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# A MIXED PROBLEM FOR SEMILINEAR WAVE EQUATIONS WITH ACOUSTIC BOUNDARY CONDITIONS IN DOMAINS WITH NON-LOCALLY REACTING BOUNDARY

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Abstract. In this article we study the existence, uniqueness and asymptotic stability of solution to the mixed problem for the semilinear wave equation with acoustic boundary conditions in domains with non-locally reacting boundary. We also prove the existence and uniqueness of solution to a problem with nonmonotone dissipative term.

## 1. Introduction

This article is devoted to the study of the existence, uniqueness and uniform stabilization of solutions $(u,\delta)$ for the problem

$$\begin{gathered}
u'' - \Delta u + \rho(u') = F \quad \text{in } \Omega \times (0,\infty), \\
u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
\frac{\partial u}{\partial \nu} = \delta' \quad \text{on } \Gamma_1 \times (0,\infty), \\
u' + f\delta'' - c^2 \Delta_\Gamma \delta + g\delta' + h\delta = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\
\delta = 0 \quad \text{on } \partial\Gamma_1 \times (0,\infty), \\
u(x,0) = \phi(x), \quad u'(x,0) = \psi(x) \quad x \in \Omega, \\
\delta(x,0) = \theta(x), \quad \delta'(x,0) = \frac{\partial \phi}{\partial \nu}(x), \quad x \in \Gamma_1,
\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded and connected set with smooth boundary $\Gamma$; $\Gamma_1$ is an open and connected set of $\Gamma$ with smooth boundary, $\partial\Gamma_1$, and $\Gamma_0 = \Gamma \backslash \Gamma_1$. Here $' = \frac{\partial}{\partial t}$, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $\Delta_\Gamma$ are the spatial Laplace and Laplace-Beltrami operators, respectively, $\nu$ is the unit outward normal vector to $\Gamma$, $c$ is a positive constant, $\rho : \mathbb{R} \to \mathbb{R}$, $F : \Omega \times (0,\infty) \to \mathbb{R}$, $f, g, h : \overline{\Gamma_1} \to \mathbb{R}$, $\phi, \psi : \Omega \to \mathbb{R}$ and $\theta : \Gamma_1 \to \mathbb{R}$ are given functions.

When $c = 0$ the boundary conditions $(1.1)_3$–$(1.1)_4$ are the classical acoustic boundary conditions which were introduced by Beale and Rosencrans [2] in wave

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propagation literature. Precisely, they derived

$$\begin{gathered}
u'' - \Delta u = 0 \quad \text{in } \Omega \times (0,\infty), \\
\frac{\partial u}{\partial \nu} = \delta' \quad \text{on } \Gamma \times (0,\infty), \\
\rho_0 u' + m\delta'' + d\delta' + k\delta = 0 \quad \text{on } \Gamma \times (0,\infty),
\end{gathered} \tag{1.2}$$

as a theoretical model to describe the acoustic wave motion into a fluid in $\Omega \subset \mathbb{R}^3$. Here $\rho_0, m, d, k$ are physical constants. The function $u(x,t)$ is the velocity potential of the fluid and $\delta(x,t)$ models the normal displacement of the point $x \in \Gamma$ in the time $t$. To obtain the model, Beale and Rosencrans assumed that each point of the surface $\Gamma$ acts like a spring in response to the excess pressure and that each point of $\Gamma$ does not influence each other. Surfaces with this characteristic are called *locally reacting*, see Morse and Ingard [22]. Acoustic boundary conditions has been studied by several authors [8, 9, 10, 11, 19, 25, 26]. Frota, Cousin and Larkin [9] obtained decay results to a nonlinear wave equation when $n = 1$ and Frota, Cousin and Larkin [10] and Park and Park [25] obtained decay results when $m = 0$ in $(1.2)_3$. The results of Liu and Sun [19] and Park and Park [25] are about general decay rates.

Recently, in [12], we introduced a new physical formulation to the acoustic conditions and we proved the existence, uniqueness and uniform stabilization to (1.1) with

$$u'' - M\Big(\int_\Omega u^2\,dx\Big)\Delta u + \alpha u' + \beta |u'|^p u' = 0 \quad \text{in } \Omega \times (0,\infty) \tag{1.3}$$

instead of $(1.1)_1$, where $\alpha, \beta$ are positive constants, $p > 1$ if $n = 2$, $1 < p \le 2$ if $n = 3$ and $M : (0,\infty) \to \mathbb{R}$ is a given function. The boundary conditions $(1.1)_3$–$(1.1)_4$ were called *acoustic boundary conditions to non-locally reacting boundary*, because for its formulation we assumed that the surface $\Gamma_1$ acts like an elastic membrane in response to the excess pressure. The exponential decay was obtained by Nakao's Lemma [23, Theorem 1], and the assumption

$$c > 4k_1k_2, \tag{1.4}$$

was necessary. Here $c$ is the same of $(1.1)_4$, $k_1$ is the constant of continuity of the trace map and $k_2$ is the Poincaré's constant. This is a strong assumption and physically it means that the velocity of wave propogation on the surface $\Gamma_1$ is higher than a known constant. The case $\beta = 0$ in (1.3) was considered by Vicente and Frota [30], where the authors assumed sufficiently small data to prove the existence of solutions. The assumption (1.4) also was necessary to prove the asymptotic stability. See also the recent work due Vicente and Frota [31] were the authors considered nonlinear boundary equation. See also the papers of Coclite, Goldstein and Goldstein [4, 5, 6, 7] were the authors studied problems with Wentzell boundary conditions. An interesting work is due Gal, G. Goldstein and J. Goldstein [13] were they proved, in some special case, the acoustic and Wentzell boundary conditions are closely related.

On this direction it is important to mention the works of Graber and Said-Houari [14] and Graber [15] were the authors also studied problems with nonhomogeneous boundary condition. Precisely in [14] the authors studied the interaction between dissipative and sources terms in the domain as well as in the boundary, for the strongly damped wave equation with boundary conditions like General Wentzell Boundary Condition (GWBC) type, which in some special case agree with Acoustic

Boundary Conditions (ABC), see section 4 of [13]. Moreover in contrast with our results on stability and decay rates, due to the source term the authors did an analysis of blow up of solutions. On the other hand in [15] the author proved the existence, uniqueness and uniform decay of finite energy solutions when the boundary is taken to be porous and the coupling between the interior and boundary dynamics is assumed to be nonlinear, furthermore was added to the boundary a frictional damping term as a feedback control. Theorem 3 of [15], which gives decay rates result, assumed a geometrical condition on a the set $\Gamma_0$.

Our purpose in this article is to study the existence, uniqueness and asymptotic behaviour of solution to (1.1). The existence is done by employing Galerkin's procedure and compactness arguments. Our main goal is to prove the exponential decay of solution to (1.1) without the assumption (1.4). This is possible because we use an alternative method based on an integral inequality instead of Nakao's Lemma. This technics are used by many authors [17, 20, 27]. We observe that the estimates we made here can be adapted to the problem treated by Frota, Medeiros and Vicente [12] therefore here we also improve their results. Moreover, as an application of (1.1), we study the existence and uniqueness of global solution to (1.1) without the assumption of monotonicity on $\rho$ ($\rho' \geq 0$). In some related papers, when $\rho$ is non-monotone, the authors prove the existence of solution by semigroups theory [20, 24, 28]. Here we solve the problem building an appropriate sequence of functions which converge to the solution of the problem with non-monotone dissipative term, it is an adaptation of the ideas of Li and Tsai [21].

Our paper is organized as follows: In Section 2 we present some notations. In Section 3 we prove the existence, uniqueness and uniform decay of solution to (1.1). Finally, in Section 4 we give the proof of the existence and uniqueness of solution to (1.1) for the nonmonotone case.

## 2. Notation

The inner product and norm in $L^2(\Omega)$ and $L^2(\Gamma_1)$ are denoted, respectively, by

$$
\begin{gathered}
(u,v)=\int_\Omega u(x)v(x)\,dx,\quad |u|=\Big(\int_\Omega (u(x))^2\,dx\Big)^{1/2},\\
(\delta,\theta)_{\Gamma_1}=\int_{\Gamma_1}\delta(x)\theta(x)\,d\Gamma\,,\quad |\delta|_{\Gamma_1}=\Big(\int_{\Gamma_1}(\delta(x))^2\,d\Gamma\Big)^{1/2}.
\end{gathered}
$$

Let $O\subset\Omega$ and $p\geq 1$, we denote by

$$
|u|_{L^p(O)}=\Big(\int_O |u(x)|_{\mathbb{R}}^p\,dx\Big)^{1/p}
$$

the norm in $L^p(O)$. We define a closed subspace of the Sobolev space $H^1(\Omega)$ as

$$
V=\{u\in H^1(\Omega);\gamma_0(u)=0 \text{ a.e. on } \Gamma_0\},
$$

where $\gamma_0 : H^1(\Omega)\to H^{1/2}(\Gamma)$ is the trace map of order zero and $H^{1/2}(\Gamma)$ is the Sobolev space of order $\frac{1}{2}$ defined over $\Gamma$, as introduced by Lions and Magenes [18]. We define the following inner product and norm in $V$:

$$
((u,v))=\sum_{i=1}^n\int_\Omega \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)\,dx,\quad \|u\|=\Big(\sum_{i=1}^n\int_\Omega\Big(\frac{\partial u}{\partial x_i}(x)\Big)^2dx\Big)^{1/2},
$$

respectively. The Poincaré's Inequality holds in $V$, thus there exists a constant $k_0$ such that

$$|u| \le k_0\|u\|, \quad \text{for all } u \in V. \tag{2.1}$$

From (2.1) we can prove the equivalence, in $V$, of the norm $\|\cdot\|$ and the usual norm $\|\cdot\|_{H^1(\Omega)} = \big(|u|^2 + \|u\|^2\big)^{1/2}$ of $H^1(\Omega)$.

As $\Gamma_1$ is a compact Riemannian manifold, with boundary, endowed with the natural metric inherited from $\mathbb{R}^n$, it is possible to give an intrinsic definition of the space $H^k(\Gamma_1)$ by using the covariant derivative operator. In fact, the Sobolev space $H^k(\Gamma_1)$ is the completion of $C^\infty(\Gamma_1)$ with respect to the norm

$$\|u\|_{H^k(\Gamma_1)} = \Big[\sum_{l=0}^{k} |\nabla^l u|^2_{\Gamma_1}\Big]^{1/2},$$

where $\nabla^l$ is the covariant derivative operator of order $l$, see Hebey [16]. We also consider the space $H^1_0(\Gamma_1)$ the closure of $C_0^\infty(\Gamma_1)$ in $H^1(\Gamma_1)$. The Poincaré's inequality holds in $H^1_0(\Gamma_1)$, thus there exists a constant $k_1$ such that

$$|\delta|_{\Gamma_1} \le k_1 |\nabla_\tau \delta|_{\Gamma_1}, \quad \text{for all } \delta \in H^1_0(\Gamma_1),$$

where $\nabla_\tau$ is the tangential gradient on $\Gamma_1$, see Taylor [29]. This allow us to consider the space $H^1_0(\Gamma_1)$ equipped with the norm and inner product:

$$\|\delta\|_{\Gamma_1} = |\nabla_\tau \delta|_{\Gamma_1} \quad \text{and} \quad ((\delta,\theta))_{\Gamma_1} = \int_{\Gamma_1} \langle \nabla_\tau \delta(x), \nabla_\tau \theta(x)\rangle \, d\Gamma.$$

We denote by $\Delta_\Gamma = \operatorname{div}_\tau \nabla_\tau$ the Laplace-Beltrami operator. We define the operator $-\Delta_\Gamma : H^1_0(\Gamma_1) \to H^{-1}(\Gamma_1)$ such that

$$\langle -\Delta_\Gamma \delta, \theta\rangle_{H^{-1}(\Gamma_1)\times H^1_0(\Gamma_1)} = \int_{\Gamma_1} \langle \nabla_\tau \delta(x)\,,\, \nabla_\tau \theta(x)\rangle \, d\Gamma,$$

for all $\delta, \theta \in H^1_0(\Gamma_1)$. We consider $(H^1_0(\Gamma_1) \cap H^2(\Gamma_1))$ endowed with the norm

$$|\delta|_{H^1_0(\Gamma_1)\cap H^2(\Gamma_1)} = |\Delta_\Gamma \delta|_{\Gamma_1} \quad \text{for all } \delta \in (H^1_0(\Gamma_1) \cap H^2(\Gamma_1)),$$

which is equivalent to $\|\cdot\|_{H^2(\Gamma_1)}$, see Biezuner [3].

We define the subspace $W$ of $V$ as

$$W = \{u \in (V \cap H^3(\Omega));\ (\gamma_1(u))_{|\Gamma_1} \in H^1_0(\Gamma_1)\},$$

here $\gamma_1 : H(\Delta,\Omega) \to H^{-\frac12}(\Gamma)$ is the Neumann trace map and $H(\Delta,\Omega) = \{u \in H^1(\Omega); \Delta u \in L^2(\Omega)\}$ is equipped with the norm

$$\|u\|_{H(\Delta,\Omega)} = \Big(\|u\|^2_{H^1(\Omega)} + |\Delta u|^2\Big)^{1/2}.$$

We equipped $W$ with the norm

$$\|u\|_W = \Big(\|u\|^2 + \|u\|^2_{H^3(\Omega)}\Big)^{1/2}.$$

The space $W$ is necessary to estimate $u''_m(0)$ in the estimate 2. We observe that $W$ is dense in $V$. As $\gamma_0$ and $\gamma_1$ are continuous there exist positive constants $k_2$ and $k_3$ such that

$$|\gamma_0(u)|_{\Gamma_1} \le k_2\|u\|_W \quad \text{and} \quad |\gamma_1(u)|_{\Gamma_1} \le k_3\|u\|_W, \quad \text{for all } u \in W. \tag{2.2}$$

Results about Sobolev's Spaces on manifolds can be found in Aubin [1], Hebey [16], Lions and Magenes [18] and Taylor [29].

## 3. Main Results

We assume

$$f, g, h \in C(\overline{\Gamma_1}) \text{ with } f, g > 0 \text{ and } h \geq 0. \tag{3.1}$$

We suppose that $\rho \in C^1(\mathbb{R})$ and there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$|\rho(x)| \leq \kappa_1 |x|^q \quad \text{if } |x| > 1, \text{ and } |\rho(x)| \leq \kappa_2 |x| \text{ if } |x| \leq 1, \tag{3.2}$$

where $q$ satisfies $1 \leq q \leq \frac{n+2}{n-2}$ if $n \geq 3$, or $q \geq 1$ if $n = 2$. We also assume

$$\rho'(x) \geq \kappa_3 > 0, \quad \text{for all } x \in \mathbb{R}. \tag{3.3}$$

**Theorem 3.1.** *Suppose that (3.1)–(3.3) hold. Let $(\phi, \psi, \theta) \in W \times (V \cap L^{2q}(\Omega)) \times (H_0^1(\Gamma_1) \cap H^2(\Gamma_1))$ and $F, F' \in L^2_{\rm loc}(0, \infty; L^2(\Omega))$. Then, there exists a unique pair $(u, \delta)$ such that*

$$u, u' \in L^\infty_{\rm loc}(0, \infty; V), \quad u'' \in L^\infty_{\rm loc}(0, \infty; L^2(\Omega)), \quad u(t) \in H(\Delta, \Omega) \text{ a.e. in } [0, \infty),$$
$$\delta \in L^\infty_{\rm loc}(0, \infty; H_0^1(\Gamma_1) \cap H^2(\Gamma_1)), \quad \delta' \in L^\infty_{\rm loc}(0, \infty; H_0^1(\Gamma_1)),$$
$$\delta'' \in L^\infty_{\rm loc}(0, \infty; L^2(\Gamma_1))$$

*and, for all $T > 0$,*

$$\begin{gathered} u'' - \Delta u + \rho(u') = F \quad \text{a.e. in } \Omega \times (0, T), \\ \langle \gamma_1(u(t)), \gamma_0(w) \rangle = \big(\delta'(t), \gamma_0(w)\big)_{\Gamma_1} \quad \text{a.e. in } [0, T], \text{ for all } w \in V, \\ u' + f\delta'' - c^2 \Delta_\Gamma \delta + g\delta' + h\delta = 0 \quad \text{a.e. in } \Gamma_1 \times (0, T), \\ u(0) = \phi, \quad u'(0) = \psi, \quad \delta(0) = \theta. \end{gathered} \tag{3.4}$$

*Proof.* Let $(w_j)_{j \in \mathbb{N}}$ be an orthonormal basis in $W$ and let $(z_j)_{j \in \mathbb{N}}$ be the orthonormal basis in $L^2(\Gamma_1)$ given by eigenfunctions of the operator $-\Delta_\Gamma$. Since the embedding $H_0^1(\Gamma_1) \hookrightarrow L^2(\Gamma_1)$ is compact the existence of the special basis $(z_j)_{j \in \mathbb{N}}$ is a consequence of the Spectral Theory and it is an orthogonal basis of $(H_0^1(\Gamma_1) \cap H^2(\Gamma_1))$ and $H_0^1(\Gamma_1)$. For each $m \in \mathbb{N}$, we consider $u_m : \Omega \times [0, T_m] \to \mathbb{R}$ and $\delta_m : \Gamma_1 \times [0, T_m] \to \mathbb{R}$ of the form

$$u_m(x, t) = \sum_{j=1}^m \alpha_{jm}(t) w_j(x) \quad \text{and} \quad \delta_m(x, t) = \sum_{j=1}^m \beta_{jm}(t) z_j(x)$$

the local solution of the approximate problem

$$(u_m''(t) + \rho(u_m'(t)) - F(t), w_j) + ((u_m(t), w_j)) - (\delta_m'(t), \gamma_0(w_j))_{\Gamma_1} = 0, \tag{3.5}$$

$$(\gamma_0(u_m'(t)) + f\delta_m''(t) + g\delta_m'(t) + h\delta_m(t), z_j)_{\Gamma_1} + c^2 \left((\delta_m(t), z_j)\right)_{\Gamma_1} = 0, \tag{3.6}$$

$$u_m(0) = \sum_{i=1}^m \phi^i w_i \to \phi \text{ in } W;\ u_m'(0) = \sum_{i=1}^m \psi^i w_i \to \psi \text{ in } V \cap L^{2q}(\Omega), \tag{3.7}$$

$$\delta_m(0) = \sum_{i=1}^m \theta^i z_i \to \theta \text{ in } H_0^1(\Gamma_1) \cap H^2(\Gamma_1), \delta_m'(0) = (\gamma_1(u_{0m}))_{|\Gamma_1}. \tag{3.8}$$

Here $1 \leq j \leq m$ and $\phi^i, \psi^i, \theta^i$, $i = 1, \dots, m$, are known scalars. The estimate 1 will allow us to extend the local solution to the whole interval $[0, T]$, for all $T > 0$. From (3.5) and (3.6) we have the following approximate equations

$$(u_m''(t) + \rho(u_m'(t)) - F(t), w) + ((u_m(t), w)) - (\delta_m'(t), \gamma_0(w))_{\Gamma_1} = 0, \tag{3.9}$$

$$(\gamma_0(u_m'(t)) + f\delta_m''(t) + g\delta_m'(t) + h\delta_m(t), z)_{\Gamma_1} + c^2 \left((\delta_m(t), z)\right)_{\Gamma_1} = 0, \tag{3.10}$$

which hold for all $w \in \operatorname{span}\{w_1, \dots, w_m\}$ and $z \in \operatorname{span}\{z_1, \dots, z_m\}$, respectively.

**Estimate 1:** Setting $w = u'_m(t)$ and $z = \delta'_m(t)$ in (3.9) and (3.10), respectively, we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[|u'_m(t)|^2 + \|u_m(t)\|^2 + |f^{1/2}\delta'_m(t)|^2_{\Gamma_1} + |h^{1/2}\delta_m(t)|^2_{\Gamma_1} + c^2\|\delta_m(t)\|^2_{\Gamma_1}\Big]\\ &+ (\rho(u'_m(t)), u'_m(t)) + |g^{1/2}\delta'_m(t)|^2_{\Gamma_1}\\ &= (F(t), u'_m(t)).\end{aligned}$$

Employing the inequality $ab \le \frac{a^2}{2} + \frac{b^2}{2}$ and integrating over $(0,t)$, we have

$$\begin{aligned}&|u'_m(t)|^2 + \|u_m(t)\|^2 + |f^{1/2}\delta'_m(t)|^2_{\Gamma_1} + |h^{1/2}\delta_m(t)|^2_{\Gamma_1} + c^2\|\delta_m(t)\|^2_{\Gamma_1}\\ &+ \int_0^t (\rho(u'_m(\xi)), u'_m(\xi)) + |g^{1/2}\delta'_m(\xi)|^2_{\Gamma_1} d\xi\\ &\le |\psi|^2 + \|\phi\|^2 + k_2^2 \max_{x\in\overline{\Gamma_1}} |f| \|\phi\|^2_W + \max_{x\in\overline{\Gamma_1}} |h| |\theta|^2_{\Gamma_1}\\ &\quad + c^2\|\theta\|^2_{\Gamma_1} + \int_0^t |F(\xi)|^2 + |u'_m(\xi)|^2\, d\xi.\end{aligned}$$

Now, taking into account (3.2), (3.3) and employing Gronwall's lemma we conclude that there exists a positive constant $C_1$, which does not depend on $m$ and $t$, such that

$$|u'_m(t)|^2 + \|u_m(t)\|^2 + |\delta'_m(t)|^2_{\Gamma_1} + |\delta_m(t)|^2_{\Gamma_1} + \|\delta_m(t)\|^2_{\Gamma_1} \le C_1, \tag{3.11}$$

for all $t \in [0,T]$, where $T > 0$ is arbitrary.

**Estimate 2:** Substituting $t = 0$ in (3.9)–(3.10) and using Green's Formula we obtain

$$(u''_m(0) + \Delta u_m(0) + \rho(u'_m(0)) - F(0), w) = 0, \tag{3.12}$$

$$\big(\gamma_0(u'_m(0)) + f\delta''_m(0) + c^2\Delta_\Gamma\delta_m(0) + g\delta'_m(0) + h\delta_m(0), z\big)_{\Gamma_1} = 0. \tag{3.13}$$

Considering

$$\Omega_1 = \{x \in \Omega;\ |u'_m(x,0)| \le 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega;\ |u'_m(x,0)| > 1\}$$

from the assumption (3.2) and the inclusion $L^{2q}(\Omega) \hookrightarrow L^2(\Omega)$ we obtain

$$\begin{aligned}\Big(\int_{\Omega_1} |\rho(u'_m(x,0))|^2 dx\Big)^{1/2} &\le C_2\Big(\int_{\Omega_1} |u'_m(x,0)|^2 dx\Big)^{1/2}\\ &\le C_2|u'_m(0)| \le C_3|u'_m(0)|_{L^{2q}(\Omega)}\\ &\le C_3(|u'_m(0)|^q_{L^{2q}(\Omega)} + |u'_m(0)|_{L^{2q}(\Omega)})\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\Big(\int_{\Omega_2} |\rho(u'_m(x,0))|^2 dx\Big)^{1/2} &\le C_4\Big(\int_{\Omega_2} |u'_m(x,0)|^{2q} dx\Big)^{1/2}\\ &\le C_5|u'_m(0)|^q_{L^{2q}(\Omega)}\\ &\le C_5(|u'_m(0)|^q_{L^{2q}(\Omega)} + |u'_m(0)|_{L^{2q}(\Omega)}).\end{aligned} \tag{3.15}$$

Thanks to (3.7), (3.14) and (3.15) we deduce

$$(\rho(u'_m(0)), w) \le |\rho(u'_m(0))||w| \le C_6|w|. \tag{3.16}$$

Taking $w = u''_m(0)$ and $z = \delta''_m(0)$ in (3.12) and (3.13), respectively, using Hölder's inequality, noting (3.7), (3.8) and (3.16) we obtain

$$|u''_m(0)| + |\delta''_m(0)|_{\Gamma_1} \le C_7. \tag{3.17}$$

Differentiating (3.9) and (3.10) with respect to $t$ and taking $w = u''_m(t)$ and $z = \delta''_m(t)$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[|u''_m(t)|^2 + \|u'_m(t)\|^2 + |f^{1/2}\delta''_m(t)|^2_{\Gamma_1} + |h^{1/2}\delta'_m(t)|^2_{\Gamma_1} + c^2\|\delta'_m(t)\|^2_{\Gamma_1}\Big] \\ &+ (\rho'(u'_m(t))u''_m(t), u''_m(t)) + |g^{1/2}\delta''_m(t)|^2_{\Gamma_1} = (F'(t), u''_m(t)). \end{aligned} \tag{3.18}$$

Integrating from 0 to $t$, taking into account (3.3) and (3.17) we have

$$\begin{aligned} &|u''_m(t)|^2 + \|u'_m(t)\|^2 + |f^{1/2}\delta''_m(t)|^2_{\Gamma_1} + |h^{1/2}\delta'_m(t)|^2_{\Gamma_1} + c^2\|\delta'_m(t)\|^2_{\Gamma_1} + |g^{1/2}\delta''_m(\xi)|^2_{\Gamma_1} \\ &\le C_8 + \int_0^t |F'(\xi)|^2 + |u''_m(\xi)|^2\,d\xi. \end{aligned} \tag{3.19}$$

This inequality and Gronwall's lemma yield

$$|u''_m(t)|^2 + \|u'_m(t)\|^2 + |\delta''_m(t)|^2_{\Gamma_1} + |\delta'_m(t)|^2_{\Gamma_1} + \|\delta'_m(t)\|^2_{\Gamma_1} \le C_9, \tag{3.20}$$

for all $t \in [0,T]$, where $T > 0$ is arbitrary. Here $C_9$ is independent of $m$ and $t$.

**Estimate 3:** Substituting $z = \Delta_\Gamma\delta_m(t)$ in (3.10) we find

$$|\Delta_\Gamma\delta_m(t)|_{\Gamma_1} \le C_{10}\Big[|f^{1/2}\delta''_m(t)|_{\Gamma_1} + |g^{1/2}\delta'_m(t)|_{\Gamma_1} + |h^{1/2}\delta_m(t)|_{\Gamma_1} + |\gamma_0(u'_m(t))|_{\Gamma_1}\Big],$$

for all $t \in [0,T]$. As $f, g, h \in C(\overline{\Gamma_1})$ and the map $\gamma_0$ is continuous we obtain

$$|\Delta_\Gamma\delta_m(t)|_{\Gamma_1} \le C_{11}\Big[|\delta''_m(t)|_{\Gamma_1} + |\delta'_m(t)|_{\Gamma_1} + |\delta_m(t)|_{\Gamma_1} + \|u'_m(t)\|\Big].$$

Taking into account (3.11) and (3.20) we conclude that

$$|\Delta_\Gamma\delta_m(t)|_{\Gamma_1} \le C_{12}, \quad \text{for all } t \in [0,T], \tag{3.21}$$

where $T > 0$ is arbitrary.

**Passage to the Limit:** Using the estimates (3.11), (3.20)–(3.21) and compactness argument, we can see that there exist a subsequence of $(u_m)_{m\in\mathbb{N}}$ and a subsequence of $(\delta_m)_{m\in\mathbb{N}}$, which we will denote by the same notations, and functions $u$ and $\delta$, such that

$$\begin{aligned} &u_m \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V) && \delta_m \overset{*}{\rightharpoonup} \delta \text{ in } L^\infty(0,T;H_0^1(\Gamma_1)\cap H^2(\Gamma_1)) \\ &u'_m \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0,T;V) && \delta'_m \overset{*}{\rightharpoonup} \delta' \text{ in } L^\infty(0,T;H_0^1(\Gamma_1)) \\ &u''_m \overset{*}{\rightharpoonup} u'' \text{ in } L^\infty(0,T;L^2(\Omega)) && \delta''_m \overset{*}{\rightharpoonup} \delta'' \text{ in } L^\infty(0,T;L^2(\Gamma_1)). \end{aligned} \tag{3.22}$$

The convergences above and the Aubin-Lions Theorem lead us to

$$u_m \to u \text{ in } L^2(0,T;L^2(\Omega)), \quad u'_m \to u' \text{ in } L^2(0,T;L^2(\Omega)). \tag{3.23}$$

Convergences (3.22) and (3.23) are sufficient to pass to the limit in approximate problem (3.5)–(3.8) and we can conclude $(u,\delta)$ satisfies (3.4) (see Frota, Medeiros and Vicente [12]). By the regularity (3.22) we apply the energy method to obtain the uniqueness. $\square$

Now, we prove the exponential decay of the energy associated to (1.1) when $F \equiv 0$. Let $(u, \delta)$ be the solution of (1.1) given by Theorem 3.1. We define the energy by

$$E(t) = \frac{1}{2}\Big[|u'(t)|^2 + \|u(t)\|^2 + |f^{1/2}\delta'(t)|^2_{\Gamma_1} + c^2\|\delta(t)\|^2_{\Gamma_1} + |h^{1/2}\delta(t)|^2_{\Gamma_1}\Big]. \quad (3.24)$$

**Lemma 3.2.** *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and assume that there exists a constant $A > 0$ such that*

$$\int_t^\infty \varphi(\xi)\, d\xi \le A\varphi(t), \quad \text{for all } t \in \mathbb{R}^+.$$

*Then*

$$\varphi(t) \le \varphi(0) \exp\big(1 - \frac{t}{A}\big), \quad \text{for all } t \ge 0.$$

For a proof of the above lemma, see Komornik [17].

Suppose $F \equiv 0$. Multiplying $(3.4)_1$ by $u'$, $(3.4)_3$ by $\delta'$ and integrating over $\Omega$ and $\Gamma_1$, respectively, we see

$$E'(t) = -\left(\rho(u'(t)), u'(t)\right) - |g^{1/2}\delta'(t)|^2_{\Gamma_1} \le 0. \quad (3.25)$$

**Lemma 3.3.** *If all assumptions of Theorem 3.1 hold and moreover $F \equiv 0$, then we have*

$$2\int_S^T E(t)\, dt \le C_{14}E(S) - \int_S^T (u(t), \rho(u'(t)))\, dt + \int_S^T (2u(t) - g\delta(t), \delta'(t))_{\Gamma_1}\, dt, \quad (3.26)$$

*for all $0 \le S < T < \infty$.*

*Proof.* Multiplying $(3.4)_1$ by $u$, $(3.4)_3$ by $\delta$, integrating over $\Omega \times (S,T)$ and $\Gamma_1 \times (S,T)$, respectively, and integrating by parts we obtain

$$\begin{aligned}
&\int_S^T (|u'(t)|^2 - \|u(t)\|^2) dt + \int_S^T (|f^{1/2}\delta'(t)|^2_{\Gamma_1} - c^2\|\delta(t)\|^2_{\Gamma_1}) dt - \int_\Omega uu' dx\Big|_S^T \\
&- \int_{\Gamma_1} (f\delta\delta' + u\delta) d\Gamma\Big|_S^T + \int_S^T [2\left(\delta'(t), u(t)\right)_{\Gamma_1} - |h^{1/2}\delta(t)|^2_{\Gamma_1}] dt \\
&- \int_S^T (u(t), \rho(u'(t)))\, dt - \int_S^T (g\delta(t), \delta'(t))_{\Gamma_1}\, dt = 0.
\end{aligned} \quad (3.27)$$

We note that

$$\begin{aligned}
&- \int_S^T (\|u(t)\|^2 + c^2\|\delta(t)\|^2_{\Gamma_1} + |h^{1/2}\delta(t)|^2_{\Gamma_1}) dt \\
&= -2\int_S^T E(t) dt + \int_S^T (|u'(t)|^2 + |f^{1/2}\delta'(t)|^2_{\Gamma_1}) dt.
\end{aligned} \quad (3.28)$$

Combining (3.27) and (3.28), we infer that

$$\begin{aligned}
2\int_S^T E(t) dt = &- \int_\Omega uu' dx\Big|_S^T - \int_{\Gamma_1} (f\delta\delta' + \delta u) d\Gamma\Big|_S^T \\
&+ 2\int_S^T [|u'(t)|^2 + |f^{1/2}\delta'(t)|^2_{\Gamma_1} + (\delta'(t), u(t))_{\Gamma_1}] dt \\
&- \int_S^T (u(t), \rho(u'(t)))\, dt - \int_S^T (g\delta(t), \delta'(t))_{\Gamma_1}\, dt.
\end{aligned} \quad (3.29)$$

By Hölder and Young's inequalities and (3.24), we have

$$\Big|\int_\Omega uu'\,dx\Big| \le \frac{1}{2}\int_\Omega |u|^2 dx + \frac{1}{2}\int_\Omega |u'|^2 dx \le C_{15}E(t),$$

thus

$$\int_\Omega uu'\,dx\Big|_S^T \le C_{16}E(S). \tag{3.30}$$

Analogously,

$$\int_{\Gamma_1} (f\delta\delta' + \delta u)\,d\Gamma\Big|_S^T \le C_{17}E(S). \tag{3.31}$$

From (3.3) and (3.25), we obtain

$$\begin{aligned} 2\int_S^T [|u'(t)|^2 + |f^{1/2}\delta'(t)|^2_{\Gamma_1}]dt &\le \frac{2}{\kappa_3}\int_S^T \int_\Omega \rho(u')u'dx + C_{18}\int_{\Gamma_1} g(\delta')^2 d\Gamma\,dt \\ &\le C_{19}\int_S^T (\rho(u'(t)), u'(t)) + |g^{1/2}\delta'(t)|^2_{\Gamma_1} dt \\ &= -C_{19}\int_S^T E'(t)dt \le C_{20}E(S). \end{aligned} \tag{3.32}$$

Combining (3.29)–(3.32), we obtain (3.26). □

**Theorem 3.4.** *If all assumptions of Theorem 3.1 hold and $F \equiv 0$, then there exist positive constants $a$ and $b$ such that $E(t) \le a\exp(-bt)$, for all $t \ge 0$.*

*Proof.* For each $t > 0$ fixed, we set

$$\Omega_1^t = \{x \in \Omega;\ |u'(x,t)| \le 1\} \quad\text{and}\quad \Omega_2^t = \{x \in \Omega;\ |u'(x,t)| > 1\}.$$

From (3.2) and (3.25), we have

$$\begin{aligned} |\rho(u'(t))|^2_{L^2(\Omega_1^t)} &= \int_{\Omega_1^t} |\rho(u')||\rho(u')|dx \le C_{21}\int_{\Omega_1^t} |u'||\rho(u')|dx \\ &\le C_{21}\,(\rho(u'(t)), u'(t)) \\ &= -C_{21}[E'(t) + |g^{1/2}\delta'(t)|^2_{\Gamma_1}] \le C_{22}|E'(t)|; \end{aligned}$$

therefore,

$$\Big|\int_{\Omega_1^t} u\rho(u')dx\Big| \le |u(t)|_{L^2(\Omega_1^t)}|\rho(u'(t))|_{L^2(\Omega_1^t)} \le C_{23}E^{1/2}(t)|E'(t)|^{1/2}. \tag{3.33}$$

Using the Sobolev's imbedding $V \hookrightarrow L^{q+1}(\Omega)$, (3.2), (3.24) and (3.25), we obtain

$$\begin{aligned} \Big|\int_{\Omega_2^t} u\rho(u')dx\Big| &\le |u(t)|_{L^{q+1}(\Omega_2^t)}|\rho(u'(t))|_{L^{\frac{q+1}{q}}(\Omega_2^t)} \\ &\le C_{24}\|u(t)\|\Big[\int_{\Omega_2^t} |\rho(u')|^{\frac{1}{q}}|\rho(u')|\,dx\Big]^{\frac{q}{q+1}} \\ &\le C_{25}\|u(t)\|\Big[\int_{\Omega_2^t} u'\rho(u')\,dx\Big]^{\frac{q}{q+1}} \le C_{26}E^{1/2}(t)|E'(t)|^{\frac{q}{q+1}}. \end{aligned} \tag{3.34}$$

We also observe that

$$\begin{aligned} |\,(2u(t) - g\delta(t), \delta'(t))_{\Gamma_1}\,| &\le C_{27}[\|u(t)\| + |\delta(t)|_{\Gamma_1}]|\delta'(t)|_{\Gamma_1} \\ &\le C_{28}E^{1/2}(t)|g^{1/2}\delta'(t)|_{\Gamma_1} \le C_{28}E^{1/2}(t)|E'(t)|^{1/2}. \end{aligned} \tag{3.35}$$

Using Lemma 3.3, (3.33)–(3.35), we obtain

$$2\int_S^T E(t)\,dt \le C_{29}\Big[E(S) + \int_S^T E^{1/2}(t)|E'(t)|^{1/2} + E^{1/2}(t)|E'(t)|^{\frac{q}{q+1}}\,dt\Big]. \tag{3.36}$$

Now, we estimate the last term of (3.36). Applying Young's inequality and using (3.25), we have

$$\begin{aligned}
&E^{1/2}(t)|E'(t)|^{1/2} + E^{1/2}(t)|E'(t)|^{\frac{q}{q+1}} \\
&\le C_{30}\Big[\frac{1}{2}E(t) + |E'(t)| + \Big(E^{\frac{1}{q+1}}E^{\frac{q-1}{2(q+1)}}(t)|E'(t)|^{\frac{q}{q+1}}\Big)^{\frac{q+1}{q}}\Big] \\
&\le C_{31}\big[E(t) + |E'(t)| + E^{\frac{q-1}{2q}}(0)|E'(t)|\big].
\end{aligned} \tag{3.37}$$

Taking into account (3.36), (3.37) and letting $T \to \infty$, we conclude that

$$\int_S^\infty E(t)\,dt \le C_{32}E(S), \quad \text{for all } S \ge 0. \tag{3.38}$$

Hence, Lemma 3.2 yields $E(t) \le C_{33}\exp(-C_{34}t)$, for all $t \ge 0$. □

**Remark 3.5.** Suppose $p > 1$ if $n = 2$ or $1 < p \le 2$ if $n = 3$. Setting $u'' - M(|u(t)|^2)\Delta u + \alpha u' + \beta|u'|^p u' = 0$, in $\Omega \times (0,\infty)$, instead of $(1.1)_1$, using the method above we can prove the exponential decay without the assumption $c > 4k_1k_2$ made by Frota, Medeiros and Vicente [12].

## 4. Non-monotone dissipative term

Let $\lambda \in C^1(\mathbb{R})$ be such that

$$|\lambda(x)| \le \kappa_4|x|^q, \text{ if } |x| > 1 \quad \text{and} \quad |\lambda(x)| \le \kappa_5|x|, \text{ if } |x| \le 1, \tag{4.1}$$

where $q$ satisfies $1 \le q \le \frac{n+2}{n-2}$ if $n \ge 3$ or $q \ge 1$ if $n = 2$. We also assume

$$\lambda(x)x \ge 0 \text{ and } \lambda'(x) > -\alpha^2, \quad \text{for all } x \in \mathbb{R}, \tag{4.2}$$

where $\alpha$ is a constant. We consider

$$\begin{gathered}
u'' - \Delta u + \lambda(u') = 0 \quad \text{in } \Omega \times (0,\infty), \\
u = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \\
\frac{\partial u}{\partial \nu} = \delta' \quad \text{on } \Gamma_1 \times (0,\infty), \\
u' + f\delta'' - c^2\Delta_\Gamma\delta + g\delta' + h\delta = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \\
\delta = 0 \quad \text{on } \partial\Gamma_1 \times (0,\infty), \\
u(x,0) = \phi(x), \quad u'(x,0) = \psi(x) \quad x \in \Omega, \\
\delta(x,0) = \theta(x), \quad \delta'(x,0) = \frac{\partial\phi}{\partial\nu}(x) \quad x \in \Gamma_1,
\end{gathered} \tag{4.3}$$

**Theorem 4.1.** *Suppose that* (4.1)–(4.2) *hold. Let* $(\phi,\psi,\theta) \in W \times (V \cap L^{2q}(\Omega)) \times (H_0^1(\Gamma_1) \cap H^2(\Gamma_1))$. *Then there exists a unique pair* $(u,\delta)$ *which is solution of* (4.3) *and satisfies*

$$\begin{gathered}
u, u' \in L^\infty_{\rm loc}(0,\infty;V), \quad u'' \in L^\infty_{\rm loc}(0,\infty;L^2(\Omega)), \quad u(t) \in H(\Delta,\Omega) \text{ a.e. in } [0,\infty), \\
\delta \in L^\infty_{\rm loc}(0,\infty;H_0^1(\Gamma_1) \cap H^2(\Gamma_1)), \quad \delta' \in L^\infty_{\rm loc}(0,\infty;H_0^1(\Gamma_1)), \\
\delta'' \in L^\infty_{\rm loc}(0,\infty;L^2(\Gamma_1)).
\end{gathered}$$

*Proof.* Let $(u_k)_{k\geq 0}$ and $(\delta_k)_{k\geq 0}$ be the sequences defined by $u_0(x,t)=\phi(x)$, for all $(x,t)\in\Omega\times[0,T]$, $\delta_0(x,t)=\theta(x)$, for all $(x,t)\in\Gamma_1\times[0,T]$, $T>0$, and $(u_k,\delta_k)$, $k=1,2,\dots$, the solution of

$$
\begin{gathered}
u_k''-\Delta u_k+\lambda(u_k')+\alpha^2u_k'=F_{k-1}=\alpha^2u_{k-1}' \quad \text{in } \Omega\times(0,T),\\
u_k=0 \quad \text{on } \Gamma_0\times(0,T),\\
\frac{\partial u_k}{\partial\nu}=\delta_k' \quad \text{on } \Gamma_1\times(0,T),\\
u_k'+f\delta_k''-c^2\Delta_\Gamma\delta_k+g\delta_k'+h\delta_k=0 \quad \text{on } \Gamma_1\times(0,T),\\
\delta_k=0 \quad \text{on } \partial\Gamma_1\times(0,T),\\
u_k(x,0)=\phi(x),\quad u_k'(x,0)=\psi(x)\quad x\in\Omega;\\
\delta_k(x,0)=\theta(x),\quad \delta_k'(x,0)=\frac{\partial\phi}{\partial\nu}(x)\quad x\in\Gamma_1,
\end{gathered}
\tag{4.4}
$$

for $k=1,2,\dots$, here $u_1'(x,t)=\psi(x)$ for all $(x,t)\in\Omega\times[0,T]$. Since $\rho(\xi)=\lambda(\xi)+\alpha^2\xi$ satisfies (3.2)–(3.3) and $F_{k-1},F_{k-1}'\in L^2(0,T;L^2(\Omega))$, for $k=1,2,\dots$, the existence and uniqueness of solution $(u_k,\delta_k)$ for (4.4) is given by Theorem 3.1.

Multiplying $(4.4)_1$ by $u_k'$, $(4.4)_4$ by $\delta_k'$, integrating over $\Omega$ and $\Gamma_1$, respectively, and substituting the second equation into the first, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big[|u_k'(t)|^2+\|u_k(t)\|^2+|f^{1/2}\delta_k'(t)|_{\Gamma_1}^2+|h^{1/2}\delta_k(t)|_{\Gamma_1}^2+c^2\|\delta_k(t)\|_{\Gamma_1}^2\Big]\\
&\leq\alpha^2\left(u_{k-1}'(t),u_k'(t)\right).
\end{aligned}
\tag{4.5}
$$

Denoting by

$$
\begin{aligned}
e_k(t)=\frac{1}{2}\operatorname{ess\,sup}_{0<s<t}\Big\{&|u_k'(s)|^2+\|u_k(s)\|^2+|f^{1/2}\delta_k'(s)|_{\Gamma_1}^2+|h^{1/2}\delta_k(s)|_{\Gamma_1}^2\\
&+c^2\|\delta_k(s)\|_{\Gamma_1}^2\Big\},
\end{aligned}
$$

$$
e(0)=\frac{1}{2}\Big[|\psi|^2+\|\phi\|^2+k_2\max_{x\in\overline{\Gamma_1}}|f|\|\phi\|_W+\max_{x\in\overline{\Gamma_1}}|h||\theta|_{\Gamma_1}^2+c^2\|\theta\|_{\Gamma_1}^2\Big]
$$

and integrating (4.5), we have

$$
e_k(t)\leq e(0)+\alpha^4t\int_0^t|u_{k-1}'(\xi)|^2d\xi+\frac{1}{4t}\int_0^t|u_k'(\xi)|^2d\xi.
$$

However,

$$
\frac{1}{4t}\int_0^t|u_k'(\xi)|^2d\xi\leq\frac{1}{2}e_k(t),\quad \text{for all } k=1,2,\dots.
$$

Therefore,

$$
\frac{1}{2}e_k(t)\leq e(0)+2\alpha^4t^2e_{k-1}(t),\quad \text{for all } k=1,2,\dots. \tag{4.6}
$$

Let $M$ be a constant such that $M>2e(0)$. By induction, it is easy to prove

$$
e_k(t)\leq 2e(0)+4\alpha^4t^2M<M,\quad \text{for all } t\leq\tau_1=\Big(\frac{M-2e(0)}{4\alpha^4M}\Big)^{1/2}, \tag{4.7}
$$

$k=1,2,\dots$.

Differentiating the equations $(4.4)_1$ and $(4.4)_4$ with respect to $t$, proceeding as in (3.14), (3.15), (3.18), (3.19) and above we obtain

$$
|u_k''(t)|^2+\|u_k'(t)\|^2+|f^{1/2}\delta_k''(t)|_{\Gamma_1}^2+|h^{1/2}\delta_k'(t)|_{\Gamma_1}^2+c^2\|\delta_k'(t)\|_{\Gamma_1}^2\leq\overline{M}, \tag{4.8}
$$

for all $t \in [0, \tau_2]$, where $\tau_2 = \left(\frac{\overline{M}-2\overline{e}(0)}{4\overline{M}}\right)^{1/2}$ with $\overline{M} > 2\overline{e}(0)$ and

$$\begin{aligned}\overline{e}(0) = \frac{1}{2}\Big\{&[|\Delta\phi| + C_v(|\psi|^q_{L^{2q}(\Omega)} + |\psi|_{L^{2q}(\Omega)}) + \alpha^2|\psi|)^{1/2} + \|\psi\|^2 + (k_0\|\psi\|_W \\ &+ c^2|\Delta_\Gamma\theta|_{\Gamma_1} + k_2 \max_{x\in\overline{\Gamma_1}}|g|\|\phi\|_W + \max_{x\in\overline{\Gamma_1}}|h||\theta|_{\Gamma_1})^{1/2} \\ &+ k_2 \max_{x\in\overline{\Gamma_1}}|h|\|\phi\|_W + c^2k_1\|\phi\|^2_W\Big\}.\end{aligned}$$

Employing the same argument to obtain (3.21), we have

$$|\Delta_\Gamma\delta_k(t)|_{\Gamma_1} \le C_{35}, \quad \text{for all } t \in [0, \tau], \tag{4.9}$$

where $\tau = \min\{\tau_1, \tau_2\}$. Using the estimates (4.7)–(4.9) and compactness argument, we can see that there exist a subsequence of $(u_k)_{k\ge0}$ and a subsequence of $(\delta_k)_{k\ge0}$, which we will denote by the same notations, and functions $u$ and $\delta$, such that

$$\begin{aligned}
&u_k \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\tau;V), &&\delta_k \overset{*}{\rightharpoonup} \delta \text{ in } L^\infty(0,\tau;H^1_0(\Gamma_1)\cap H^2(\Gamma_1)),\\
&u'_k \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0,\tau;V), &&\delta'_k \overset{*}{\rightharpoonup} \delta' \text{ in } L^\infty(0,\tau;H^1_0(\Gamma_1)),\\
&u''_k \overset{*}{\rightharpoonup} u'' \text{ in } L^\infty(0,\tau;L^2(\Omega)), &&\delta''_k \overset{*}{\rightharpoonup} \delta'' \text{ in } L^\infty(0,\tau;L^2(\Gamma_1)).
\end{aligned} \tag{4.10}$$

These convergences are sufficient to pass to the limit in (4.4). Therefore $(u, \delta)$ is a local solution of (4.3). The proof of the uniqueness is standard. Now, we will extend the local solution to whole interval $[0, T]$, for all $T > 0$. In fact, let $(u, \delta)$ the local solution of (4.3), we consider the problem

$$\begin{gathered}
v'' - \Delta v + \lambda(v') = 0 \quad \text{in } \Omega \times (0,\infty),\\
v = 0 \quad \text{on } \Gamma_0 \times (0,\infty),\\
\frac{\partial v}{\partial\nu} = \sigma' \quad \text{on } \Gamma_1 \times (0,\infty),\\
v' + f\sigma'' - c^2\Delta_\Gamma\sigma + g\sigma' + h\sigma = 0 \quad \text{on } \Gamma_1 \times (0,\infty);\\
\sigma = 0 \quad \text{on } \partial\Gamma_1 \times (0,\infty),\\
v(x,0) = u(x,\frac{\tau}{2}), \quad v'(x,0) = u'(x,\frac{\tau}{2}) \quad x \in \Omega;\\
\sigma(x,0) = \delta(x,\frac{\tau}{2}), \quad \sigma'(x,0) = \frac{\partial u}{\partial\nu}(x,\frac{\tau}{2}) \quad x \in \Gamma_1,
\end{gathered} \tag{4.11}$$

then we have a local solution $(v, \sigma)$ of (4.11) on $[\frac{\tau}{2}, \frac{3\tau}{2}]$. By uniqueness, we obtain $(u, \delta) = (v, \sigma)$ on $[\frac{\tau}{2}, \tau]$. Therefore we can extend the solution $(u, \delta)$ to whole interval $[0, \frac{3\tau}{2}]$. Then we have a global solution of (4.3). $\square$

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