ISOCHRONOUS BIFURCATIONS IN SECOND-ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article we consider a special type of second-order delay differential equations. More precisely, we take an equation of a conservative mechanical system in one dimension with an added term that is a function of the difference between the value of the position at time t minus the position at the delayed time $t-\tau$. For this system, we show that, under certain conditions of non-degeneration and of convergence of the periodic solutions obtained by the Homotopy Analysis Method, bifurcation branches appearing in a neighbourhood of Hopf bifurcation due to the delay are isochronous; i.e., all the emerging cycles have the same frequency.

1. INTRODUCTION

Delay differential equations are a particular case of functional differential equations [11]. Functional differential equations are of the form

$$x'(t) = F(t, x^t). \tag{1.1}$$

Here $x(t) \in \mathbb{R}^n$, $x^t \in C([-\tau, 0], \mathbb{R}^n)$ is the function $x^t(\theta) = x(t + \theta)$ and $F : D \subset \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous.

In the case of the delay differential equations, the functional F is of the form $F(t, x^t) = h(t, x(t), x(t-\tau_1), x(t-\tau_2), \dots)$ for one or more delays $0 < \tau_i \leq \tau$. In this paper, we consider equations with a single delay τ. The system is in some ways simple, but the problem is still infinite-dimensional. The phase space is $C([-\tau, 0], \mathbb{R}^n)$.

In this article we consider delay differential equations related to ordinary differential equations. More precisely, we take a differential equation corresponding to a particle moving in one dimension under the influence of a conservative force. This equation is of the form

$$x''(t) = -g(x(t), \beta), \tag{1.2}$$

where β represents a parameter.

If $\partial g / \partial x(x^0, \beta) > 0$ at an equilibrium x^0 of (1.2), then this equilibrium is a center. Periodic orbits are densely distributed in a neighborhood of x^0 in the phase space x-x'. Numerous studies have considered the case in which the center is isochronous; i.e., all the orbits have the same frequency [4, 5, 9, 15, 19].

The case of an isochronous focus has also been considered in the literature [8, 9, 18]. In this case there are no periodic orbits and the study is centered at the existence of the so-called isochronous sections.

We modify equation (1.2) by adding a linear term, proportional to the difference of the value of the position at time t and at the delayed time $t - \tau$. This scheme has been used in control theory to control chaotic systems [16, 17]. Then we get an equation of the form

$$x'' + g(x, \beta) = \gamma f(x - x_\tau), \tag{1.3}$$

where we have introduced the real parameter γ and x_τ is the delayed position $x_\tau(t) = x(t - \tau)$.

The presence of the delayed term usually breaks the center. In some cases, a Hopf bifurcation occurs when one of the parameters γ or β changes. In this work, we show that the generic situation is that the cycles emerging from the bifurcation have the same frequency. In this case we say that the bifurcation is isochronic.

We might ask if a focus were considered instead of a center. For example one coming from the equation

$$x'' + g(x, x', \beta) = 0. \tag{1.4}$$

In this case the result that we show in this work is not true. The branches from the Hopf bifurcations, when the term with the delay is added, are not isochronous, see for example [6].

Many of the techniques used to study delay differential equations are generalizations of the corresponding techniques for differential equations, especially in the study of bifurcations [1].

The Homotopy Analysis Method (HAM) [13, 14] is a nonperturbative method initially developed to solve differential equations. It has also been applied to find periodic solutions of delay differential equations [2].

In this article we use the HAM as a theoretical tool to prove the main result; namely, that the bifurcation branches that appear due to the delay are isochronous, at least in a neighborhood of the bifurcation and under certain generic conditions. The result is stated in Theorem 3.5.

This article is organized as follows. Section 2 gives the bifurcation conditions which appear due to the delay. Section 3 provides the main result. In the first part the HAM, as used in this work, is described. Then the corresponding theorems are given. Section 4 discusses two examples, in the first the mechanical system is an anharmonic oscillator with a cubic nonlinearity and in the second a rotating pendulum.

2. Bifurcation conditions

As stated in the introduction we consider equation (1.3):

$$x'' + g(x, \beta) = \gamma f(x - x_\tau),$$

where $x(t) \in \mathbb{R}$, $\beta \in \mathbb{R}$ is a parameter, $\gamma \neq 0$, $\tau > 0$ is the time delay and $x_\tau(t) = x(t - \tau)$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an analytic function, with $f(0) = 0$ and $f'(0) \neq 0$, meanwhile $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^1.

The equilibrium points x^0 of the previous system are the solutions of the equation $g(x, \beta) = 0$. The characteristic equation for x^0 is

$$s^2 + \frac{\partial g}{\partial x}(x^0, \beta) - \gamma f'(0)(1 - e^{-s\tau}) = 0. \tag{2.1}$$

The necessary condition for the existence of a static bifurcation ($s = 0$), is verified if $\partial g / \partial x(x^0, \beta) = 0$. Furthermore, the necessary condition for the existence of Hopf bifurcation ($s = i\omega$) leads to

$$-\omega^2 + \frac{\partial g}{\partial x}(x^0, \beta) - \gamma f'(0)(1 - \cos \omega \tau) = 0,$$
$$\gamma f'(0) \sin \omega \tau = 0. \tag{2.2}$$

Since $f'(0) \neq 0$, from the previous system is obtained

$$\tau = \frac{k\pi}{\omega}, \quad \omega = \sqrt{\frac{\partial g}{\partial x}(x^0, \beta) - \gamma f'(0)(1 - (-1)^k)}, \quad k \in \mathbb{Z}. \tag{2.3}$$

From (2.1) we can calculate the derivative of $\operatorname{Re} s$ with respect to the parameter β or γ. In the first case the derivative evaluated at points in (2.3) is

$$\frac{d \operatorname{Re} s}{d \beta}\Big|_{s=i\omega} = \frac{(-1)^k \gamma \tau f'(0)}{(\gamma \tau f'(0))^2 + 4\omega^2} \frac{d}{d\beta}\left(\frac{\partial g}{\partial x}(x^0, \beta)\right). \tag{2.4}$$

In the other case, the derivative is

$$\frac{d \operatorname{Re} s}{d \gamma}\Big|_{s=i\omega} = \frac{(1 - (-1)^k)\gamma \tau f'(0)^2}{(\gamma \tau f'(0))^2 + 4\omega^2}. \tag{2.5}$$

We use μ for parameter β or γ, which we consider as a bifurcation parameter. If the derivative with respect to μ is positive (negative) then conjugate complex eigenvalues associated with Hopf bifurcation cross the imaginary axis from left to right (right to left). In both cases, we can apply Hopf theorem for delay differential equations and ensure the existence of periodic solutions in the vicinity of the bifurcation point [11]. On the other hand, if the derivative is zero then the bifurcation, if any, will be degenerate. We denote with μ_0 the critical value at which bifurcation occurs, and ω_{μ_0} the frequency associated with that value.

Conditions (2.3) define curves in the μ-τ space. Different branches can intersect themselves leading to possible double Hopf points.

3. Isochronous bifurcations

This section establishes the main result of the paper. In Theorem 3.5 it is shown that, under certain conditions, periodic solutions of equation (1.3) have the same frequency. In particular, the conditions stated in the theorem imply that small amplitude solutions arising from Hopf bifurcations are part of (what we call) isochronous branches.

The following subsection describes HAM as shown in [13]. This method allows us to construct analytical expressions for the periodic solutions. If the series found in the neighborhood of a Hopf bifurcation are convergent, then HAM can be used as a tool to prove the theorem 3.5.

3.1. **Description of the Homotopy Analysis Method.** Consider (1.3). By a change of coordinates we move the origin to x^0. Suppose that there is a solution of the equation with frequency ω and amplitude a. Then replace t for ωt and x for ax, the equation becomes

$$a\omega^2 x'' + g(x^0 + ax, \beta) = \gamma f(a(x - x_{\omega\tau})). \tag{3.1}$$

In the new variables, the periodic solution x_P has frequency and amplitude equal to one. We set the phase of the solution by imposing two conditions, $x(0) = 1$ and $x'(0) = 0$.

To find x_P we consider the family of operators $\mathcal{H}_q$ which depends on a deformation parameter $q \in [0, 1]$

$$\mathcal{H}_q[\phi] = (1 - q)\mathcal{L}[\phi - x_0] - qh\mathcal{N}_q[\phi], \tag{3.2}$$

where $\phi(t, q)$ is a homotopy that it is built with the method, $h \neq 0$ is a real parameter and x_0 is an initial approximation of the periodic solution which verifies $x_0(0) = 1$ and $x_0'(0) = 0$. $\mathcal{L}$ is the linear operator

$$\mathcal{L}[\phi] = \frac{\partial^2 \phi}{\partial t^2} + \phi, \tag{3.3}$$

and $\mathcal{N}_q$ is the following non-linear operator defined from (3.1)

$$\mathcal{N}_q[\phi] = A\Omega^2 \frac{\partial^2 \phi}{\partial t^2} + g(x^0 + A\phi, \beta) - \gamma f(A(\phi - \phi_{\Omega\tau})). \tag{3.4}$$

The procedure is based on the search of analytic functions $\phi(t, q)$, $\Omega(q)$ and $A(q)$, of q, so that

(i) $\mathcal{H}_q[\phi] = 0$ for $q \in [0, 1]$,
(ii) $\phi(t, q)$ verifies $\phi(0, q) = 1$ and $\partial\phi/\partial t(0, q) = 0$ for $q \in [0, 1]$.

If these functions exist, then taking $q = 0$, we obtain

$$\mathcal{H}_0[\phi] = \mathcal{L}[\phi(t, 0) - x_0(t)] = 0. \tag{3.5}$$

As ϕ and x_0 verify the same conditions in $t = 0$ we have $\phi(t, 0) = x_0(t)$. Moreover, if $q = 1$,

$$\mathcal{H}_1[\phi] = -h\,\mathcal{N}_1[\phi(t, 1)] = 0; \tag{3.6}$$

therefore, $x_P(t) = \phi(t, 1)$, $\omega = \Omega(1)$ and $a = A(1)$ will be solutions of the equation (3.1). Thus, when the parameter q varies from 0 to 1, the function $\phi(t, q)$ varies from the initial approximation x_0 to the desired solution x_P.

To find functions ϕ, Ω and A we consider their series expansions as follow

$$\phi(t, q) = \sum_{k=0}^{+\infty} x_k(t)q^k, \quad \Omega(q) = \sum_{k=0}^{+\infty} \omega_k q^k, \quad A(q) = \sum_{k=0}^{+\infty} a_k q^k. \tag{3.7}$$

Substituting these series in $\mathcal{H}_q[\phi] = 0$, and evaluating the k-th derivatives with respect to q at $q = 0$ it is obtained

$$\mathcal{L}[x_k(t) - x_{k-1}(t) + \delta_{1k}x_0(t)] = \frac{h}{(k-1)!} \frac{\partial^{k-1}\mathcal{N}_q[\phi]}{\partial q^{k-1}}\Big|_{q=0}. \tag{3.8}$$

Similarly, by taking the series expansion (3.7) of ϕ and whereas $\phi(t, 0) = x_0(t)$ verifies the conditions $x_0(0) = 1$ and $x_0'(0) = 0$, at $t = 0$, it follows that $x_k(0) = x_k'(0) = 0$, for $k \geq 1$.

We impose, as an additional condition, that each term of the solution must be periodic. For the operator $\mathcal{L}$ defined above, we obtain some conditions to ensure that the k-th term does not contain non-periodic functions (such as $t \cos t$ or $t \sin t$). Solving the equations (3.8) with the above mentioned conditions we can calculate the functions x_k in the series expansion of $\phi(t, 1)$.

Because $x_0(0) = 1$ and $x_0'(0) = 0$ must be verified, we choose the initial guess $x_0(t) = \cos t$. Replacing x_0 in (3.8) it gives

$$\begin{aligned}
\mathcal{L}[x_1] &= x_1''(t) + x_1(t) \\
&= h(-a_0\omega_0^2\cos t + g(x^0 + a_0\cos t, \beta) - \gamma f(a_0(\cos t - \cos(t - \omega_0\tau)))),
\end{aligned} \tag{3.9}$$

with initial conditions $x_1(0) = x_1'(0) = 0$. Assume $a_0 \neq 0$, since otherwise the right-hand side of the above equation vanishes.

In this case, solution x_1 will be periodic if the coefficients of $\sin t$ and $\cos t$ in the right-hand side of (3.9) vanishes. Then, ω_0 and a_0, must be solutions of the following non-linear system of equations

$$-a_0\omega_0^2 + \frac{1}{\pi}\int_0^{2\pi} g(x^0 + a_0\cos t, \beta)\cos t\, dt$$

$$-\frac{\gamma}{\pi}\int_0^{2\pi} f(a_0(\cos t - \cos(t - \omega_0\tau)))\cos t\, dt = 0, \tag{3.10}$$

$$-\frac{\gamma}{\pi}\int_0^{2\pi} f(a_0(\cos t - \cos(t - \omega_0\tau)))\sin t\, dt = 0.$$

In the coefficient of $\sin(t)$ only appears the function f because $g(x^0 + a_0\cos t, \beta)$ is even.

Remark 3.1. We note that above equations reduce to bifurcation equations (2.2) in the limit $a_0 \to 0$.

If $k \geq 2$ the obtained system is linear and we can easily calculate ω_{k-1} and a_{k-1}. For each value of k, once solved the corresponding system, the term x_k is calculated. This procedure is repeated until the desired order.

The obtained series expansions depend on the parameter h, we need to determine an appropriate value for this parameter to find solution x_P. The approximations of ω and a are polynomials in h and so are the approximations of x_P and its derivatives for fixed values of t. The observation of the behavior of these polynomials permit us to select an appropriate value of h [13]. For parameter values that result in convergent series, polynomials must converge to a value which is independent of h when the order goes to infinity. Then, the graph of these polynomials, for large order, give us a rough idea of where the convergence regions are, and we can determine an appropriate value of h.

3.2. Main results.

Lemma 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be analytic with $f(0) = 0$, $f'(0) \neq 0$ and $a_0 \neq 0$, small enough. Then it holds

$$\int_0^{2\pi} f(a_0(\cos t - \cos(t - \omega_0\tau)))\sin t\, dt = 0 \tag{3.11}$$

if and only if $\omega_0\tau = k\pi$, $k \in \mathbb{Z}$.

Proof. If $\omega_0\tau = k\pi$ and $k \in \mathbb{Z}$, then

$$f(a_0(\cos t - \cos(t - \omega_0\tau))) = f(a_0(1 - (-1)^k)\cos t), \tag{3.12}$$

from which it results that integral (3.11) is 0.

To prove the other implication, we consider the power series of the analytic function f

$$f(a_0(\cos t - \cos(t - \omega_0 \tau))) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} (a_0(\cos t - \cos(t - \omega_0 \tau)))^n. \quad (3.13)$$

By replacing this expression in (3.11) we have

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} a_0^n \int_0^{2\pi} (\cos t - \cos(t - \omega_0 \tau))^n \sin t \, dt = 0 \quad (3.14)$$

If n is even the above integral vanishes. If $n = 2k + 1$, it can be proved that the integral take the value $-\binom{2k+1}{k+1} \sin(\omega_0 \tau) \sin(\omega_0 \tau / 2)^{2k}$. Then (3.11) leads to

$$- \sin(\omega_0 \tau) a_0 \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(k+1)! k!} \left(a_0 \sin \left(\frac{\omega_0 \tau}{2} \right) \right)^{2k} = 0. \quad (3.15)$$

The series in the above equation does not vanish for small enough values of a_0, because $f'(0) \neq 0$ and the series is uniformly convergent on an interval containing the origin. Therefore, as $a_0 \neq 0$ we obtain $\omega_0 \tau = k\pi$, $k \in \mathbb{Z}$. $\square$

We consider cycles arising from a Hopf bifurcation point μ_0. The previous lemma implies that for a sufficiently small initial amplitude a_0, initial frequency ω_0 depends only on the delay τ and the chosen bifurcation branch. From the above it follows that, for small initial amplitudes, ω_0 is equal to frequency ω_{μ_0} at the bifurcation point.

In the following lemma we will show some special features that present the series constructed with HAM.

Lemma 3.3. *Consider the system of (infinite) linear equations* (3.8) *with initial conditions* $x_k(0) = x'_k(0) = 0$, *for all* $k \geq 1$. *In these equations* $\mathcal{L}$ *is the linear operator* (3.3), $\mathcal{N}$ *is given by expression* (3.4), *and* ϕ, Ω *and* A *are the series* (3.7). *Suppose* f *is analytic and* g *is* C^1. *Suppose further that* $f(0) = 0$ *and* $f'(0) \neq 0$. *Then taking* $x_0(t) = \cos t$, *imposing the cancelation of terms corresponding to the first harmonic in the right-hand side of equations* (3.8) *and assuming that initial value* a_0 *is small enough, it results that all subsequent obtained solutions* x_k *are sum of cosines, and* $\omega_k = 0$ *for all* $k \geq 1$.

Proof. If $x_0(t) = \cos t$, we obtain the system of conditions (3.10) to find ω_0 and a_0. According to the above lemma for a_0 small we have $\omega_0 \tau = k\pi$, $k \in \mathbb{Z}$. Then, $x_0(t - \omega_0 \tau) = (-1)^k \cos t$. It follows that

$$\mathcal{N}_0[\phi] = -a_0 \omega_0^2 \cos t + g(x^0 + a_0 \cos t, \beta) - \gamma f(a_0(\cos t - (-1)^k \cos t)). \quad (3.16)$$

Note that $\mathcal{N}_0[\phi]$ can be expressed as a sum of cosines, depending on nonlinearity of f this may be a finite or infinite sum. Then, by solving differential equation (3.9) we observe that x_1 has the form

$$x_1(t) = c_1 \cos t + c_2 \sin t + \sum_{m=0. \ m \neq 1}^{\infty} b_m \cos mt, \quad (3.17)$$

but $x_1(0) = x'_1(0) = 0$, then $c_2 = 0$, and so x_1 is sum of cosines. Note that functions $x_1(t - k\pi)$ and $x''_1(t)$ also are sum of cosines, this will be useful later.

Now let us see that $\omega_1 = 0$. The condition for finding ω_1 come from considering the coefficients of $\cos t$ and $\sin t$ in

$$\frac{\partial \mathcal{N}}{\partial q}\Big|_{q=0} = a_1 \omega_0^2 x_0''(t) + 2a_0 \omega_0 \omega_1 x_0''(t) + a_0 \omega_0^2 x_1''(t)$$
$$+ (a_1 x_0(t) + a_0 x_1(t))\frac{\partial g}{\partial x}(x^0 + a_0 x_0(t), \beta) - \gamma \frac{\partial f}{\partial q}\Big|_{q=0}.$$

Because of parity of x_0 and x_1 the term that contains $\sin t$ is in

$$\frac{\partial f}{\partial q}\Big|_{q=0} = f'(a_0(x_0(t) - x_0(t - \omega_0 \tau)))(a_0 x_1(t) + a_1 x_0(t)$$
$$- a_0 x_1(t - \omega_0 \tau) - a_1 x_0(t - \omega_0 \tau) + a_0 \omega_1 \tau x_0'(t - \omega_0 \tau)),$$

other terms are sums of cosines. Replacing x_0 and ω_0 is obtained

$$\frac{\partial f}{\partial q}\Big|_{q=0} = f'(a_0(1 - (-1)^k)\cos t)(a_0 x_1(t) + a_1 \cos t$$
$$- a_0 x_1(t - k\pi) - a_1(-1)^k \cos t - a_0 \omega_1 \tau(-1)^k \sin t),$$

In particular, the coefficient of $\sin t$ vanishes if

$$\frac{1}{2} a_0 \omega_1 \tau \gamma (-1)^k \int_0^{2\pi} f'(a_0(1 - (-1)^k)\cos(t))\sin^2 t \, dt = 0. \tag{3.18}$$

It can be shown that the integral in the above equation is

$$2\pi f'(0) + 2\pi a_0^2 \sum_{n=1}^{\infty} \frac{2^{4n} a_0^{2(n-1)} f^{(2n)}(0)(\frac{1}{2})_n (\frac{3}{2})_n}{(1)_{2n+2}(1)_{2n}}, \tag{3.19}$$

where we have used Pochhammer symbols $(a)_n = \Gamma(a + n)/\Gamma(a)$. Therefore, as $f'(0) \neq 0$, it follows that for sufficiently small a_0 the integral in (3.18) is not zero, then it should be $\omega_1 = 0$. Again, solving the equation (3.8) for $k = 2$ and taking into account the initial conditions is obtained that term x_2 is a sum of cosines.

Suppose that $\omega_k = 0$ for $1 \leq k \leq n - 1$ and x_k is sum of cosines for $1 \leq k \leq n$.

Let us prove that $\omega_n = 0$. As in case $n = 1$, we consider the coefficients of $\cos t$ and $\sin t$ in $\partial^n \mathcal{N}/\partial q^n|_{q=0}$. The term that contains $\sin t$ is in

$$\frac{\partial^n f}{\partial q^n}\Big|_{q=0}$$
$$= \sum_{\mathbf{m}} \frac{n! f^{(m_1 + \cdots + m_n)}(a_0(1 - (-1)^k)\cos t)}{m_1! m_2! \ldots m_n!} \prod_{j=1}^{n} \left(\frac{1}{j!} \frac{\partial^j (A\phi - A\phi_{\Omega\tau})}{\partial q^j}\Big|_{q=0}\right)^{m_j}, \tag{3.20}$$

where sum is over all n-tuples $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ such that $1m_1 + 2m_2 + \cdots + n\,m_n = n$ (Faà di Bruno's formula).

Derivatives of any order of $A\phi$ on q evaluated at $q = 0$, only contain terms x_k, $0 \leq k \leq n$. Therefore, these derivatives do not provide terms containing $\sin t$. If we write $\omega = \omega_0 + \sum_{k=n}^{\infty} \omega_k q^k$, derivatives of $A\phi_{\Omega\tau}$ are

$$\frac{\partial^j (A\phi_{\Omega\tau})}{\partial q^j}\Big|_{q=0} = j! \sum_{m=0}^{j} a_{j-m} x_m(t - \omega_0 \tau), \quad \text{if } 1 \leq j < n,$$
$$\frac{\partial^n (A\phi_{\Omega\tau})}{\partial q^n}\Big|_{q=0} = n! \sum_{m=0}^{n} a_{n-m} x_m(t - \omega_0 \tau) - n! a_0 \omega_n \tau x_0'(t - \omega_0 \tau). \tag{3.21}$$

Then, in (3.20) the only term that contains $\sin t$ is that with the n-th derivative of $A\phi_{\Omega\tau}$ on q, corresponding to the case $m_i = 0$, $i \neq n$ and $m_n = 1$. Then, we have the condition

$$\frac{1}{2} n! a_0 \omega_n \tau \gamma (-1)^k \int_0^{2\pi} f'(a_0(1-(-1)^k)\cos t) \sin^2 t \, dt = 0, \qquad (3.22)$$

from which, similarly to the case of equation (3.18), it follows $\omega_n = 0$. □

Remark 3.4. Condition that a_0 is small enough is verified in a neighborhood of a Hopf bifurcation. If f is linear the previous lemmas are valid for any value of a_0.

Using the previous lemma in neighborhoods of Hopf bifurcation points of the original system (1.3) it straightforward proves the following theorem.

Theorem 3.5. *Consider the differential equation* (1.3) *with* $f : \mathbb{R} \to \mathbb{R}$ *analytic and* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of class* C^1, $\gamma \neq 0$ *and* $\tau > 0$. *Suppose that the equation has a Hopf bifurcation at the value* μ_0 *(*μ *is bifurcation parameter* β *or* γ*) and that* $f(0) = 0$ *and* $f'(0) \neq 0$. *Suppose further that for some value of* h *the series obtained for periodic solutions with HAM converges. Then, in a neighborhood of the bifurcation, the emerging cycles are isochronic. Furthermore, the frequency of these solutions coincides with the frequency at the point of bifurcation* ω_{μ_0}.

Remark 3.6. If periodic solution exists and k is even, the delay τ is a multiple of the period of the solution. In control theory, this situation is known as non-invasive monitoring scheme, used for stabilizing cycles [12].

4. Some examples

4.1. Anharmonic oscillator.
Consider the equation

$$x'' + x + \beta x^3 = \gamma(x - x_\tau), \qquad (4.1)$$

corresponding to an anharmonic oscillator to which it has been added a delayed term.

In this case $g(x,\beta) = x + \beta x^3$, and $f(x - x_\tau) = x - x_\tau$. The point $x^0 = 0$ is an equilibrium of the system for all β. If $\beta < 0$ there are two new equilibria $x^0 = \pm\sqrt{-1/\beta}$.

We study the cycles around $x^0 = 0$. The characteristic equation (2.1) for this equilibrium does not depend of β. For this reason we consider the gain γ as a bifurcation parameter. For fixed τ, at points $\gamma_0 = (\tau^2 - k^2\pi^2)/(2\tau^2)$ a Hopf bifurcation exists if $\gamma_0\tau \neq 0$ and $k = 2n + 1$, $n \in \mathbb{N}$. The frequency at the bifurcation point is $\omega_{\gamma_0} = \sqrt{1 - 2\gamma_0}$.

We used the HAM as developed in section 3.1 to find periodic solutions arising from the Hopf bifurcation points. Because f is a linear function, the conditions of theorem 3.5 are verified for all a_0. Therefore, the expressions of periodic solutions arising of a Hopf bifurcation point γ_0, have frequency $\omega = \sqrt{1 - 2\gamma_0} = (2n+1)\pi/\tau$.

Equations (3.10) for finding the initial conditions are

$$\begin{aligned} 1 - \omega_0^2 + \frac{3}{4}\beta a_0^2 - \gamma + \gamma \cos \omega_0 \tau &= 0, \\ \gamma \sin \omega_0 \tau &= 0. \end{aligned} \qquad (4.2)$$

Solving the previous system in neighbourhoods of Hopf bifurcation points, we obtain $\omega_0 = (2n+1)\pi/\tau$, and

$$a_0 = \sqrt{\frac{4}{3\beta}(\omega_0^2 - 1 + 2\gamma)} = \sqrt{\frac{4}{3\beta}\Big(\Big(\frac{(2n+1)\pi}{\tau}\Big)^2 - 1 + 2\gamma\Big)}. \qquad (4.3)$$

Then, a_0 exists if $\beta > 0$ and $\gamma > \gamma_0$, or $\beta < 0$ and $\gamma < \gamma_0$. In the case that the solutions found with HAM are convergent, there are in the regions of the plane γ-τ mentioned, at right (left) of Hopf curves if $\beta > 0$ ($\beta < 0$). This information along with the stability of the equilibrium allows us to determine the stability of the periodic solutions of small amplitude. The stability can change due to cycle bifurcations when the parameter continues varying.

In Figure 1 we consider $\beta < 0$, the trivial equilibrium is stable in the shaded region of the plane γ-τ. Also we plot Hopf curves, continuous line corresponding to supercritical bifurcation points (arising cycle is stable) and the dashed line corresponding to subcritical bifurcations. The results were corroborated with the package DDE-Biftool [7]. Setting $\beta = -1$ and $\tau = 2.5$, we have a Hopf point in $\gamma_0 \approx -0.289568$ (black point in Figure 1). For several values of the parameter $\gamma < \gamma_0$ we show the profiles of the solutions obtained with HAM until order 10, it is clear the isochronicity of them. Also we plot some polynomials in variable h mentioned at the end of the section 3.1, for the solution corresponding to $\gamma = -1$.

Figure 1 also shows the points $\tau = 2n\pi$, $n \in \mathbb{N}$, resulting from considering k even. At these points there is a complex conjugate pair of eigenvalues $\pm i$ but derivative in (2.5) vanishes. The system of initial conditions has solution $\omega_0 = 1$ and $a_0 = 0$, and therefore the HAM can not be applied as in the previous section.
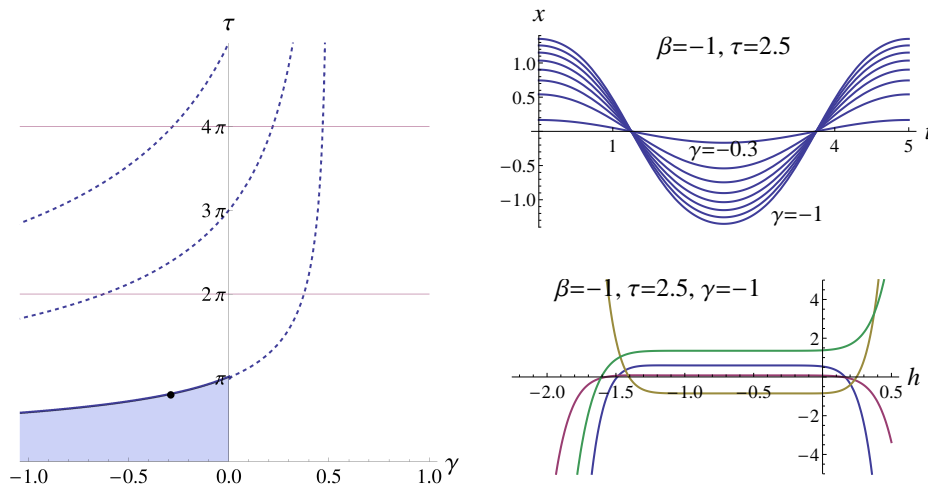

FIGURE 1. Left: Supercritical (continuous line) and subcritical (dashed line) Hopf bifurcation points. Right up: The profiles from Hopf point with $\tau = 2.5$. Right bottom: Some polynomials in h of the cycle to $\beta = -1$, $\tau = 2.5$ and $\gamma = -1$.

4.2. **Rotating pendulum with delay.** Consider a pendulum of mass m and length l restricted to oscillate in a plane. Besides, suppose that the plane rotates around a vertical axis with constant angular velocity ω_r. Let x be the angular deviation of the pendulum from vertical. Performing a normalization in time it is obtained

$$x'' + (\beta - \cos x)\sin x = 0. \tag{4.4}$$

where $\beta = g/(l\,\omega_r^2)$. Pendulum behavior described in the above equation is well known [10]. If $\beta > 1$ there are two equilibria $x^0 = 0$ and $x^0 = \pi$, for the rest and inverted pendulum, respectively. If $0 < \beta < 1$, there is a third equilibrium $x^0 = \arccos\beta$, which is a center; that is, there are a family of closed orbits around this equilibrium.

Consider the function $f(x - x_\tau) = \sin(x - x_\tau)$. Then we consider the equation

$$x'' + (\beta - \cos x)\sin x = \gamma \sin(x - x_\tau). \tag{4.5}$$

This feedback may be interpreted as a torque acting on the pivot. The torque is always perpendicular to the plane in which the pendulum swings.

Now, we discuss the behavior of the new system around the equilibrium $x^0 = \arccos\beta$, taking γ as bifurcation parameter. In a previous work [3] we considered β as bifurcation parameter and the isochronicity was observed using frequency domain methods in addition to the HAM.

Considering τ fixed, we obtain the following solution points of (2.3):

$$\gamma_0 = \frac{\tau^2(1 - \beta^2) - k^2\pi^2}{\tau^2(1 - (-1)^k)}, \quad k \in \mathbb{Z}, \tag{4.6}$$

where $\omega_{\gamma_0} = \sqrt{1 - \beta^2 + \gamma_0((-1)^k - 1)}$, $k \in \mathbb{Z}$. Derivative in (2.5) do not vanish if k is odd and $\gamma_0\tau \neq 0$, in this case the system exhibits a Hopf bifurcation at γ_0.

Using HAM, the initial conditions (3.10) are

$$-a_0\omega_0^2 + 2\beta^2 J_1(a_0) + (1 - 2\beta^2)J_1(2a_0)$$
$$-2\gamma \sum_{n=0}^{\infty}(-1)^n J_{2n+1}(\eta)\left(J_{2n}(\zeta) + J_{2n+2}(\zeta)\right) = 0,$$
$$\gamma \sum_{n=0}^{\infty}(-1)^n J_{2n+1}(\zeta)\left(J_{2n}(\eta) + J_{2n+2}(\eta)\right) = 0, \tag{4.7}$$

where $\eta = a_0(1 - \cos(\omega_0\tau))$ and $\zeta = a_0\sin(\omega_0\tau)$. The above equations were obtained using the Jacobi-Anger expressions for the composition of trigonometric functions. The nonlinearity of the system generates a function from which we can not easily obtain a_0 as done in the previous example.

Despite the complexity of the considered system, the conditions of the theorem 3.5 are verified. Then, given a Hopf bifurcation point γ_0, if the series obtained with HAM are convergent, it is possible to ensure isochronicity of periodic solutions associated with that bifurcation for values of γ in a neighbourhood of γ_0.

In Figure 2 we plot the maximum (in the original variable x) of the periodic solutions arising from Hopf points, together with the numerical solutions obtained with DDE-Biftool. In the left figure we consider the values $\beta = 0.5$ and $\tau = 2$ while in the right we used $\beta = 0.3$ and $\tau = 7$. In Figure 3 we show the profiles of the solutions for the two cases considered above.
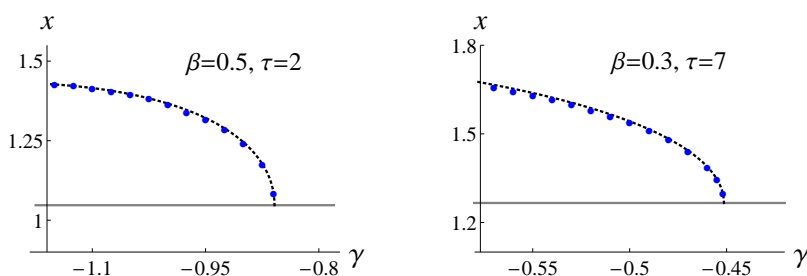
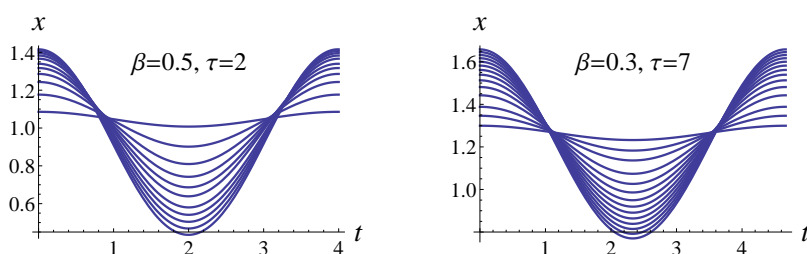FIGURE 2. Maximum of the cycles. Left: $\beta = 0.5$ and $\tau = 2$. Right: $\beta = 0.3$ and $\tau = 7$.

FIGURE 3. Left: Profiles for $\beta = 0.5$ and $\tau = 2$. Right: Profiles for $\beta = 0.3$ and $\tau = 7$.

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