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SEMICLASSICAL SOLUTIONS FOR LINEARLY COUPLED SCHRÖDINGER EQUATIONS

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Abstract. We consider the system of coupled nonlinear Schrödinger equations

$$\begin{gathered} -\varepsilon^2\Delta u + a(x)u = H_u(x,u,v) + \mu(x)v, \quad x \in \mathbb{R}^N, \\ -\varepsilon^2\Delta v + b(x)v = H_v(x,u,v) + \mu(x)u, \quad x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{gathered}$$

where $N \geq 3$, $a, b, \mu \in C(\mathbb{R}^N)$ and $H_u, H_v \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. Under conditions that $a_0 = \inf a = 0$ or $b_0 = \inf b = 0$ and $|\mu(x)|^2 \leq \theta a(x)b(x)$ with $\theta \in (0,1)$ and some mild assumptions on H, we show that the system has at least one nontrivial solution provided that $0 < \varepsilon \leq \varepsilon_0$, where the bound ε_0 is formulated in terms of N, a, b and H.

1. Introduction

In this article, we study the existence of semiclassical solutions of the system of coupled nonlinear Schrödinger equations

$$\begin{gathered} -\varepsilon^2\Delta u + a(x)u = H_u(x,u,v) + \mu(x)v, \quad x \in \mathbb{R}^N, \\ -\varepsilon^2\Delta v + b(x)v = H_v(x,u,v) + \mu(x)u, \quad x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $z := (u,v) \in \mathbb{R}^2$, $N \geq 3$, $a, b, \mu \in C(\mathbb{R}^N, \mathbb{R})$ and $H, H_u, H_v \in C(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. Systems of this type arise in nonlinear optics [1].

In the past several years, there are many papers about the semiclassical solutions of the nonlinear perturbed Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N)$$

under various hypotheses on the potential and the nonlinearity (see [2, 6, 7, 12, 13, 14, 16, 19, 22, 25]).

However, by Kaminow [17], we know that single-mode optical fibers are not really "single-mode", but actually bimodal due to the presence of birefringence. And recently, different authors focused their attention on coupled nonlinear Schrödinger

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systems (see [3, 4, 8, 9, 10, 11, 18]) which describe physical phenomena (see, e.g., [1, 5, 15]).

In a recent article, [8], Chen and Zou studied the system of nonlinear Schrödinger equations

$$
\begin{gathered}
-\varepsilon^2\Delta u + a(x)u = f(u) + \mu v, \quad x \in \mathbb{R}^N, \\
-\varepsilon^2\Delta v + b(x)v = g(v) + \mu u, \quad x \in \mathbb{R}^N, \\
u, v > 0 \quad \text{in } \mathbb{R}^N, \quad u, v \in H^1(\mathbb{R}^N),
\end{gathered} \tag{1.2}
$$

where N, a and b are the same as in (1.1). Under the assumptions

(i) there exists a constant $a_0 > 0$ such that $a(x),\ b(x) \ge a_0$ and $0 \le \mu < a_0$;

(ii) $f, g \in C(\mathbb{R}^N, \mathbb{R})$ and $\lim_{s\to 0} \frac{f(s)}{s} = \frac{g(s)}{s} = 0$;

(iii) there exists a constant $p_0 \in (1, 2^* - 1)$ such that

$$
\limsup_{s\to+\infty} \frac{f(s)}{s^{p_0}} < +\infty, \quad \limsup_{s\to+\infty} \frac{g(s)}{s^{p_0}} < +\infty;
$$

(iv) either $\limsup_{s\to+\infty} \frac{\int_0^s f(t)\,\mathrm{d}t}{s^2} = +\infty$ or $\limsup_{s\to+\infty} \frac{\int_0^s g(t)\,\mathrm{d}t}{s^2} = +\infty$.

They proved that (1.2) has a positive solution for sufficiently small $\varepsilon > 0$ and all $\mu \in (0, \mu_1]$ for some $\mu_1 \in (0, a_0)$.

Obviously, if $a_0 = 0$, their arguments become invalid due to the fact that $0 \le \mu < a_0$ can not be satisfied. To the best of our knowledge, the existence of semiclassical solutions to system (1.1), under the assumption of $a_0 = \inf a = 0$ or $b_0 = \inf b = 0$, has not ever been studied by variational methods. In addition, as the nonlinearity is non-autonomous and dependent on u and v, the problem will become more complex.

Motivated by [8, 20, 24, 26], we shall choose the case $a_0 = \inf a = 0$ or $b_0 = \inf b = 0$ as the objective of the present paper.

Before presenting the main results, we introduce the following assumptions.

(A0) $a(x) \ge a(0) = 0$, $b(x) \ge 0$ and there exist $a_0, b_0 > 0$ such that the sets $\mathcal{A}_{a_0} := \{x \in \mathbb{R}^N : a(x) < a_0\}$ and $\mathcal{B}_{b_0} := \{x \in \mathbb{R}^N : b(x) < b_0\}$ have finite measure;

(A1) there exists a constant $\theta \in (0, 1)$ such that $|\mu(x)|^2 \le \theta a(x)b(x)$, for all $x \in \mathbb{R}^N$;

(B0) $a(x) \ge 0$, $b(x) \ge b(0) = 0$ and there exist $a_0, b_0 > 0$ such that the sets $\mathcal{A}_{a_0} := \{x \in \mathbb{R}^N : a(x) < a_0\}$ and $\mathcal{B}_{b_0} := \{x \in \mathbb{R}^N : b(x) < b_0\}$ have finite measure;

(H1) there exist constants $p \in (2, 2^*)$ and $C > 0$ such that

$$
|H(x, z)| \le C(|z| + |z|^p), \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2;
$$

(H2) $H_z(x, z) \cdot z = o(|z|^2)$, as $|z| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(H3) $\lim_{|z|\to\infty} \frac{|H(x,z)|}{|z|^2} = \infty$ uniformly in $x \in \mathbb{R}^N$;

(H4) there exist $c_0 > 0$, $T_0 > 0$ and $q \in (2, 2^*)$ such that

$$
H(x, u, 0) \ge c_0|u|^q, \quad \forall x \in \mathbb{R}^N,\ u \in [-T_0, T_0]
$$

and

$$
u^{-2}h^{4-N} \int_{|x|\le h} H(\lambda^{-1/2}x, u/h, 0)\,\mathrm{d}x \ge \frac{(N^2 + 2)\omega_N}{2N(1 - 2^{-N})^2},
$$

for all $h \ge 1$, $\lambda \ge 1$, $u \ge hT_0$; here and in the sequel, $\omega_N = \mathrm{meas}(B_1(0)) = 2\pi^{N/2}/N\Gamma(N/2)$;

(H4') there exist $c_0 > 0$, $T_0 > 0$ and $q \in (2, 2^*)$ such that
$$H(x, 0, v) \geq c_0|v|^q, \quad \forall x \in \mathbb{R}^N,\ v \in [-T_0, T_0]$$
and
$$v^{-2}h^{4-N}\int_{|x|\leq h} H(\lambda^{-1/2}x, 0, v/h)\,\mathrm{d}x \geq \frac{(N^2+2)\omega_N}{2N(1-2^{-N})^2},$$
for all $h \geq 1$, $\lambda \geq 1$, $v \geq hT_0$;

(H5) $\mathcal{H}(x, z) := \frac{1}{2}H_z(x, z)\cdot z - H(x, z) \geq 0$ for all $(x, z) \in \mathbb{R}^N \times \mathbb{R}^2$, and there exist $c_1 > 0$ and $\kappa > \max\{1, N/2\}$ such that
$$\frac{H_z(x, z)\cdot z}{|z|^2} \geq \frac{(1-\theta)m_0}{3} \Rightarrow |H_z(x, z)\cdot z|^\kappa \leq c_1|z|^{2\kappa}\mathcal{H}(x, z),$$
where $m_0 := \min\{a_0, b_0\}$;

(H6') there exist $c_0 > 0$ and $q \in (2, 2^*)$ such that $H(x, u, 0) \geq c_0|u|^q$ for all $(x, u) \in \mathbb{R}^N \times \mathbb{R}$;

(H6") there exist $c_0 > 0$ and $q \in (2, 2^*)$ such that $H(x, 0, v) \geq c_0|v|^q$ for all $(x, v) \in \mathbb{R}^N \times \mathbb{R}$.

Remark 1.1. It is easy to check that (H6') and (H6") imply (H4) and (H4') with
$$T_0 = \Big[\frac{N^2+2}{2c_0(1-2^{-N})^2}\Big]^{1/(q-2)},$$
respectively, but (H4), (H4') can not yield (H6'), (H6"). We give the following nonlinear example to illustrate it. Let
$$H(x, u, v) = (|u|^2 + |v|^2)\ln(1 + |u| + |v|).$$
Clearly, H satisfies both (H4) and (H4') with
$$\ln(1 + T_0) = \frac{N^2+2}{2\big(1-2^{-N}\big)^2},$$
but neither (H6') nor (H6").

Example 1.2. Let $q \in (2, 2^*)$. Then it is easy to see that following two functions satisfy (H1)–(H3) and (H6'):
$$H(x, u, v) = a_1|u|^q + a_2|v|^q, \quad H(x, u, v) = \zeta(x)\big(|u|^2 + |v|^2\big)^{q/2},$$
where $a_1, a_2 > 0$ and $\zeta \in C(\mathbb{R}^N)$ with $0 < \inf_{\mathbb{R}^N}\zeta \leq \sup_{\mathbb{R}^N}\zeta < +\infty$.

Since $(q-2)N - 2q < 0$, we can let $h_0 \geq 1$ be such that
$$\begin{aligned}&\frac{(q-2)\omega_N}{2Nq(qc_0)^{2/(q-2)}}\Big\{\frac{N^2+2(N+2)}{(N+2)(1-2^{-N})^2}\Big\}^{q/(q-2)} h_0^{[(q-2)N-2q]/(q-2)}\\ &= \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N},\end{aligned} \tag{1.3}$$
where γ_0 and γ_{2^*} are embedding constants, see (2.1) and (2.2). If a and b satisfy (A0), we can choose $\lambda_0 > 1$ such that
$$\sup_{\lambda^{1/2}|x|\leq 2h_0}|a(x)| \leq h_0^{-2}, \quad \forall \lambda \geq \lambda_0, \tag{1.4}$$

if a and b satisfy (B0), we can choose $\lambda_0 > 1$ such that

$$\sup_{\lambda^{1/2}|x|\le 2h_0} |b(x)| \le h_0^{-2}, \quad \forall \lambda \ge \lambda_0. \tag{1.5}$$

Letting $\varepsilon^{-2} = \lambda$, (1.1) is rewritten as

$$\begin{gathered} -\Delta u + \lambda a(x) u = \lambda H_u(x,u,v) + \lambda \mu(x) v, \quad x \in \mathbb{R}^N, \\ -\Delta v + \lambda b(x) v = \lambda H_v(x,u,v) + \lambda \mu(x) u, \quad x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N). \end{gathered} \tag{1.6}$$

Let

$$\begin{aligned} \Phi_\lambda(z) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + \lambda a(x)|u|^2 + \lambda b(x)|v|^2)\,\mathrm{d}x \\ - \lambda \int_{\mathbb{R}^N} H(x,z)\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \mu(x) uv\,\mathrm{d}x, \quad z = (u,v). \end{aligned} \tag{1.7}$$

Obviously, the solutions of (1.1) are the critical points of $\Phi_{\varepsilon^{-1/2}}(z)$; the solutions of (1.6) are the critical points of $\Phi_\lambda(z)$.

We are now in a position to state the main results of this paper.

Theorem 1.3. *Assume that a, b, μ and H satisfy* (A0), (A1), (H1)–(H5). *Then for $0 < \varepsilon \le \lambda_0^{-1/2}$,* (1.1) *has a solution $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ such that*

$$0 < \Phi_{\varepsilon^{-1/2}}(z_\varepsilon) \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\varepsilon^{N-2},$$

$$\int_{\mathbb{R}^N} \mathcal{H}(x, z_\varepsilon)\,\mathrm{d}x \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\varepsilon^{N}.$$

Theorem 1.4. *Assume that a, b, μ and H satisfy* (A1), (B0), (H1)–(H3), (H4'), (H5). *Then for $0 < \varepsilon \le \lambda_0^{-1/2}$,* (1.1) *has a solution $z_\varepsilon = (u_\varepsilon, v_\varepsilon)$ such that*

$$0 < \Phi_{\varepsilon^{-1/2}}(z_\varepsilon) \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\varepsilon^{N-2},$$

$$\int_{\mathbb{R}^N} \mathcal{H}(x, z_\varepsilon)\,\mathrm{d}x \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\varepsilon^{N}.$$

Theorem 1.5. *Assume that a, b, μ and H satisfy* (A0), (A1), (H1)–(H5). *Then for $\lambda \ge \lambda_0$,* (1.6) *has a solution $z_\lambda = (u_\lambda, v_\lambda)$ such that*

$$0 < \Phi_\lambda(z_\lambda) \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{1-N/2},$$

$$\int_{\mathbb{R}^N} \mathcal{H}(x, z_\lambda)\,\mathrm{d}x \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{-N/2}.$$

Theorem 1.6. *Assume that a, b, μ and H satisfy* (A1), (B0), (H1)–(H3), (H4'), (H5). *Then for $\lambda \ge \lambda_0$,* (1.6) *has a solution $z_\lambda = (u_\lambda, v_\lambda)$ such that*

$$0 < \Phi_\lambda(z_\lambda) \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{1-N/2},$$

$$\int_{\mathbb{R}^N} \mathcal{H}(x, z_\lambda)\,\mathrm{d}x \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{-N/2}.$$

The rest of the article is organized as follows. In Section 2, we provide some preliminaries and lemmas. In Section 3, we give the proofs of Theorems 1.3–1.6.

2. Preliminaries

Let

$$E = \{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [a(x)|u|^2 + b(x)|v|^2]\,\mathrm{d}x < +\infty\},$$

$$\|z\|_{\lambda\dagger} = \Big\{\int_{\mathbb{R}^N} [|\nabla u|^2 + \lambda a(x)|u|^2 + |\nabla v|^2 + \lambda b(x)|v|^2]\,\mathrm{d}x\Big\}^{1/2}, \quad \forall z = (u,v) \in E.$$

Analogous to the proof of [23, Lemma 1], by using (A0) or (B0) and the Sobolev inequality, one can show that there exists a constant $\gamma_0 > 0$ independent of λ such that

$$\|z\|_{H^1(\mathbb{R}^N)} \le \gamma_0 \|z\|_{\lambda\dagger}, \quad \forall z \in E,\ \lambda \ge 1. \tag{2.1}$$

This shows that $(E, \|\cdot\|_{\lambda\dagger})$ is a Banach space for $\lambda \ge 1$. Furthermore, by the Sobolev embedding theorem, we have

$$\|z\|_s \le \gamma_s \|z\|_{H^1(\mathbb{R}^N)} \le \gamma_s\gamma_0 \|z\|_{\lambda\dagger}, \quad \forall z \in E,\ \lambda \ge 1,\ 2 \le s \le 2^*, \tag{2.2}$$

here and in the sequel, we denote by $\|\cdot\|_s$ the usual norm in space $L^s(\mathbb{R}^N)$.

In view of the definition of the norm $\|\cdot\|_{\lambda\dagger}$, we can re-write Φ_λ in the form

$$\Phi_\lambda(z) = \frac{1}{2}\|z\|_{\lambda\dagger}^2 - \lambda\int_{\mathbb{R}^N} H(x,z)\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} \mu(x)uv\,\mathrm{d}x, \quad \forall z \in E. \tag{2.3}$$

It is easy to see that $\Phi_\lambda \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\langle \Phi_\lambda'(z), \tilde z\rangle &= \int_{\mathbb{R}^N} [\nabla u\cdot\nabla\tilde u + \nabla v\cdot\nabla\tilde v + \lambda a(x)u\tilde u + \lambda b(x)v\tilde v]\,\mathrm{d}x\\ &\quad - \lambda\int_{\mathbb{R}^N} [H_u(x,z)\tilde u + H_v(x,z)\tilde v]\,\mathrm{d}x\\ &\quad - \lambda\int_{\mathbb{R}^N} \mu(x)(u\tilde v + v\tilde u)]\,\mathrm{d}x, \quad \forall z = (u,v),\ \tilde z = (\tilde u,\tilde v) \in E.\end{aligned} \tag{2.4}$$

As in [20], we let

$$\vartheta(x) := \begin{cases} \frac{1}{h_0}, & |x| \le h_0,\\ \frac{h_0^{N-1}}{1-2^{-N}}[|x|^{-N} - (2h_0)^{-N}], & h_0 < |x| \le 2h_0,\\ 0, & |x| > 2h_0.\end{cases} \tag{2.5}$$

Then $\vartheta \in H^1(\mathbb{R}^N)$, moreover,

$$\|\nabla\vartheta\|_2^2 = \int_{\mathbb{R}^N} |\nabla\vartheta(x)|^2\,\mathrm{d}x \le \frac{N^2\omega_N}{(N+2)(1-2^{-N})^2} h_0^{N-4}, \tag{2.6}$$

$$\|\vartheta\|_2^2 = \int_{\mathbb{R}^N} |\vartheta(x)|^2\,\mathrm{d}x \le \frac{2\omega_N}{(1-2^{-N})^2 N} h_0^{N-2}. \tag{2.7}$$

Let $e_\lambda(x) = \vartheta(\lambda^{1/2}x)$. Then we can prove the following lemma which is used for our proofs.

Lemma 2.1. *Let $H(x,z)\ge 0$, for all $(x,z)\in\mathbb{R}^N\times\mathbb{R}^2$. Suppose that* (A0), (A1), (H1)–(H4) *are satisfied. Then*

$$\sup\{\Phi_\lambda(se_\lambda,0):s\ge 0\}\le\frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{1-N/2},\quad\forall\lambda\ge\lambda_0.\tag{2.8}$$

Proof. From (H4), (1.3), (1.4) ,(1.7), (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned}
&\Phi_\lambda(se_\lambda,0)\\
&=\frac{s^2}{2}\int_{\mathbb{R}^N}(|\nabla e_\lambda|^2+\lambda a(x)|e_\lambda|^2)\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}H(x,se_\lambda,0)\,\mathrm{d}x\\
&=\lambda^{1-N/2}\Big[\frac{s^2}{2}\int_{\mathbb{R}^N}(|\nabla\vartheta|^2+a(\lambda^{-1/2}x)|\vartheta|^2)\,\mathrm{d}x-\int_{\mathbb{R}^N}H(\lambda^{-1/2}x,s\vartheta,0)\,\mathrm{d}x\Big]\\
&\le\lambda^{1-N/2}\Big[\frac{s^2}{2}\Big(\|\nabla\vartheta\|_2^2+\|\vartheta\|_2^2\sup_{|x|\le 2h_0}|a(\lambda^{-1/2}x)|\Big)\\
&\quad-\int_{|x|\le h_0}H(\lambda^{-1/2}x,s/h_0,0)\,\mathrm{d}x\Big]\\
&\le\lambda^{1-N/2}\Big[\frac{s^2}{2}(\|\nabla\vartheta\|_2^2+h_0^{-2}\|\vartheta\|_2^2)-\int_{|x|\le h_0}H(\lambda^{-1/2}x,s/h_0,0)\,\mathrm{d}x\Big],\\
&\quad\forall s\ge 0,\ \lambda\ge\lambda_0,
\end{aligned}\tag{2.9}$$

$$\begin{aligned}
&\frac{s^2}{2}(\|\nabla\vartheta\|_2^2+h_0^{-2}\|\vartheta\|_2^2)-\int_{|x|\le h_0}H(\lambda^{-1/2}x,s/h_0,0)\,\mathrm{d}x\\
&\le\frac{s^2}{2}[\|\nabla\vartheta\|_2^2+h_0^{-2}\|\vartheta\|_2^2-\frac{(N^2+2)\omega_N}{N(1-2^{-N})^2}h_0^{N-4}]\le 0,\quad\forall s\ge h_0T_0,\ \lambda\ge\lambda_0
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
&\frac{s^2}{2}(\|\nabla\vartheta\|_2^2+h_0^{-2}\|\vartheta\|_2^2)-\int_{|x|\le h_0}H(\lambda^{-1/2}x,s/h_0,0)\,\mathrm{d}x\\
&\le\frac{s^2}{2}(\|\nabla\vartheta\|_2^2+h_0^{-2}\|\vartheta\|_2^2)-\frac{c_0\omega_N}{N}s^qh_0^{N-q}\\
&\le\frac{(q-2)(\|\nabla\vartheta\|_2^2+h_0^{-2}\|\vartheta\|_2^2)^{q/(q-2)}}{2q(\frac{qc_0\omega_N}{N}h_0^{N-q})^{2/(q-2)}}\\
&\le\frac{(q-2)\omega_N}{2Nq(qc_0)^{2/(q-2)}}\Big\{\frac{N^2+2(N+2)}{(N+2)(1-2^{-N})^2}\Big\}^{q/(q-2)}h_0^{[(q-2)N-2q]/(q-2)}\\
&=\frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N},\quad\forall 0\le s\le h_0T_0,\quad\lambda\ge\lambda_0.
\end{aligned}\tag{2.11}$$

The conclusion of Lemma 2.1 follows from (2.9), (2.10) and (2.11). $\square$

We can prove the following lemma in the same way as Lemma 2.1.

Lemma 2.2. *Let $H(x,z)\ge 0$ for all $(x,z)\in\mathbb{R}^N\times\mathbb{R}^2$. Suppose that* (A1), (B0), (H1)–(H3), (H4') *are satisfied. Then*

$$\sup\{\Phi_\lambda(0,se_\lambda):s\ge 0\}\le\frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{1-N/2},\quad\forall\lambda\ge\lambda_0.\tag{2.12}$$

Applying the mountain-pass lemma without the (PS) condition, by standard arguments, we can prove the following two lemmas.

Lemma 2.3. *Let $H(x,z) \geq 0$ for all $(x,z) \in \mathbb{R}^N \times \mathbb{R}^2$. Suppose that* (A0), (A1), (H1)–(H4) *are satisfied. Then there exist a constant $d_\lambda \in (0, \sup_{s\geq 0} \Phi_\lambda(se_\lambda, 0)]$ and a sequence $\{z_n\} \subset E$ satisfying*

$$\Phi_\lambda(z_n) \to d_\lambda, \quad \|\Phi'_\lambda(z_n)\|_{E^*}(1 + \|z_n\|_{\lambda\dagger}) \to 0. \tag{2.13}$$

Lemma 2.4. *Let $H(x,z) \geq 0$ for all $(x,z) \in \mathbb{R}^N \times \mathbb{R}^2$. Suppose* (A1), (B0), (H1)–(H3), (H4') *are satisfied. Then there exist a constant $d_\lambda \in (0, \sup_{s\geq 0} \Phi_\lambda(0, se_\lambda)]$ and a sequence $\{z_n\} \subset E$ satisfying*

$$\Phi_\lambda(z_n) \to d_\lambda, \quad \|\Phi'_\lambda(z_n)\|_{E^*}(1 + \|z_n\|_{\lambda\dagger}) \to 0. \tag{2.14}$$

Lemma 2.5. *Suppose that* (A0), (A1), (H1)–(H5) *are satisfied. Then any sequence $\{z_n\} \subset E$ satisfying* (2.13) *is bounded in E.*

Proof. We argue by contradiction for proving boundedness of $\{z_n\}$. Suppose that $\|z_n\|_{\lambda\dagger} \to \infty$. Let $\tilde{z}_n = z_n/\|z_n\|_{\lambda\dagger} := (\tilde{u}_n, \tilde{v}_n)$. Then $\|\tilde{z}_n\|_{\lambda\dagger} = 1$. In view of (A1), we obtain

$$\begin{aligned} 2\lambda \int_{\mathbb{R}^N} \mu(x)\tilde{u}_n\tilde{v}_n \,\mathrm{d}x &\leq 2\theta\lambda \int_{\mathbb{R}^N} \sqrt{a(x)b(x)}|\tilde{u}_n\tilde{v}_n| \,\mathrm{d}x \\ &\leq \theta\lambda \int_{\mathbb{R}^N} [a(x)\tilde{u}_n^2 + b(x)\tilde{v}_n^2] \,\mathrm{d}x \leq \theta. \end{aligned} \tag{2.15}$$

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |\tilde{z}_n|^2 \,\mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [21] or [27, Lemma 1.21], $\tilde{z}_n \to (0,0)$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Set

$$\Omega_n := \big\{x \in \mathbb{R}^N : \frac{z_n \cdot H_z(x,z_n)}{|z_n|^2} \leq \frac{(1-\theta)m_0}{3}\big\}, \quad \mathcal{D} := \mathcal{A}_{a_0} \cup \mathcal{B}_{b_0}.$$

Hence, from (A0) and the Hölder inequality it follows that

$$\begin{aligned} &\lambda \int_{\Omega_n} \frac{|H_z(x,z_n)\cdot z_n|}{\|z_n\|_{\lambda\dagger}^2} \,\mathrm{d}x \\ &= \lambda \int_{\Omega_n} \frac{|H_z(x,z_n)\cdot z_n|}{|z_n|^2} |\tilde{z}_n|^2 \,\mathrm{d}x \\ &\leq \frac{(1-\theta)\lambda m_0}{3} \int_{\Omega_n} |\tilde{z}_n|^2 \,\mathrm{d}x \\ &\leq \frac{(1-\theta)\lambda m_0}{3} \int_{\mathbb{R}^N\setminus\mathcal{D}} |\tilde{z}_n|^2 \,\mathrm{d}x + \frac{(1-\theta)\lambda m_0}{3} \int_{\mathcal{D}} |\tilde{z}_n|^2 \,\mathrm{d}x \\ &\leq \frac{1-\theta}{3}\|\tilde{z}_n\|_{\lambda\dagger}^2 + \frac{(1-\theta)\lambda m_0[\mathrm{meas}(\mathcal{D})]^{1/(N+1)}}{3} \\ &\quad \times \Big(\int_{\mathcal{D}} |\tilde{z}_n|^{2(N+1)/N} \,\mathrm{d}x\Big)^{N/(N+1)} \\ &= \frac{1-\theta}{3} + o(1). \end{aligned} \tag{2.16}$$

From (2.3), (2.4) and (2.13), there holds

$$d_\lambda + o(1) = \lambda \int_{\mathbb{R}^N} \mathcal{H}(x, z_n)\,\mathrm{d}x. \tag{2.17}$$

Let $\kappa' = \kappa/(\kappa - 1)$, then $2 < 2\kappa' < 2^*$. By (H5), (2.17) and the Hölder inequality, one obtains

$$\begin{aligned}
&\lambda \int_{\mathbb{R}^N\setminus\Omega_n} \frac{|H_z(x, z_n)\cdot z_n|}{\|z_n\|_{\lambda\dagger}^2}\,\mathrm{d}x \\
&= \lambda \int_{\mathbb{R}^N\setminus\Omega_n} \frac{|H_z(x, z_n)\cdot z_n|}{|z_n|^2}|\tilde{z}_n|^2\,\mathrm{d}x \\
&\le \lambda\Big[\int_{\mathbb{R}^N\setminus\Omega_n} \Big(\frac{|H_z(x, z_n)\cdot z_n|}{|z_n|^2}\Big)^\kappa \mathrm{d}x\Big]^{1/\kappa}\Big(\int_{\mathbb{R}^N\setminus\Omega_n} |\tilde{z}_n|^{2\kappa'}\,\mathrm{d}x\Big)^{1/\kappa'} \\
&\le \lambda\Big(c_1 \int_{\mathbb{R}^N\setminus\Omega_n} \mathcal{H}(x, z_n)\,\mathrm{d}x\Big)^{1/\kappa}\Big(\int_{\mathbb{R}^N} |\tilde{z}_n|^{2\kappa'}\,\mathrm{d}x\Big)^{1/\kappa'} \\
&\le \lambda^{1-1/\kappa}[c_1 d_\lambda + o(1)]^{1/\kappa}\|\tilde{z}_n\|_{2\kappa'}^2 = o(1).
\end{aligned} \tag{2.18}$$

Combining (2.17) with (2.18) and using (2.4), (2.13) and (2.15), we have

$$\begin{aligned}
1 + o(1) &\le \frac{\|z_n\|_{\lambda\dagger}^2 - \langle \Phi_\lambda'(z_n), z_n\rangle}{\|z_n\|_{\lambda\dagger}^2} \\
&= \lambda \int_{\mathbb{R}^N} \frac{|H_z(x, z_n)\cdot z_n|}{\|z_n\|_{\lambda\dagger}^2} + 2\lambda \int_{\mathbb{R}^N} \mu(x)\tilde{u}_n\tilde{v}_n\,\mathrm{d}x \\
&\le \lambda \int_{\Omega_n} \frac{|H_z(x, z_n)\cdot z_n|}{\|z_n\|_{\lambda\dagger}^2}\,\mathrm{d}x + \lambda \int_{\mathbb{R}^N\setminus\Omega_n} \frac{|H_z(x, z_n)\cdot z_n|}{\|z_n\|_{\lambda\dagger}^2}\,\mathrm{d}x + \theta \\
&\le \frac{1 + 2\theta}{3} + o(1).
\end{aligned} \tag{2.19}$$

This contradiction shows that $\delta > 0$.

Going to a subsequence if necessary, we assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |\tilde{z}_n|^2 dx > \frac{\delta}{2}$. Let $w_n(x) = \tilde{z}_n(x + k_n)$; then

$$\int_{B_{1+\sqrt{N}}(0)} |w_n|^2 dx > \frac{\delta}{2}. \tag{2.20}$$

Now we define $\hat{z}_n(x) = z_n(x + k_n)$, then $\hat{z}_n/\|z_n\|_{\lambda\dagger} = w_n$ and $\|w_n\|_{H^1(\mathbb{R}^N)}^2 = \|\tilde{z}_n\|_{H^1(\mathbb{R}^N)}^2$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \to w$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $w_n \to w$ a.e. on $\mathbb{R}^N$. Obviously, (2.20) implies that $w \ne (0, 0)$. For a.e. $x \in \{z \in \mathbb{R}^N : w(z) \ne (0, 0)\}$, we have $\lim_{n\to\infty} |\hat{z}_n(x)| = \infty$. Hence, it follows from (H3), (2.3), (2.13), (2.15) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{d_\lambda + o(1)}{\|z_n\|_{\lambda\dagger}^2} = \lim_{n\to\infty} \frac{\Phi_\lambda(z_n)}{\|z_n\|_{\lambda\dagger}^2} \\
&= \lim_{n\to\infty} \Big[\frac{1}{2}\|\tilde{z}_n\|_{\lambda\dagger}^2 - \lambda \int_{\mathbb{R}^N} \frac{H(x + k_n, \hat{z}_n)}{|\hat{z}_n|^2}|w_n|^2\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} \mu(x)\tilde{u}_n\tilde{v}_n\,\mathrm{d}x\Big] \\
&\le \frac{1 + \theta}{2} - \lambda \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{H(x + k_n, \hat{z}_n)}{|\hat{z}_n|^2}|w_n|^2\,\mathrm{d}x = -\infty.
\end{aligned}$$

This contradiction shows that $\{\|z_n\|_{\lambda\dagger}\}$ is bounded. □

We can prove the following lemma in the same way as Lemma 2.5.

Lemma 2.6. *Suppose that* (A1), (B0), (H1)–(H3), (H4'), (H5) *are satisfied. Then any sequence* $\{z_n\} \subset E$ *satisfying* (2.14) *is bounded in* E.

3. Proofs of main results

In this section, we give the proofs of Theorems 1.3–1.6.

Proof of Theorem 1.5. Applying Lemmas 2.1, 2.3 and 2.5, we deduce that there exists a bounded sequence $\{z_n\} \subset E$ satisfying (2.13) with

$$d_\lambda \le \frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{1-N/2}, \quad \forall \lambda \ge \lambda_0. \tag{3.1}$$

Going to a subsequence, if necessary, we can assume that $z_n \rightharpoonup z_\lambda$ in $(E, \|\cdot\|_{\lambda\dagger})$ and $\Phi_\lambda'(z_n) \to 0$. Next, we prove that $z_\lambda \ne (0,0)$.

Arguing by contradiction, suppose that $z_\lambda = (0,0)$, i.e. $z_n \rightharpoonup (0,0)$ in E, and so $z_n \to (0,0)$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $z_n \to (0,0)$ a.e. on $\mathbb{R}^N$. Since $\mathcal{D}$ is a set of finite measure, there holds

$$\|z_n\|_2^2 = \int_{\mathbb{R}^N\setminus\mathcal{D}} |z_n|^2\,\mathrm{d}x + \int_{\mathcal{D}} |z_n|^2\,\mathrm{d}x \le \frac{1}{\lambda m_0}\|z_n\|_{\lambda\dagger}^2 + o(1). \tag{3.2}$$

For $s \in (2, 2^*)$, it follows from (2.2), (3.2) and the Hölder inequality that

$$\begin{aligned} \|z_n\|_s^s &\le \|z_n\|_2^{2(2^*-s)/(2^*-2)}\|z_n\|_{2^*}^{2^*(s-2)/(2^*-2)} \\ &\le (\gamma_{2^*}\gamma_0)^{2^*(s-2)/(2^*-2)}(\lambda m_0)^{-(2^*-s)/(2^*-2)}\|z_n\|_{\lambda\dagger}^s + o(1). \end{aligned} \tag{3.3}$$

According to (3.2), one can obtain that

$$\begin{aligned} \lambda\int_{\Omega_n} H_z(x,z_n)\cdot z_n\,\mathrm{d}x &= \lambda\int_{\Omega_n}\frac{H_z(x,z_n)\cdot z_n}{|z_n|^2}|z_n|^2\,\mathrm{d}x \\ &\le \frac{(1-\theta)\lambda m_0}{3}\|z_n\|_2^2 \le \frac{1-\theta}{3}\|z_n\|_{\lambda\dagger}^2 + o(1). \end{aligned} \tag{3.4}$$

By (2.3), (2.4) and (2.13), we have

$$\Phi_\lambda(z_n) - \frac{1}{2}\langle\Phi_\lambda'(z_n), z_n\rangle = \lambda\int_{\mathbb{R}^N}\mathcal{H}(x,z_n)\,\mathrm{d}x = d_\lambda + o(1). \tag{3.5}$$

Using (H5), (3.1), (3.3) with $s = 2\kappa/(\kappa-1)$ and (3.5), we obtain

$$
\begin{aligned}
&\lambda \int_{\mathbb{R}^N\setminus\Omega_n} H_z(x,z_n)\cdot z_n \,\mathrm{d}x \\
&\le \lambda\Big(\int_{\mathbb{R}^N\setminus\Omega_n}\Big(\frac{|H_z(x,z_n)\cdot z_n|}{|z_n|^2}\Big)^{\kappa}\mathrm{d}x\Big)^{1/\kappa}\|z_n\|_s^2 \\
&\le (\gamma_{2^*}\gamma_0)^{2\cdot 2^*(s-2)/s(2^*-2)}\lambda\Big(c_1\int_{\mathbb{R}^N\setminus\Omega_n}\mathcal{H}(x,z_n)\,\mathrm{d}x\Big)^{1/\kappa} \\
&\quad\times(\lambda m_0)^{-2(2^*-s)/s(2^*-2)}\|z_n\|_{\lambda\dagger}^2+o(1) \\
&\le c_1^{1/\kappa}(\gamma_{2^*}\gamma_0)^{N/\kappa}\lambda^{1-1/\kappa}d_\lambda^{1/\kappa}(\lambda m_0)^{-2(2^*-s)/s(2^*-2)}\|z_n\|_{\lambda\dagger}^2+o(1) \\
&= \frac{c_1^{1/\kappa}(\gamma_{2^*}\gamma_0)^{N/\kappa}}{m_0^{(2\kappa-N)/2\kappa}}\big[\lambda^{(N-2)/2}d_\lambda\big]^{1/\kappa}\|z_n\|_{\lambda\dagger}^2+o(1) \\
&\le \frac{c_1^{1/\kappa}(\gamma_{2^*}\gamma_0)^{N/\kappa}}{m_0^{(2\kappa-N)/2\kappa}}\big[\frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\big]^{1/\kappa}\|z_n\|_{\lambda\dagger}^2+o(1) \\
&= \frac{1-\theta}{3}\|z_n\|_{\lambda\dagger}^2+o(1),
\end{aligned}
\tag{3.6}
$$

which, together with (2.4), (2.13) and (3.4), yields

$$
\begin{aligned}
o(1) &= \langle\Phi_\lambda'(z_n),z_n\rangle \\
&= \|z_n\|_{\lambda\dagger}^2-\lambda\int_{\mathbb{R}^N}H_z(x,z_n)\cdot z_n\,\mathrm{d}x-2\lambda\int_{\mathbb{R}^N}\mu(x)u_nv_n\,\mathrm{d}x \\
&\ge (1-\theta)\|z_n\|_{\lambda\dagger}^2-\lambda\int_{\Omega_n}H_z(x,z_n)\cdot z_n\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N\setminus\Omega_n}H_z(x,z_n)\cdot z_n\,\mathrm{d}x \\
&\ge \frac{1-\theta}{3}\|z_n\|_{\lambda\dagger}^2+o(1).
\end{aligned}
\tag{3.7}
$$

Consequently, it follows from (2.3) and (2.13) that

$$
0<d_\lambda=\lim_{n\to\infty}\Phi_\lambda(z_n)\le\frac{1+\theta}{2}\lim_{n\to\infty}\|z_n\|_{\lambda\dagger}^2=0,
$$

since $H(x,z)\ge 0,\ \forall(x,z)\in\mathbb{R}^N\times\mathbb{R}^2$. This contradiction shows $z_\lambda\ne(0,0)$. By a standard argument, we easily certify that $\Phi_\lambda'(z_\lambda)=0$ and $\Phi_\lambda(z_\lambda)\le d_\lambda$. Then z_λ is a nontrivial solution of (1.7), moreover

$$
d_\lambda\ge\Phi_\lambda(z_\lambda)=\Phi_\lambda(z_\lambda)-\frac{1}{2}\langle\Phi_\lambda'(z_\lambda),z_\lambda\rangle=\lambda\int_{\mathbb{R}^N}\mathcal{H}(x,z_\lambda)\,\mathrm{d}x. \tag{3.8}
$$

□

Proof of Theorem 1.6. Applying Lemmas 2.2, 2.4 and 2.6, we deduce that there exists a bounded sequence $\{z_n\}\subset E$ satisfying (2.14) with

$$
d_\lambda\le\frac{(1-\theta)^\kappa m_0^{(2\kappa-N)/2}}{3^\kappa c_1(\gamma_{2^*}\gamma_0)^N}\lambda^{1-N/2},\quad\forall\lambda\ge\lambda_0.
$$

The rest of the proof is the same as Theorem 1.5, so we omit it. □

Theorem 1.3 and 1.4 are direct consequences of Theorem 1.5 and 1.6, respectively. We omit their proofs.

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