frac{1}{2}\Big(1-\delta^2|A|\Big)\|u\|^2 - 2\int_{\mathbb{R}} V(t,u(t))dt,\end{aligned} \tag{3.12}$$

with $1-\delta^2|A| > 0$ by (2.2). On the other hand, by the mean value theorem and (V3), we have

$$\begin{aligned}\int_{\mathbb{R}} V(t,u(t))dt &= \int_{\mathbb{R}} (\nabla V(t,\theta(t)u(t)),u(t))dt \\ &\leq a_4\int_{\mathbb{R}} a(t)|u(t)|^\beta dt + a_4\int_{\mathbb{R}} a(t)|u(t)|dt\end{aligned} \tag{3.13}$$

where $\theta(t) \in (0,1)$. Since the function a is bounded, by (3.13) there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}} V(t,u(t))dt \leq c_1\Big(\|u\|^\beta_{L^\beta} + \|u\|_{L^1}\Big). \tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$\Phi_\lambda(u) \geq \frac{1}{2}\Big(1-\delta^2|A|\Big)\|u\|^2 - 2c_1\Big(\|u\|^\beta_{L^\beta} + \|u\|_{L^1}\Big). \tag{3.15}$$

For $k \in \mathbb{N}$, define

$$l_1(k) := \sup_{u\in Z_k,\|u\|=1} \|u\|_{L^1}, \quad l_\beta(k) := \sup_{u\in Z_k,\|u\|=1} \|u\|_{L^\beta}.$$

Since E is compactly embedded into L^1 and L^β respectively,

$$l_1(k) \to 0, \quad l_\beta(k) \to 0, \quad \text{as } k\to\infty. \tag{3.16}$$

Consequently, for any $k \geq \bar{n}+1$, (3.15) implies

$$\Phi_\lambda(u) \geq \frac{1}{2}\Big(1-\delta^2|A|\Big)\|u\|^2 - 2c_1\Big(l_\beta^\beta(k)\|u\|^\beta + l_1(k)\|u\|\Big), \tag{3.17}$$

for all $(\lambda, u) \in [1,2] \times Z_k$. Let

$$\rho_k = \frac{8c_1}{1-\delta^2|A|}\Big(l_\beta^\beta(k) + l_1(k)\Big), \quad \forall k \in \mathbb{N}.$$

From (3.16), we obtain

$$\rho_k \to 0 \quad \text{as } k \to \infty, \tag{3.18}$$

and there exists $k_0 > \bar{n}+1$ such that

$$\rho_k < 1, \quad \forall k \geq k_0. \tag{3.19}$$

Combining (3.17)-(3.19) and the definition of ρ_k, a straightforward computation shows that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq \frac{1-\delta^2|A|}{4}\rho_k^2 > 0, \quad \forall k \geq k_0.$$

Furthermore, by (3.17), for any $k \geq k_0$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\Phi_\lambda(u) \geq -2c_1\Big(l_\gamma^\beta(k)\rho_k^\beta + l_1(k)\rho_k\Big).$$

Then

$$0 \geq \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \geq -2c_1\Big(l_\beta^\beta(k)\rho_k^\beta + l_1(k)\rho_k\Big), \quad \forall k \geq k_0.$$

Combining (3.16) and (3.18), we obtain

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \text{as } k \to \infty, \text{ uniformly for } \lambda \in [1,2].$$

□

Lemma 3.4. *Under the assumptions of Theorem 1.1, there exists $0 < r_k < \rho_k$ for all $k \in \mathbb{N}$ such that*

$$b_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N},$$

where the sequence $\{\rho_k\}_{k\in\mathbb{N}}$ is obtained in Lemma 3.3 and $Y_k = \oplus_{j=1}^k X_j$.

Proof. For $u = u^- + u^0 + u^+ \in Y_k$ with $\|u\| \leq \frac{\omega}{C_\infty}$ where C_∞ is the constant given by (2.3), one has $\|u\|_{L^\infty} \leq \omega$ and by (V2), we have

$$\begin{aligned}
\Phi_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 + \frac{1}{2}|A|\|u\|_{L^2}\|\dot{u}\|_{L^2} - \int_{\mathbb{R}} V(t,u(t))dt \\
&\leq \frac{1}{2}\Big(1+\delta^2|A|\Big)\|u\|^2 - a_3\|u\|_{L^\nu(a)}^\nu \\
&\leq \frac{1}{2}\Big(1+\delta^2|A|\Big)\|u\|^2 - \delta_k\|u\|^\nu
\end{aligned} \tag{3.20}$$

where the last inequality is obtained by the equivalence of norms $\|\cdot\|_{L^\nu(a)}$ and $\|\cdot\|$ on the finite dimensional space Y_k and $\delta_k > 0$ depending on Y_k. Now, choosing

$$0 < r_k < \min\{\rho_k, \frac{\omega}{C_\infty}, \delta_k^{1/(2-\nu)}\}, \quad \forall k \in \mathbb{N}.$$

By (3.20), a direct computation gives

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u) \le \frac{\delta^2 |A| - 1}{2} r_k^2 < 0, \quad \forall k \in \mathbb{N}.$$

□

Proof of Theorem 1.1. Combining Lemma 2.1, lemma 2.2, (3.1) and (3.14), it is easy to see that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Moreover, by (V4), $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus the condition (T1) of Theorem 2.5 holds. Lemma 3.2 shows that the condition (T2) holds, while Lemma 3.3 together with Lemma 3.4 imply that the condition (T3) holds for all $k \ge k_0$, where k_0 is given in Lemma 3.3. Therefore, by Theorem 2.5, for each $k \ge k_0$, there exist $\lambda_n \to 1, u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \to f_k \in [d_k(2), b_k(1)] \quad \text{as } n \to \infty. \tag{3.21}$$

It remains to prove that the sequence $\{u_{\lambda_n}\}$ is bounded. Otherwise, we suppose, up to a subsequence, that

$$\|u_{\lambda_n}\| \to \infty, \quad \text{as } n \to \infty. \tag{3.22}$$

Let $u_n := u_{\lambda_n} = u_n^- + u_n^0 + u_n^+$ in $E = E^- \oplus E^0 \oplus E^+$ and assume that

$$u_n/\|u_n\| \rightharpoonup w, \quad u_n^{\pm}/\|u_n\| \rightharpoonup w^{\pm}, \quad u_n^0/\|u_n\| \rightharpoonup w^0.$$

By (3.21), we have

$$(u_n^+, v_n) - \lambda_n(u_n^-, v_n) + \int_{\mathbb{R}} (Au_n(t), \dot v_n(t))dt - \lambda_n \int_{\mathbb{R}} (\nabla V(t, u_n(t)), v_n(t))dt = 0, \tag{3.23}$$

where $v_n = v|_{Y_n}$, $v = \sum_{i=1}^{\infty} s_i e_i$. Using (V3) and Lemma 2.2 we can find a constant $d > 0$ such that

$$\begin{aligned} \Big| \int_{\mathbb{R}} (\nabla V(t, u_n(t)), v_n(t))dt \Big| &\le a_4 \int_{\mathbb{R}} a(t)|u_n(t)|^{\beta-1}|v_n(t)|dt + a_4 \int_{\mathbb{R}} a(t)|v_n(t)|dt \\ &\le d\Big(\|u_n\|^{\beta-1} + 1\Big)\|v_n\| \end{aligned} \tag{3.24}$$

Since $\beta - 1 < 1$, from (3.22) and (3.24), we obtain

$$\frac{1}{\|u_n\|} \int_{\mathbb{R}} (\nabla V(t, u_n(t)), v_n(t))dt \to 0, \quad \text{as } n \to \infty. \tag{3.25}$$

Also, dividing by $\|u_n\|$ in (3.23) and passing to the limit, we obtain

$$(w^+, v) - (w^-, v) + \int_{\mathbb{R}} (Aw(t), \dot v(t))dt = 0. \tag{3.26}$$

If $w \ne 0$, (3.26) is equivalent to $0 \in \sigma\Big(-(d^2/dt^2) + L(t) - A(d/dt)\Big)$ which contradicts assumption (L3).

If $w = 0$. From (3.21), we have

$$\begin{aligned} 0 = \lambda_n\Big(\|u_n\|^2 - \|u_n^0\|^2\Big) &+ \int_{\mathbb{R}} (Au_n, \lambda_n \dot u_n^+ - \dot u_n^-)dt \\ &- \lambda_n \int_{\mathbb{R}} (\nabla V(t, u_n), \lambda_n u_n^+ - u_n^-)dt. \end{aligned} \tag{3.27}$$

Arguing as in (3.24)-(3.25), we obtain

$$\frac{1}{\|u_n\|^2}\int_{\mathbb{R}}(\nabla V(t,u_n),\lambda_n u_n^+ - u_n^-)dt \to 0 \quad \text{as } n\to\infty. \tag{3.28}$$

Combining (3.22), (3.27) and (3.28) we obtain

$$\frac{1}{\|u_n\|^2}\int_{\mathbb{R}}(Au_n,\lambda_n \dot{u}_n^+ - \dot{u}_n^-)dt \to -1 \quad \text{as } n\to\infty. \tag{3.29}$$

On the other hand, by Lemma 2.2, passing if necessary to a subsequence, we have $\frac{u_n}{\|u_n\|}\to 0$ in L^2. Also, by Lemma 2.1, the sequence $\{\frac{\lambda_n\dot{u}_n^+ - \dot{u}_n^-}{\|u_n\|}\}$ is bounded in L^2, so it is obvious that

$$\frac{1}{\|u_n\|^2}\int_{\mathbb{R}}(Au_n,\lambda_n \dot{u}_n^+ - \dot{u}_n^-)dt \to 0 \quad \text{as } n\to\infty.$$

This is in contradiction with (3.29). Therefore, $\{u_n\}$ is bounded and by a standard argument it possesses a strong convergent subsequence in E (see [18, 21]).

Now, from the last assertion of Theorem 2.5, we know that $\Phi=\Phi_1$ has infinitely many nontrivial critical points and by Lemma 3.1, system (1.1) possesses infinitely many nontrivial homoclinic solutions. This completes the proof. $\square$

4. Proof of Theorem 1.4

The proof is based on the following two lemmas.

Lemma 4.1. *Under the conditions of Theorem 1.4, $\Phi\in C^1(E,\mathbb{R})$ and*

$$\begin{aligned}\Phi'(u)v &= \int_{\mathbb{R}}\Big[(\dot{u}(t),\dot{v}(t)) + (L(t)u(t),v(t))\Big]dt\\ &\quad + \int_{\mathbb{R}}(Au(t),\dot{v}(t))dt - \int_{\mathbb{R}}(\nabla V(t,u(t)),v(t))dt.\end{aligned}$$

for all $u=u^-+u^0+u^+$, $v=v^-+v^0+v^+$ in $E=E^-\oplus E^0\oplus E^+$. Moreover, any critical point of Φ on E is a homoclinic solution of (1.1).

Proof. Using the notation of Lemma 3.1, we need to prove that $\Psi_3\in C^1(E,\mathbb{R})$ and

$$\Psi_3'(q)v = \int_{\mathbb{R}}(\nabla V(t,q(t)),v(t))dt, \quad \forall q,v\in E.$$

Let $u\in E$, from Lemma 2.1, we know that $u\in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ and hence there exists $T_0>0$ such that

$$|u(t)|\le r/2, \quad \forall |t|\ge T_0. \tag{4.1}$$

By (2.3), for any $v\in E$ with $\|v\|\le \frac{r}{2C_\infty}$, we have

$$\|v\|_{L^\infty}\le r/2. \tag{4.2}$$

Combining (4.1), (4.2) and (V4'), by the mean value theorem and the Hölder inequality, for any $T > T_0$ and $v \in E$ with $\|v\| \le \frac{r}{2C_\infty}$, we have

$$\begin{aligned} &\Big|\int_{|t|>T}\Big[V(t,u+v)-V(t,u)-(\nabla V(t,u),v)\Big]dt\Big| \\ &= \Big|\int_{|t|>T}\Big[\int_0^1(\nabla V(t,u+sv)-\nabla V(t,u),v)ds\Big]dt\Big| \\ &\le 2a_2\int_{|t|>T}(|u|+|v|)^\beta|v|dt \\ &\le 2a_2\Big(\int_{|t|>T}(|u|+|v|)dt\Big)^\beta\|v\|_{L^{\frac{1}{1-\beta}}} \\ &\le 2a_2C_{\frac{1}{1-\beta}}\Big(\int_{|t|>T}(|u|+|v|)dt\Big)^\beta\|v\|. \end{aligned} \tag{4.3}$$

In view of Lemma 2.2, for any $\varepsilon > 0$, there exist $0 < \delta_1 \le \frac{r}{2C_\infty}$ and $T_\varepsilon > T_0$ such that

$$2a_2C_{\frac{1}{1-\beta}}\Big(\int_{|t|>T_\varepsilon}(|u|+|v|)dt\Big)^\beta \le \varepsilon/2, \quad \forall v \in E,\ \|v\| \le \delta_1. \tag{4.4}$$

Define $\Psi_T : W^{1,2}([-T,T],\mathbb{R}^N) \to \mathbb{R}$ by

$$\Psi_T(u) = \int_{-T}^{T} V(t,u)dt, \quad \forall u \in W^{1,2}([-T,T],\mathbb{R}^N).$$

It is known (see, e.g., [14]) that $\Psi_T \in C^1(W^{1,2}([-T,T],\mathbb{R}^N))$ for any $T > 0$. Combining this with the fact E is continuously embedded in $W^{1,2}(\mathbb{R},\mathbb{R}^N)$ from Lemma 2.1, for the ε and T_ε given above, there exists $\delta_2 = \delta_2(u,\varepsilon,T_\varepsilon)$ such that

$$\Big|\int_{-T_\varepsilon}^{T_\varepsilon}\big[V(t,u+v)-V(t,u)-(\nabla V(t,u),v)\big]dt\Big| \le \frac{\varepsilon}{2}\|v\|, \quad \forall v \in E,\ \|v\| \le \delta_2. \tag{4.5}$$

Combining (4.3)-(4.5) and taking $\delta = \min\{\delta_1,\delta_2\}$, we obtain

$$\Big|\int_{\mathbb{R}}\big[V(t,u+v)-V(t,u)-(\nabla V(t,u),v)\big]dt\Big| \le \varepsilon\|v\|, \quad \forall v \in E,\ \|v\| \le \delta.$$

Thus Ψ_3 is Fréchet differentiable and

$$\Psi_3'(q)v = \int_{\mathbb{R}}(\nabla V(t,q(t)),v(t))dt, \quad \forall q,v \in E.$$

Next we prove that Ψ_3' is weakly continuous. Let $u_n \rightharpoonup u_0$ in E. Again, using Lemma 2.2, $u_n \to u_0$ in L^p for all $1 \le p \le \infty$. By the Hölder inequality,

$$\begin{aligned} \|\Psi_3'(u_n)-\Psi_3'(u_0)\|_{E^*} &= \sup_{\|v\|=1}\|(\Psi_3'(u_n)-\Psi_3'(u_0))v\| \\ &= \sup_{\|v\|=1}\Big|\int_{\mathbb{R}}(\nabla V(t,u_n)-\nabla V(t,u_0),v)dt\Big| \\ &\le \sup_{\|v\|=1}\Big[\Big(\int_{\mathbb{R}}|\nabla V(t,u_n)-\nabla V(t,u_0)|^3dt\Big)^{1/3}\|v\|_{3/2}\Big] \\ &\le C_{3/2}\Big(\int_{\mathbb{R}}|\nabla V(t,u_n)-\nabla V(t,u_0)|^3dt\Big)^{1/3}, \quad \forall n \in \mathbb{N}, \end{aligned} \tag{4.6}$$

Since $u_n \to u_0$ in L^1, there exists a constant $M_0 > 0$ such that

$$\|u_n\|_{L^1} \le M_0, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

By (V4'), for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\nabla V(t,u)| \le \frac{\varepsilon}{2(M_0^{1/3} + \|u_0\|_{L^1}^{1/3})}|u|^{1/3}, \quad \forall u \in \mathbb{R}, |u| \le \eta. \tag{4.8}$$

Due to (4.8), the fact that $u_0 \in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ and $u_n \to u_0$ in L^∞, there exist $T'_\varepsilon > 0$ and $N_1 \in \mathbb{N}$ such that for all $n > N_1$ and $|t| \ge T'_\varepsilon$,

$$\begin{gathered}
|\nabla V(t,u_n)| \le \frac{\varepsilon}{2(M_0^{1/3} + \|u_0\|_{L^1}^{1/3})}|u_n|^{1/3}, \\
|\nabla V(t,u_0)| \le \frac{\varepsilon}{2(M_0^{1/3} + \|u_0\|_{L^1}^{1/3})}|u_0|^{1/3}.
\end{gathered} \tag{4.9}$$

By (4.7) and (4.9), we have

$$\begin{aligned}
&\Big(\int_{|t|\ge T'_\varepsilon} |\nabla V(t,u_n) - \nabla V(t,u_0)|^3 dt\Big)^{1/3} \\
&\le \frac{\varepsilon}{2(M_0^{1/3} + \|u_0\|_{L^1}^{1/3})}(\|u_n\|_{L^1}^{1/3} + \|u_0\|_{L^1}^{1/3}) \\
&\le \frac{\varepsilon}{2}, \quad \forall n \ge \mathbb{N}.
\end{aligned} \tag{4.10}$$

On the other hand, using $u_n \to u_0$ in L^∞ and (V3'), by Lebesgue's Dominated Convergence Theorem,

$$\Big(\int_{-T'_\varepsilon}^{T'_\varepsilon} |\nabla V(t,u_n) - \nabla V(t,u_0)|^3 dt\Big)^{1/3} \to 0 \quad \text{as } n \to \infty.$$

Thus there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$,

$$\Big(\int_{-T'_\varepsilon}^{T'_\varepsilon} |\nabla V(t,u_n) - \nabla V(t,u_0)|^3 dt\Big)^{1/3} \le \varepsilon/2.$$

Combining the last inequality with (4.10) and taking $N_\varepsilon = \max\{N_1, N_2\}$, we obtain

$$\Big(\int_{\mathbb{R}} |\nabla V(t,u_n) - \nabla V(t,u_0)|^3 dt\Big)^{1/3} \le \varepsilon, \quad \forall n \ge N_\varepsilon. \tag{4.11}$$

Inequality (4.11) with (4.6) imply the continuity of Ψ'_3 and therefore $\Psi_3 \in C^1(E,\mathbb{R})$. The rest of the proof is similar to that of Lemma 3.1. $\square$

Lemma 4.2. *Under the assumption* (V1'), $\mathcal{B}(u) \ge 0$ *and* $\mathcal{B}(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* E.

Proof. Evidently, $\mathcal{B}(u) \ge 0$. An argument similar to but easier than the proof of (3.6) allows to claim that for any finite dimensional subspace $F \subset E$, there exists $\epsilon > 0$ such that

$$\operatorname{meas}(\{t \in \mathbb{R}; |u(t)| \ge \epsilon\|u\|\}) \ge \epsilon, \quad \forall u \in F\backslash\{0\}. \tag{4.12}$$

By (V1'), for any $A > 0$, there exists $B > 0$ such that

$$V(t,x) \ge A/\epsilon, \quad \forall\, t \in \mathbb{R} \text{ and } |x| \ge B. \tag{4.13}$$

where ϵ is given in (4.12). Let

$$\Lambda_u = \{t \in \mathbb{R} : |u(t)| \ge \epsilon\|u\|\}, \quad \forall u \in F\backslash\{0\}.$$

Then by (4.12),

$$\text{meas}(\Lambda_u) \geq \epsilon, \quad \forall u \in F \backslash \{0\}. \tag{4.14}$$

Observing that for $u \in F$ with $\|u\| \geq B/\epsilon$, there holds

$$|u(t)| \geq B, \quad \forall\, t \in \Lambda_u. \tag{4.15}$$

Combining (4.13)-(4.15), for any $u \in F$ with $\|u\| \geq B/\epsilon$, we have

$$\begin{aligned} \mathcal{B}(u) &= \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} V(t,u(t))dt \\ &\geq \int_{\Lambda_u} V(t,u(t))dt \\ &\geq \text{meas}(\Lambda_u)A/\epsilon \geq A, \end{aligned}$$

which implies that $\mathcal{B}(u) \to \infty$ as $\|u\| \to \infty$ on F. $\square$

To complete the proof of Theorem 1.4, we observe that since (V3') is the particular case of (V3) where $\beta = 1$, then Lemma 3.3 remains true under the assumption (V3'). Also, it is obvious that Lemma 3.4 still holds with (V2') replacing (V2). The remainder of the proof is analogous to Theorem 1.1.

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