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ASYMPTOTIC STABILITY OF SOLUTIONS TO ELASTIC SYSTEMS WITH STRUCTURAL DAMPING

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Abstract. In this article, we study the asymptotic stability of solutions for the initial value problems of second order evolution equations in Banach spaces, which can model elastic systems with structural damping. The discussion is based on exponentially stable semigroups theory. Applications to the vibration equation of elastic beams with structural damping are also considered.

1. Introduction

The study of elastic systems with damping seems to have been initiated by Chen and Russell [1] in 1981. They considered the linear elastic systems with structural damping,

$$\begin{gathered} \ddot{u}(t) + B\dot{u}(t) + Au(t) = 0, \\ u(0) = x_0, \quad \dot{u}(0) = y_0 \end{gathered} \tag{1.1}$$

in a Hilbert space $\mathbb{H}$ with inner product $(\cdot, \cdot)$, where A (the elastic operator) and B (the damping operator) are unbounded positive definite self-adjoint operators in $\mathbb{H}$. Let $x_1 = A^{1/2}u$, $x_2 = \dot{u}$, we get the equivalent first-order linear systems

$$\begin{gathered} \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = L_B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \\ x_1(0) = A^{1/2}x_0, \quad x_2(0) = y_0. \end{gathered}$$

Chen and Russell [1] proved that

$$L_B = \begin{pmatrix} 0 & A^{1/2} \\ -A^{1/2} & -B \end{pmatrix}$$

generates an analytic semigroup on $\mathbb{W} = \mathbb{H} \oplus \mathbb{H}$, if some additional conditions are satisfied. In the same paper, they pose the following conjecture proved by Huang [9, 10]: Let $D(B) \supset D(A^{1/2})$; then either of the following conditions (1) and (2) implies that L_B generates an analytic semigroup on $\mathbb{W}$:

(1) $\rho_1(A^{1/2}x, x) \le (Bx, x) \le \rho_2(A^{1/2}x, x)$ for all $x \in D(A^{1/2})$ or (not, in general, equivalent)
(2) $\rho_1(Ax, x) \le (B^2x, x) \le \rho_2(Ax, x)$ for all $x \in D(A)$

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for some $\rho_1, \rho_2 > 0$ with $\rho_1 \le \rho_2$. In addition, the semigroup generated by L_B is exponentially stable.

But these results do not contain the case $B = \rho A$, which could possibly appear in engineering applications. For this situation, Massatt [16] shows that if $B = \rho A$ with $\rho > 0$, then

$$\mathscr{A}_\rho = \begin{pmatrix} 0 & 1 \\ -A & -\rho A \end{pmatrix}$$

generates an analytic semigroup which is exponentially stable.

Huang [11] investigated the more widely used linear elastic systems (1.1) with damping B related in various ways to $A^\alpha(\frac{1}{2} \le \alpha \le 1)$, so that the C_0-semigroups associated with them are analytic and exponentially stable. Meanwhile, the spectral property and some fundamental results for the analytic property and the exponential stability of the semigroups associated with the systems were discussed. Then other sufficient conditions for L_B generates an analytic semigroup were discussed in [5, 6, 8, 9, 10, 11, 12] and the references therein.

Recently, the present authors [5] studied the linear second-order evolution equation

$$\begin{gathered} \ddot{u}(t) + \rho\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = 0, \quad t > 0, \\ u(0) = x_0, \quad \dot{u}(0) = y_0, \end{gathered} \tag{1.2}$$

in a frame of Banach spaces, which can model the elastic systems with structural damping. New forms of the corresponding first-order evolution equations were introduced and sufficient conditions for analyticity and exponential stability of the associated semigroups were given.

In [7] and [6], existence results of mild solutions for the elastic systems with structural damping were established by the fixed point theorems and monotone iterative technique in the presence of lower and upper solutions, respectively. However, the theory of the elastic systems with structural damping remains to be developed.

In this paper, we concentrate on the asymptotic behavior of solutions for the linear elastic systems with structural damping

$$\begin{gathered} \ddot{u}(t) + \rho\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = h(t), \quad t > 0, \\ u(0) = x_0, \quad \dot{u}(0) = y_0 \end{gathered} \tag{1.3}$$

and the semilinear elastic systems with structural damping

$$\begin{gathered} \ddot{u}(t) + \rho\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = f(t, u(t)), \quad t > 0, \\ u(0) = x_0, \quad \dot{u}(0) = y_0, \end{gathered} \tag{1.4}$$

in a Banach space $\mathbb{X}$, where "$\cdot$" means d/dt, ρ is the damping coefficient; $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ is a sectorial operator and $-\mathscr{A}$ generates an analytic and exponentially stable semigroup $S(t)(t \ge 0)$ on $\mathbb{X}$; $f \in C(J \times \mathbb{X}, \mathbb{X})$, $x_0 \in \mathscr{D}(\mathscr{A})$, $y_0 \in \mathbb{X}$.

2. Preliminaries

Definition 2.1 ([4]). A semigroup $T(t)(t \ge 0)$ on a Banach space $\mathbb{X}$ is called exponentially stable if there exist constants $\delta > 0$, $M \ge 1$ such that

$$\|T(t)\| \le Me^{-\delta t}, \quad t \ge 0.$$

First we present a simple result on the asymptotic behavior of mild solutions for the inhomogeneous initial value problem of the first-order linear evolution equation

$$\begin{gathered} u'(t) = Au(t) + h(t), \quad t > 0, \\ u(0) = x. \end{gathered} \tag{2.1}$$

Lemma 2.2 ([17, Page 119, Theorem 4.4]). *Let $\mu > 0$ and let A be the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ satisfying $\|T(t)\| \leq Me^{-\mu t}$. Let h be bounded and measurable on $[0, +\infty)$. If*

$$\lim_{t \to +\infty} h(t) = b,$$

then, $u(t)$, the mild solution of (2.1) satisfies

$$\lim_{t \to +\infty} u(t) = -A^{-1}b.$$

Next we recall some basic facts and conclusions on the elastic systems (1.3) and (1.4), which can be found in [5, 7] in order to prove our main results.

Since $\mathscr{A}$ is a sectorial operator on $\mathbb{X}$. It follows from the definition that there exist $\alpha \in (0, \frac{\pi}{2})$ and $K > 0$ satisfying

$$\Sigma_\alpha := \{\lambda || \arg \lambda| < \frac{\pi}{2} + \alpha\} \subset \rho(-\mathscr{A}), \tag{2.2}$$

$$\|(\lambda I + \mathscr{A})^{-1}\| \leq \frac{K}{1 + |\lambda|}, \quad \lambda \in \Sigma_\alpha. \tag{2.3}$$

For the second-order equation

$$\ddot{u}(t) + \rho \mathscr{A} \dot{u}(t) + \mathscr{A}^2 u(t) = h(t), \quad t > 0,$$

it has the decomposition

$$\big(\frac{\partial}{\partial t} + \sigma_1 \mathscr{A}\big)\big(\frac{\partial}{\partial t} + \sigma_2 \mathscr{A}\big) u = h(t), \quad t > 0.$$

Let

$$\frac{\partial u}{\partial t} + \sigma_2 \mathscr{A} u = v(t),$$

which means $v(0) = y_0 + \sigma_2 \mathscr{A} x_0 := v_0$. Then the elastic systems (1.3) can be transformed into the following two abstract Cauchy problems in $\mathbb{X}$:

$$\begin{gathered} \frac{\partial v}{\partial t} + \sigma_1 \mathscr{A} v = h(t), \quad t > 0, \\ v(0) = v_0 \end{gathered} \tag{2.4}$$

and

$$\begin{gathered} \frac{\partial u}{\partial t} + \sigma_2 \mathscr{A} u = v(t), \quad t > 0, \\ u(0) = x_0, \end{gathered} \tag{2.5}$$

where

$$\sigma_1 + \sigma_2 = \rho, \quad \sigma_1 \sigma_2 = 1. \tag{2.6}$$

Lemma 2.3 ([5]). *Let $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator, if the damping coefficient $\rho > 2\cos\alpha$, then $-\sigma_1 \mathscr{A}$, $-\sigma_2 \mathscr{A}$ generate analytic and exponentially stable semigroups on $\mathbb{X}$, where α is defined in (2.2) and σ_1, σ_2 are specified in (2.6).*

For the convenience of the reader, throughout this paper we assume that $-\sigma_1\mathscr{A}$ and $-\sigma_2\mathscr{A}$ generate analytic and exponentially stable semigroups $S_1(t)(t \geq 0)$ and $S_2(t)(t \geq 0)$ on $\mathbb{X}$, respectively. By Definition 2.1, there exist constants $\delta_1 > 0, \delta_2 > 0$ and $M_1 \geq 1, M_2 \geq 1$ such that

$$\|S_1(t)\| \leq M_1 e^{-\delta_1 t}, \quad \|S_2(t)\| \leq M_2 e^{-\delta_2 t}, \quad t \geq 0. \tag{2.7}$$

Definition 2.4 ([7])**.** Let $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator, $\rho > 2\cos\alpha$, and $f : J \times \mathbb{X} \to \mathbb{X}$ be a continuous function, $x_0 \in \mathscr{D}(\mathscr{A})$, $y_0 \in \mathbb{X}$. A continuous solution of the integral equation

$$\begin{aligned} u(t) = S_2(t)x_0 &+ \int_0^t S_2(t-s)S_1(s)v_0\,ds \\ &+ \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f(\tau, u(\tau))d\tau\,ds \end{aligned}$$

is said to be a mild solution of the initial-value problem (1.4), where α is defined in (2.2).

3. Main results

In this section it is our aim to introduce the asymptotic behavior of solutions for the elastic systems (1.3) and (1.4), which can be given by the following theorems.

Theorem 3.1. *Let $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator, the damping coefficient $\rho > 2\cos\alpha$, where α is defined in* (2.2)*, $x_0 \in \mathscr{D}(\mathscr{A})$, $y_0 \in \mathbb{X}$, $h : [0, +\infty) \to \mathbb{X}$ is continuous. If*

$$\lim_{t\to+\infty} h(t) = b,$$

then, the mild solution $u(t)$ of the initial value problem (1.3) *satisfies*

$$\lim_{t\to+\infty} u(t) = \mathscr{A}^{-2}b.$$

Proof. Since $S_1(t)$ $(t \geq 0)$ is exponentially stable on $\mathbb{X}$. By Definition 2.1 and Lemma 2.2, the mild solution $v(t)$ of the initial-value problem (2.4) satisfies

$$\lim_{t\to+\infty} v(t) = (\sigma_1\mathscr{A})^{-1}b. \tag{3.1}$$

Similarly, since $S_2(t)(t \geq 0)$ is also exponentially stable on $\mathbb{X}$. By Definition 2.1, Lemma 2.2 and (3.1), the mild solution $u(t)$ of the initial value problem (2.5) satisfies

$$\begin{aligned} \lim_{t\to+\infty} u(t) &= (\sigma_2\mathscr{A})^{-1} \lim_{t\to+\infty} v(t) \\ &= (\sigma_2\mathscr{A})^{-1}(\sigma_1\mathscr{A})^{-1}b \\ &= \frac{1}{\sigma_1\sigma_2}\mathscr{A}^{-2}b. \end{aligned} \tag{3.2}$$

Combining this fact with (2.6), it follows that $\lim_{t\to+\infty} u(t) = \mathscr{A}^{-2}b$. $\square$

We now show that if the semigroups $S_1(t)(t \geq 0)$ and $S_2(t)(t \geq 0)$ are exponentially stable on $\mathbb{X}$, then, we can choose the constants δ_1, δ_2 in (2.7) satisfying $0 < \delta_1 < \delta_2$. If, on the contrary, let $\delta_1 = \delta_2 := \delta$ and let $\delta = \delta' + \delta''$, where $\delta' > 0$, $\delta'' > 0$, then for all $t \geq 0$, we have

$$\|S_2(t)\| \leq M_2 e^{-\delta t},$$

$$\|S_1(t)\| \le M_1 e^{-\delta t} = M_1 e^{-(\delta'+\delta'')t} = M_1 e^{-\delta' t} e^{-\delta'' t} \le M_1 e^{-\delta' t}.$$

It is evident that $\delta > \delta' > 0$. Hence, in what follows, we always assume that the constants δ_1 and δ_2 in (2.7) satisfying $0 < \delta_1 < \delta_2$.

Next we establish the globally asymptotic stability result of the zero solution for the initial value problem (1.4).

Theorem 3.2. *Let $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ be a sectorial operator, the damping coefficient $\rho > 2\cos\alpha$, where α is defined in* (2.2)*, $x_0 \in \mathscr{D}(\mathscr{A})$, $y_0 \in \mathbb{X}$, $f : [0,+\infty) \times \mathbb{X} \to \mathbb{X}$ is continuous and satisfies the following conditions:*

(H1) *There exists $L > 0$, such that*

$$\|f(t,u_2) - f(t,u_1)\| \le L\|u_2 - u_1\|, \quad t \in [0,+\infty),\ u_1, u_2 \in \mathbb{X}.$$

(H2) *$f(t,\theta) = \theta$ (θ is the zero element of $\mathbb{X}$) for $t \ge 0$.*

(H3) $0 < L < \frac{\delta_1(\delta_2-\delta_1)}{M_1 M_2}$.

Then the mild solution $u(t)$ of the initial value problem (1.4) *satisfies*

$$\lim_{t\to+\infty} u(t) = \theta.$$

Proof. By assumption (H1) and [7, Theorem 4], the initial value problem (1.4) has a unique global mild solution $u(t)$, then by the semigroup representation of the mild solution, $u(t)$ satisfies the integral equation

$$\begin{aligned} u(t) = S_2(t)x_0 &+ \int_0^t S_2(t-s)S_1(s)v_0\, ds \\ &+ \int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))d\tau\, ds, \quad t \ge 0. \end{aligned} \tag{3.3}$$

Using this, we conclude that

$$\begin{aligned} \|u(t)\| \le \|S_2(t)x_0\| &+ \Big\|\int_0^t S_2(t-s)S_1(s)v_0\, ds\Big\| \\ &+ \Big\|\int_0^t \int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))d\tau\, ds\Big\|, \quad t \ge 0. \end{aligned} \tag{3.4}$$

From the inequality (2.7), it follows that

$$\|S_2(t)x_0\| \le \|S_2(t)\|\|x_0\| \le M_2 e^{-\delta_2 t}\|x_0\| \le M_2 e^{-\delta_1 t}\|x_0\| \tag{3.5}$$

and

$$\begin{aligned} \Big\|\int_0^t S_2(t-s)S_1(s)v_0\, ds\Big\| &\le \int_0^t \|S_2(t-s)\|\|S_1(s)\|\|v_0\|\, ds \\ &\le M_1M_2 e^{-\delta_2 t}\|v_0\| \int_0^t e^{(\delta_2-\delta_1)s}\, ds \\ &\le \frac{M_1M_2\|v_0\|}{\delta_2-\delta_1} e^{-\delta_1 t}. \end{aligned} \tag{3.6}$$

By assumption (H2) and (2.7), we have

$$\begin{aligned}
&\Big\|\int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))d\tau\,ds\Big\| \\
&\le \int_0^t\int_0^s \|S_2(t-s)\|\|S_1(s-\tau)\|\|f(\tau,u(\tau))\|d\tau\,ds \\
&\le \int_0^t\int_0^s \|S_2(t-s)\|\|S_1(s-\tau)\|\|f(\tau,u(\tau))-f(\tau,\theta)\|d\tau\,ds \\
&\quad+\int_0^t\int_0^s \|S_2(t-s)\|\|S_1(s-\tau)\|\|f(\tau,\theta)\|d\tau\,ds \\
&\le LM_1M_2e^{-\delta_2 t}\int_0^t e^{(\delta_2-\delta_1)s}\int_0^s e^{\delta_1\tau}\|u(\tau)\|d\tau\,ds.
\end{aligned} \tag{3.7}$$

From integration by parts, we get

$$\begin{aligned}
&\int_0^t e^{(\delta_2-\delta_1)s}\int_0^s e^{\delta_1\tau}\|u(\tau)\|d\tau\,ds \\
&=\frac{1}{\delta_2-\delta_1}\int_0^t d[e^{(\delta_2-\delta_1)s}]\int_0^s e^{\delta_1\tau}\|u(\tau)\|d\tau \\
&=\frac{1}{\delta_2-\delta_1}\Big[e^{(\delta_2-\delta_1)t}\int_0^t e^{\delta_1 s}\|u(s)\|\,ds-\int_0^t e^{\delta_2 s}\|u(s)\|\,ds\Big] \\
&\le\frac{1}{\delta_2-\delta_1}e^{(\delta_2-\delta_1)t}\int_0^t e^{\delta_1 s}\|u(s)\|\,ds,
\end{aligned} \tag{3.8}$$

and therefore

$$\Big\|\int_0^t\int_0^s S_2(t-s)S_1(s-\tau)f(\tau,u(\tau))d\tau\,ds\Big\| \le \frac{LM_1M_2}{\delta_2-\delta_1}e^{-\delta_1 t}\int_0^t e^{\delta_1 s}\|u(s)\|\,ds. \tag{3.9}$$

Together with (3.4), (3.5), (3.6) and (3.9) this gives

$$\|u(t)\|\le M_2e^{-\delta_1 t}\|x_0\|+\frac{M_1M_2\|v_0\|}{\delta_2-\delta_1}e^{-\delta_1 t}+\frac{LM_1M_2}{\delta_2-\delta_1}e^{-\delta_1 t}\int_0^t e^{\delta_1 s}\|u(s)\|\,ds,$$

$t\ge 0$. Hence

$$e^{\delta_1 t}\|u(t)\|\le M_2\|x_0\|+\frac{M_1M_2\|v_0\|}{\delta_2-\delta_1}+\frac{LM_1M_2}{\delta_2-\delta_1}\int_0^t e^{\delta_1 s}\|u(s)\|\,ds,\quad t\ge 0.$$

According to Growall inequality, we obtain that

$$e^{\delta_1 t}\|u(t)\|\le\big[M_2\|x_0\|+\frac{M_1M_2\|v_0\|}{\delta_2-\delta_1}\big]e^{\frac{LM_1M_2}{\delta_2-\delta_1}t},\quad t\ge 0.$$

Which means

$$\|u(t)\|\le\big[M_2\|x_0\|+\frac{M_1M_2\|v_0\|}{\delta_2-\delta_1}\big]e^{\big(\frac{LM_1M_2}{\delta_2-\delta_1}-\delta_1\big)t},\quad t\ge 0.$$

By the assumption (H3), we have

$$\frac{LM_1M_2}{\delta_2-\delta_1}-\delta_1<0. \tag{3.10}$$

This implies $u(t)\to\theta$ as $t\to+\infty$. Consequently, the zero solution is globally asymptotically stable and it exponentially attracts every mild solution of the initial-value problem (1.4). □

4. Applications

In this section, we will apply the abstract results in Section 3 to the vibration equation of elastic beams with structural damping, to obtain the results of asymptotic stability of mild solutions.

The vibration state of an elastic beam with structural damping, whose two ends are simply supported, can be described by the initial-boundary value problem (IBVP)

$$\begin{gathered} u_{tt} - 4u_{xxt} + u_{xxxx} = h(x,t), \quad x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, \quad t > 0, \\ u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t > 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,1), \end{gathered} \tag{4.1}$$

where u_{xxxx} denotes the elastic effect, u_{xxt} is the damping term, $\rho = 4$ is the damping coefficient and the non-homogeneous term $h(x,t)$ be defined by

$$h(x,t) = \begin{cases} \frac{2x^2t^2}{1+3x^2t^2}, & x \in (0,1),\ t \geq 0, \\ 2/3, & x = 0,\ t \geq 0. \end{cases} \tag{4.2}$$

Let $I = [0,1]$ and choose $\mathbb{X} = L^p(I)(2 \leq p < +\infty)$. Define a linear operator $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ by

$$\mathscr{D}(\mathscr{A}) = W^{2,p}(I) \cap W_0^{1,p}(I), \quad \mathscr{A}u = -\Delta u, \tag{4.3}$$

where Δ is the Laplace operator acting on functions on the interval I. Choosing $\alpha = \arccos 2/5 \in (0, \pi/2)$, by [5], $\mathscr{A}$ is a sectorial operator for the region $\sum_\alpha$ defined by (2.2).

Let $h(t) = h(\cdot, t)$, then the problem (4.1) can be rewritten into the abstract form

$$\begin{gathered} \ddot{u}(t) + 4\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = h(t), \quad t > 0, \\ u(0) = \varphi, \quad \dot{u}(0) = \psi. \end{gathered} \tag{4.4}$$

Theorem 4.1. *Let $2 \leq p < +\infty$, for every $\varphi \in W^{2,p}(I) \cap W_0^{1,p}$ and $\psi \in L^p(I)$, the mild solution $u(t)$ of the equation (4.1) satisfying $\lim_{t\to+\infty} u(t) = \Delta^{-2}\frac{2}{3}$.*

Proof. By setting $\rho = 4$ and $\alpha = \arccos 2/5$, it is easy to verify that the damping coefficient ρ satisfies $\rho > 2\cos\alpha$. From (4.2), it follows that $h(t)$ is continuous on $[0, +\infty)$ and $\lim_{t\to+\infty} h(t) = \frac{2}{3}$. Hence by Theorem 3.1, the mild solution $u(t)$ of the equation (4.1) satisfying $\lim_{t\to+\infty} u(t) = \Delta^{-2}\frac{2}{3}$. □

In what follows, we consider the nonlinear vibration equation of elastic beams with structural damping, namely the following initial-boundary value problem

$$\begin{gathered} u_{tt} - 4u_{xxt} + u_{xxxx} = \frac{1}{2}\sin u(x,t), \quad x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0, \quad t > 0, \\ u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t > 0, \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in (0,1), \end{gathered} \tag{4.5}$$

Let $u(t) = u(\cdot, t)$, $f(t, u(t)) = \frac{1}{2}\sin u(\cdot, t)$. Then the initial-boundary value problem (4.5) can be rewritten to the Cauchy problem of the second order evolution equation in the Banach space $\mathbb{X}$

$$\begin{aligned} \ddot{u}(t) + 4\mathscr{A}\dot{u}(t) + \mathscr{A}^2 u(t) = f(t, u(t)), \quad t > 0, \\ u(0) = \varphi, \quad \dot{u}(0) = \psi, \end{aligned} \tag{4.6}$$

where $\mathscr{A}$ is defined in (4.3) and $\mathscr{A}$ is a sectorial operator for the region $\Sigma_\alpha(\alpha = \arccos 2/5)$ defined by (2.2). We assume that $\varphi \in \mathscr{D}(\mathscr{A})$ and $\psi \in \mathbb{X}$, Then the equation (4.6) has the following decomposition form

$$\begin{aligned} (\frac{\partial}{\partial t} + \sigma_1\mathscr{A})(\frac{\partial}{\partial t} + \sigma_2\mathscr{A})u = f(t, u(t)), \quad t > 0, \\ u(0) = \varphi, \quad \dot{u}(0) = \psi, \end{aligned} \tag{4.7}$$

where $\sigma_1 = 2 - \sqrt{3}$, $\sigma_2 = 2 + \sqrt{3}$ are defined by (2.6).

It is well-known [8, 17], $-\mathscr{A}$ generates an analytic and exponentially stable semigroup $S(t)(t \geq 0)$ satisfying

$$\|S(t)\| \leq e^{-t}, \quad t \geq 0.$$

By Lemma 2.3 and the characterization of the infinitesimal generators of C_0-semigroups, $-\sigma_1\mathscr{A}$ and $-\sigma_2\mathscr{A}$ generate analytic and exponentially stable semigroups $S_1(t)(t \geq 0)$ and $S_2(t)(t \geq 0)$ respectively, which satisfy

$$\|S_i(t)\| = \|S(\sigma_i t)\| \leq e^{-\sigma_i t}, \quad t \geq 0, \quad i = 1, 2.$$

Now take $M_1 = M_2 = 1$, $\delta_1 = \sigma_1 = 2 - \sqrt{3}$ and $\delta_2 = \sigma_2 = 2 + \sqrt{3}$, we obtain that

$$\frac{1}{2} < \frac{\delta_1(\delta_2 - \delta_1)}{M_1 M_2} = 4\sqrt{3} - 6. \tag{4.8}$$

Theorem 4.2. *Let $2 \leq p < +\infty$, for every $\varphi \in W^{2,p}(I) \cap W_0^{1,p}$ and $\psi \in L^p(I)$, the mild solution $u(t)$ of the equation (4.5) satisfying $\|u(t)\|_p \to 0$ as $t \to \infty$.*

Proof. By $\rho = 4$ and $\alpha = \arccos 2/5$, we can easily obtain that the damping coefficient ρ satisfies $\rho > 2\cos\alpha$. Since $f(x, t, u(x, t)) = \frac{1}{2}\sin u(x, t)$ is continuous on $[0, 1] \times [0, +\infty) \times \mathbb{X}$ and satisfying

$$|f'_u(x, t, u)| = \frac{1}{2}|\cos u(x, t)| \leq \frac{1}{2}, \quad (x, t, u) \in [0, 1] \times [0, +\infty) \times \mathbb{X}; \tag{4.9}$$

$$f(x, t, 0) = \sin 0 = 0, \quad (x, t) \in [0, 1] \times [0, +\infty). \tag{4.10}$$

From (4.9), for $u_1, u_2 \in \mathbb{X}$, we have

$$|f(x, t, u_2) - f(x, t, u_1)| \leq \frac{1}{2}|u_2 - u_1|, \quad (x, t) \in [0, 1] \times [0, +\infty). \tag{4.11}$$

Which implies

$$\|f(t, u_2) - f(t, u_1)\|_p \leq \frac{1}{2}\|u_2 - u_1\|_p, \quad t \in [0, +\infty),\ u_1, u_2 \in \mathbb{X}. \tag{4.12}$$

Then assumptions (H1) and (H2) hold. According to (4.8) and (4.12), we obtain that $(H3)$ is satisfied. Hence by Theorem 3.2, we conclude that the mild solution $u(t)$ of (4.5) satisfying $\lim_{t\to+\infty} u(t) = 0$, which implies $\|u(t)\|_p \to 0$ as $t \to +\infty$. □

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