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HÖLDER CONTINUITY OF BOUNDED WEAK SOLUTIONS TO GENERALIZED PARABOLIC p-LAPLACIAN EQUATIONS II: SINGULAR CASE

SUKJUNG HWANG, GARY M. LIEBERMAN

ABSTRACT. Here we generalize quasilinear parabolic p-Laplacian type equations to obtain the prototype equation

$$u_t - \operatorname{div}\Big(\frac{g(|Du|)}{|Du|}Du\Big) = 0,$$

where g is a nonnegative, increasing, and continuous function trapped in between two power functions $|Du|^{g_0-1}$ and $|Du|^{g_1-1}$ with $1 < g_0 \le g_1 \le 2$. Through this generalization in the setting from Orlicz spaces, we provide a uniform proof with a single geometric setting that a bounded weak solution is locally Hölder continuous with some degree of commonality between degenerate and singular types. By using geometric characters, our proof does not rely on any of alternatives which is based on the size of solutions.

1. Introduction

This article is intended as a companion paper to [12], which proved the Hölder continuity of solutions to degenerate parabolic equations satisfying a generalized p-Laplacian structure. Here, we examine the same question for singular equations, but we refer the reader to [12] for a more detailed description of the history of this problem.

Our interest here is in the parabolic equation

$$u_t - \operatorname{div}\mathcal{A}(x,t,u,Du) = 0 \tag{1.1}$$

when there is an increasing function g such that

$$\mathcal{A}(x,t,u,\xi)\cdot\xi \ge C_0 G(|\xi|), \tag{1.2a}$$

$$|\mathcal{A}(x,t,u,\xi)| \le C_1 g(|\xi|) \tag{1.2b}$$

for some positive constants C_0 and C_1, where G is defined by

$$G(\sigma) = \int_0^\sigma g(s)\,ds,$$

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and we assume that there are constants g_0 and g_1 satisfying $1 < g_0 \le g_1$ such that

$$g_0 G(\sigma) \le \sigma g(\sigma) \le g_1 G(\sigma) \tag{1.3}$$

for all $\sigma > 0$. For the most part, we are only concerned here with the case that $g_1 \le 2$, but some of our results do not need this additional restriction, so we shall always state it explicitly when it is used. Our results generalized those of Ladyzhenskaya and Ural'tseva [15] and Chen and DiBenedetto [2, 3], who proved Hölder continuity under the structure conditions

$$\mathcal{A}(x,t,u,\xi) \cdot \xi \ge C_0 |\xi|^p, \ |\mathcal{A}(x,u,\xi)| \le C_1 |\xi|^{p-1} \tag{1.4}$$

with $p = 2$ and $p < 2$, respectively. The structure (1.4) is contained in this model as the special case $g(s) = s^{p-1}$, in which case we may take $g_0 = g_1 = p$, but we consider a class of structure functions g much wider than that of just power functions. In this way, we obtain a uniform proof of Hölder continuity (with appropriate uniformity of constants) for all $p \in (1,2]$ at once under the structure condition (1.4) as well as a proof of Hölder continuity under more general structure conditions.

In [12], we have discussed our approach for a generalization of the case $p \ge 2$, so we concern ourselves here with the points relevant to the generalization of the case $p \le 2$. It is known that solutions of this problem generally become zero in finite time when $p < 2$ (see [7, Sections VII.2 and VII.3] for a more complete discussion of this phenomenon) but not when $p = 2$ (because of the Harnack inequality, first proved by Moser [19]), so our proof needs to take this behavior into account. In addition, [8, Section 4] gives a Hölder exponent which degenerates as p approaches 2; the proof must be further modified for p close to 2 if the Hölder exponent is to remain positive near $p = 2$. Our method manages the whole range $1 < p \le 2$ uniformly for p away from 1. Although, as the authors point out in [9], this method is more complicated analytically, it does handle the whole range easily and it is quite simple geometrically.

We use the definition of weak solution given in [12], which we now present. For an arbitrary open set $\Omega \subset \mathbb{R}^{n+1}$, we introduce the generalized Sobolev space $W^{1,G}(\Omega)$, which consists of all functions u defined on Ω with weak derivative Du satisfying

$$\iint_\Omega G(|Du|)\,dx\,dt < \infty.$$

We say that $u \in C_{\mathrm{loc}}(\Omega) \cap W^{1,G}(\Omega)$ is a *weak supersolution* of (1.1) if

$$0 \le -\iint_\Omega u\varphi_t\,dx\,dt + \iint_\Omega \mathcal{A}(x,t,u,Du) \cdot D\varphi\,dx\,dt$$

for all $\varphi \in C^1(\bar\Omega)$ which vanish on the parabolic boundary of Ω; a *weak subsolution* is defined by reversing the inequality; and a *weak solution* is a function which is both a weak supersolution and a weak subsolution. In fact, we shall use a larger class of φ's which we discuss in a later section.

Our method of proof uses some recent geometric ideas of Gianazza, Surnachev, and Vespri [10], who gave a different proof for the Hölder continuity in [2, 3]. While [2, 3] examine an alternative based on the size of the set on which $|u|$ is close to its maximum, the method in [10] use a geometric approach from regularity theory and Harnack estimates. Here, we use this geometric approach along with some elements of the analytic approach in [3].

The proof is based on studying two cases separately. Either a bounded weak solution u is close to its maximum at least half of a cylinder around (x_0, t_0) or not.

In either case, the conclusion is that the essential oscillation of u is smaller in a subcylinder centered at (x_0, t_0). Basically, our goal is reached using the geometric character of u with two integral estimates, local and logarithmic estimates (5.2), (5.3).

In the next section, we provide some preliminary results, mostly involving notation for our geometric setting. Section 3 states the main lemma and uses that lemma to prove the Hölder continuity of the weak solutions. The main lemma is proved in Section 4, based on some integral inequalities which are proved in Section 5.

2. Preliminaries

Notation. (1) The parameters g_0, g_1, N, C_0, and C_1 are the data. When we make the additional assumption that $g_1 \le 2$, we use the word "data" to denote the constants g_0, N, C_0, and C_1.

(2) Let K_ρ^y denote the $N-$dimensional cube centered at $y \in \mathbb{R}^N$ with the side length 2ρ, i.e.,

$$K_\rho^y := \{x \in \mathbb{R}^N : \max_{1 \le i \le N} |x^i - y^i| < \rho\}.$$

(Here, we use superscripts to denote the coordinates of x; we'll use subscripts to indicate different points.) For simpler notation, let $K_\rho := K_\rho^0$. We also define the spatial distance $|\cdot|_\infty$ by

$$|x - y|_\infty = \max_{1 \le i \le N} |x^i - y^i|.$$

In fact, all of our work can be recast with the ball

$$B_\rho^y = \{x \in \mathbb{R}^N : |x - y| < \rho\},$$

where $|x - y|$ is the usual Euclidean distance, in place of K_ρ^y with only slight notational changes. There is no significant reason to use cubes rather than balls in the degenerate case, but the method used in [2, 3] requires that cubes be subdivided into congruent smaller subcubes, and the corresponding decomposition for balls is much more complicated. In this work, no such decomposition is needed.

(3) For given $(x_0, t_0) \in \mathbb{R}^{N+1}$, and given positive constants θ, ρ and k, we say

$$T_{k,\rho}(\theta) := \theta k^2 G\big(\frac{k}{\rho}\big)^{-1},$$
$$Q_{k,\rho}^{x_0,t_0}(\theta) := K_\rho^{x_0} \times [t_0 - T_{k,\rho}, t_0],$$
$$Q_{k,\rho}(\theta) := Q_{k,\rho}^{0,0}(\theta).$$

The point (x_0, t_0) is called the *top-center point of* $Q_{k,\rho}^{x_0,t_0}(\theta)$. We also abbreviate

$$T_{k,\rho} = T_{k,\rho}(1), \quad Q_{k,\rho}^{x_0,t_0} = Q_{k,\rho}^{x_0,t_0}(1), \quad Q_{k\rho} = Q_{k,\rho}(1).$$

Geometry. We refer the reader to [12] for a discussion of our choices of notation, but we do recall that if u is any function defined on an open set Ω, then for any positive number ω and any $(x_0, t_0) \in \Omega$, there a number R such that

$$Q_{\omega,4R}^{x_0,t_0} \subset \Omega.$$

Useful inequalities. Because of the generalized functions g and G, we are not able to apply Hölder's inequality or typical Young's inequality. Here we present essential inequalities which will be used through out the paper, all of which were proved in [12].

Lemma 2.1. *For a nonnegative and nondecreasing function $g \in C[0,\infty)$, let G be the antiderivative of g. Suppose that g and G satisfy (1.3). Then for all nonnegative real numbers σ, σ_1, and σ_2, we have*

(a) *$G(\sigma)/\sigma$ is a monotone increasing function.*
(b) *For $\beta > 1$,*
$$\beta^{g_0} G(\sigma) \le G(\beta\sigma) \le \beta^{g_1} G(\sigma).$$
(c) *For $0 < \beta < 1$,*
$$\beta^{g_1} G(\sigma) \le G(\beta\sigma) \le \beta^{g_0} G(\sigma).$$
(d) *$\sigma_1 g(\sigma_2) \le \sigma_1 g(\sigma_1) + \sigma_2 g(\sigma_2)$.*
(e) (Young's inequality) *For any $\epsilon \in (0,1)$,*
$$\sigma_1 g(\sigma_2) \le \epsilon^{1-g_1} g_1 G(\sigma_1) + \epsilon g_1 G(\sigma_2).$$

Lemma 2.2. *For any $\sigma > 0$, let*
$$h(\sigma) = \frac{1}{\sigma}\int_0^\sigma g(s)\,ds, \quad H(\sigma) = \int_0^\sigma h(s)\,ds.$$
Then we have
$$\begin{gathered}
g_0 h(\sigma) \le g(\sigma) \le g_1 h(\sigma),\\
g_0 H(\sigma) \le G(\sigma) \le g_1 H(\sigma),\\
(g_0 - 1)h(\sigma) \le \sigma h'(\sigma) \le (g_1 - 1)h(\sigma),\\
\frac{1}{g_1}\sigma h(\sigma) \le H(\sigma) \le \frac{1}{g_0}\sigma h(\sigma),\\
\beta^{g_0} H(\sigma) \le H(\beta\sigma) \le \beta^{g_1} H(\sigma)
\end{gathered}$$
for any $\beta > 1$.

Our next result concerns some inequalities about integration of a function over various intervals. We shall use these inequalities in the proof of the Main Lemma. This lemma is probably well-known, but we are unaware of any reference for it.

Lemma 2.3. *Let f be a continuous, decreasing, positive function defined on $(0,\infty)$. Then, for all δ and $\sigma \in (0,1)$, we have*
$$\int_0^1 f(\delta+s)\,ds \le \frac{1}{\sigma}\int_0^\sigma f(\delta+s)\,ds. \tag{2.1}$$
If, in addition, for all $\beta > 1$ and $\sigma > 0$, we have
$$\beta f(\beta\sigma) \ge f(\sigma), \quad f(\beta\sigma) \le f(\sigma), \tag{2.2}$$
then, for all $\delta \in (0,1)$, we have
$$\int_0^\delta f(\delta+s)\,ds \le \frac{2}{2+\ln(1/\delta)}\ int_0^1 f(\delta+s)\,ds. \tag{2.3}$$

Proof. To prove (2.1), we define the function

$$F(\sigma) = \sigma \int_0^1 f(\delta + s)\, ds - \int_0^\sigma f(\delta + s)\, ds.$$

Since

$$F'(\sigma) = \int_0^1 f(\delta + s)\, ds - f(\delta + \sigma),$$

and f is decreasing, it follows that F' is increasing so F is convex. Moreover

$$F(0) = F(1) = 0,$$

so $F(\sigma) \le 0$ for all $\sigma \in (0,1)$. Simple algebra then yields (2.1).

To prove (2.3), we first use a change of variables to see that, for any $j \ge 1$, we have

$$\int_{j\delta}^{2j\delta} f(\delta + s)\, ds = \int_0^{j\delta} f((j+1)\delta + \sigma)\, d\sigma = j \int_0^\delta f((j+1)\delta + js))\, ds.$$

Since $(j+1)\delta + js \le (j+1)(\delta + s)$ and f is decreasing, we have

$$\int_{j\delta}^{2j\delta} f(\delta + s)\, ds \ge j \int_0^\delta f((j+1)(\delta + s))\, ds$$

and then (2.2) gives

$$\int_{j\delta}^{2j\delta} f(\delta + s)\, ds \ge \frac{j}{j+1} \int_0^\delta f(\delta + s)\, ds.$$

We now let J be the unique positive integer such that $2^{-J} < \delta \le 2^{1-J}$ and we take $j = 2^i$ with $i = 0, \dots, J-1$. Since $j/(j+1) \ge 1/2$, it follows that

$$\int_0^\delta f(\delta + s)\, ds \le 2 \int_{2^i\delta}^{2^{i+1}\delta} f(\delta + s)\, ds.$$

Since

$$\int_0^{2^J\delta} f(\delta + s)\, ds = \int_0^\delta f(\delta + s)\, ds + \sum_{i=0}^{J-1} \int_{2^i\delta}^{2^{i+1}\delta} f(\delta + s)\, ds,$$

we infer that

$$\int_0^{2^J\delta} f(\delta + s)\, ds \ge [1 + \frac{1}{2}J] \int_0^\delta f(\delta + s)\, ds.$$

The proof is completed by noting that $J > \ln(1/\delta)$ and that

$$\int_0^1 f(\delta + s)\, ds \ge \int_0^{2^J\delta} f(\delta + s)\, ds.$$

$\square$

Note that condition (2.2) is satisfied if $f(\sigma) = \sigma^{-p}$ with $0 \le p \le 1$, in which case this lemma can be proved by computing the integrals directly.

3. Basic results and the proof of Hölder continuity

In this section, we prove the Hölder continuity of solutions of (1.1) for singular equations (that is, equations with $g_1 \le 2$) and for degenerate equations (that is, equations with $g_0 \ge 2$). Our proof is based on some estimates for nonnegative supersolutions of the equation, and these estimates will be proved in the next section.

Our Main Lemma states that a nonnegative supersolution u of a singular equation is strictly positive in a subcylinder if u is near to the maximum value in more than a half of cylinder.

Lemma 3.1 (Main Lemma). *Let ω and R be positive constants. Then there are positive constants δ and μ, both less than one and determined only by the data such that, if u is a nonnegative solution of* (1.1) *in*

$$Q = Q_{\delta\omega,2R}\big(\frac{3}{4}\big)$$

with $g_1 \le 2$, and

$$\big|Q \cap \{u \le \frac{\omega}{2}\}\big| \le \frac{1}{2}|Q|, \tag{3.1}$$

then

$$\operatorname{ess\,inf}_{\mathcal{Q}} u \ge \mu\omega, \tag{3.2}$$

with $\mathcal{Q} = Q_{\mu\omega,R/2}$.

We shall prove this lemma in the next section. Here we show first how to infer a decay estimate for the oscillation of any bounded solution of (1.1).

Lemma 3.2. *Let C_0, C_1, g_0, g_1, ρ, and ω be positive constants with $C_0 \le C_1$ and $1 < g_0 \le g_1 \le 2$. Then there are positive constants σ and λ, both less than one and determined only by data such that, if u is a bounded weak solution of* (1.1) *in $Q_{\omega,\rho}$ with*

$$\operatorname{ess\,osc}_{Q_{\omega,\rho}} u \le \omega,$$

then

$$\operatorname{ess\,osc}_{Q_{\sigma\omega,\lambda\rho}} u \le \sigma\omega. \tag{3.3}$$

Proof. We begin by taking δ and μ to be the constants from Lemma 3.1 and we set

$$\sigma = 1 - \mu, \quad \lambda = \frac{1}{4}\left(\frac{\mu}{\sigma}\right)^{(2-g_0)/g_0}.$$

From the proof of Lemma 3.1, it follows that $\mu \le 1/4$, so $\mu/\sigma \le 1$. We also introduce the functions u_1 and u_2 by

$$u_1 = u - \inf_{Q_{\omega,\rho}} u, \quad u_2 = \omega - u_1. \tag{3.4}$$

It follows from Lemma 2.1(b) that

$$3\big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega}{\rho}\big)^{-1} \le \omega^2 G\big(\frac{\omega}{\rho}\big)^{-1},$$

and hence the cylinder Q from Lemma 3.1 is a subset of $Q_{\omega,\rho}$ provided $R = \rho/2$.

There are now two cases. First, if

$$\big|Q \cap \{u_1 \le \frac{\omega}{2}\}\big| \le \frac{1}{2}|Q|,$$

then we apply Lemma 3.1 to u_1 and hence

$$\operatorname{ess\,inf}_{\mathcal{Q}} u_1 \geq \mu\omega.$$

Since

$$\operatorname{ess\,sup}_{\mathcal{Q}} u_1 \leq \omega,$$

it follows that

$$\operatorname{ess\,osc}_{\mathcal{Q}} u = \operatorname{ess\,osc}_{\mathcal{Q}} u_1 \leq (1-\mu)\omega = \sigma\omega.$$

On the other hand if

$$\big|Q \cap \{u_1 \leq \frac{\omega}{2}\}\big| \geq \frac{1}{2}|Q|,$$

then

$$\big|Q \cap \{u_2 \leq \frac{\omega}{2}\}\big| \leq \frac{1}{2}|Q|,$$

and an application of Lemma 3.1 to u_2 implies once again that

$$\operatorname{ess\,osc}_{\mathcal{Q}} u \leq \sigma\omega.$$

Since $4\lambda\mu/\sigma \leq 1$, we infer from Lemma 2.1(c) that

$$(\sigma\omega)^2 G\big(\frac{\sigma\omega}{\lambda\rho}\big)^{-1} \leq (\mu\omega)^2 G\big(\frac{4\mu\omega}{\rho}\big)^{-1}.$$

Since $\lambda \leq 1/4$, it follows that $Q_{\sigma\omega,\lambda\rho}$ is a subset of the cylinder $\mathcal{Q}$ from Lemma 3.1, and (3.3) follows. □

For any real number τ and any function u defined on an open subset Ω of $\mathbb{R}^{N+1}$, we define

$$|\tau|_G = \frac{U}{G^{-1}(U^2/|\tau|)}, \tag{3.5a}$$

where

$$U = \operatorname{ess\,osc}_{\Omega} u. \tag{3.5b}$$

With this time scale, we define the parabolic distance between two sets such $\mathcal{K}_1$ and $\mathcal{K}_2$ by

$$\operatorname{dist}_P(\mathcal{K}_1;\mathcal{K}_2) := \inf_{\substack{(x,t)\in\mathcal{K}_1\\ (y,s)\in\mathcal{K}_2,\, s\leq t}} \max\{|x-y|_\infty, |t-s|_G\}.$$

(Note that, strictly speaking, this quantity is not a distance because it is not symmetric with respect to the order in which we write the sets. Nonetheless, the terminology of distance is useful as a suggestion of the technically correct situation.)

The proof of [12, Theorem 2.4] immediately yields a modulus of continuity in terms of G and a Hölder continuity estimate.

Theorem 3.3. *Let u be a bounded weak solution of* (1.1) *with* (1.2) *in Ω, and suppose $1 < g_0 \leq g_1 \leq 2$. Then u is locally continuous. Moreover, there exist constants γ and $\alpha \in (0,1)$ depending only upon the data such that, for any two distinct points (x_1,t_1) and (x_2,t_2) in any subset Ω' of Ω with $\operatorname{dist}_P(\Omega';\partial_p\Omega)$ positive, we have*

$$|u(x_1,t_1)-u(x_2,t_2)| \leq \gamma U\Big(\frac{|x_1-x_2|+|t_1-t_2|_G}{\operatorname{dist}_P(\Omega';\partial_P\Omega)}\Big)^\alpha. \tag{3.6}$$

In addition (with the same constants),

$$|u(x_1,t_1)-u(x_2,t_2)| \le \gamma U\Big(\frac{|x_1-x_2|+|1|_G \max\{|t_1-t_2|^{1/g_0},|t_1-t_2|^{1/g_1}\}}{\operatorname{dist}_P(\Omega';\partial_P\Omega)}\Big)^{\alpha}. \tag{3.7}$$

For initial regularity, we have the following variant of Lemma 3.1. Note that this lemma is essentially the same as [7, Proposition IV.13.1] (the result is mentioned only indirectly in [3]), but the proof is much simpler. To simplify notation, we define the cylinders

$$Q^{+,x_0,t_0}_{k,R}(\theta) = K^{x_0}_R \times \Big(t_0, t_0+\theta k^2 G\big(\frac{k}{R}\big)^{-1}\Big), \quad Q^+_{k,R}(\theta) = Q^{+,0,0}_{k,R}(\theta),$$

and we set $Q^+_{k,R} = Q^+_{k,R}(1)$. With ν_0 the constant from Proposition 4.4 and U a given constant, we also define $Q_R(U)$ to be the cylinder $Q^+_{U,R}(\nu_0/9)$.

Lemma 3.4. *Let C_0, C_1, g_0, g_1, ρ, and ω be positive constants with $C_0 \le C_1$ and $1 < g_0 \le g_1$. Suppose also that u is a bounded weak solution of (1.1) in $Q^+_{\omega,\rho}$ with*

$$\operatorname{ess\,osc}_{Q^+_{\omega,\rho}} u \le \omega.$$

Then there is a constant $\lambda \in (0,1)$, determined only by data, such that

$$\operatorname{ess\,osc}_{Q^+_{\omega',\lambda\rho}} u \le \omega', \tag{3.8a}$$

where

$$\omega' = \max\{\frac{5}{6}\omega, 3\operatorname{ess\,osc}_{K_\rho\times\{0\}} u\}. \tag{3.8b}$$

Proof. We begin by setting

$$\omega_0 = \operatorname{ess\,osc}_{K_\rho\times\{0\}} u.$$

If $\omega < 3\omega_0$, then $\omega' = 3\omega_0$. We now perform some elementary calculations to show that $Q^+_{\omega',\lambda\rho} \subset Q^+_{\omega,\rho}$ if λ is small enough. First, since $\omega \le \omega'$ and $\lambda \le 1$, we can use Lemma 2.1(c) to infer that

$$G\big(\frac{\omega}{\rho}\big) \le \big(\frac{\lambda\omega}{\omega'}\big)^{g_0} G\big(\frac{\omega'}{\lambda\rho}\big) \le \lambda^{g_0} G\big(\frac{\omega'}{\lambda\rho}\big).$$

If $\lambda \le 9^{-1/g_0}$, then we have

$$G\big(\frac{\omega}{\rho}\big) \le \frac{1}{9} G\big(\frac{\omega'}{\lambda\rho}\big).$$

Since $\omega_0 \le \omega$, it follows that $\omega \ge \frac{1}{3}\omega'$ and therefore

$$\omega^2 G\big(\frac{\omega}{\rho}\big)^{-1} \ge \frac{1}{9}(\omega')^2 G\big(\frac{\omega}{\rho}\big)^{-1},$$

so

$$\omega^2 G\big(\frac{\omega}{\rho}\big)^{-1} \ge (\omega')^2 G\big(\frac{\omega'}{\lambda\rho}\big)^{-1}.$$

Hence $\lambda \le 9^{-1/g_0}$ implies that $Q^+_{\omega',\lambda\rho} \subset Q^+_{\omega,\rho}$ and therefore (3.8) is valid.

If $\omega \ge 3\omega_0$, we take ν_0 be the constant from Proposition 4.4. This time, we set $R = \frac{1}{6}\rho$, and we note that $Q^+_{\omega/3,2R}(\nu_0) \subset Q^+_{\omega,\rho}$. To proceed, we define

$$u_1 = u - \operatorname{ess\,inf}_{Q^+_{\omega/3,2R}} u, \quad u_2 = \omega - u_1,$$

and we set $k=\omega/3$.

We consider two cases. First, if

$$\operatorname{ess\,sup}_{K_{2R}\times\{0\}} u_1 \le \frac{2}{3}\omega, \tag{3.9}$$

then $u_1 \ge k$ on $K_{2R}\times\{0\}$. We then apply Proposition 4.4 to u_1 in $Q^+_{k,2R}(\nu_0)$ to infer that

$$\operatorname{ess\,inf}_{Q^+_{k,R}(\nu_0)} u_1 \ge \frac{k}{2}.$$

It follows that

$$\operatorname{ess\,osc}_{Q^+_{k,R}(\nu_0)} u \le \omega - \frac{k}{2} = \frac{5}{6}\omega. \tag{3.10}$$

If (3.9) does not hold, then some straightforward algebra shows that

$$\operatorname{ess\,sup}_{K_{2R}\times\{0\}} u_2 \le \frac{2}{3}\omega,$$

so we can apply Proposition 4.4 to u_2, again obtaining (3.10).

To see that (3.10) implies (3.8), we examine separately the cases $\omega' = \frac{5}{6}\omega$ and $\omega' = \omega_0$. In both cases, we see that $\lambda\rho \le R$ if $\lambda \le \frac{1}{3}$.

In the first case, we observe that $\lambda \le \frac{1}{3}$ implies that $5/(6\lambda) \ge \frac{5}{2} \ge 1$. If, in addition,

$$\lambda \le \frac{5}{6}\Big(\frac{4\nu_0}{25}\Big)^{1/g_0},$$

we conclude that

$$G\Big(\frac{\omega}{\rho}\Big) \le \Big(\frac{6\lambda}{5}\Big)^{g_0} G\Big(\frac{5\omega}{6\lambda}\Big) \le \frac{4\nu_0}{25} G\Big(\frac{5\omega}{6\lambda}\Big),$$

and hence $Q^+_{\omega',\lambda\rho} \subset Q^+_{k,R}(\nu_0)$ in this case. Combining this observation with (3.10) gives (3.8).

In the second case, we observe that $\omega' \ge \frac{5}{6}\omega$, so $\lambda \le \frac{5}{6}$ implies that

$$G\Big(\frac{\omega}{\rho}\Big) \le \Big(\frac{\lambda\omega}{\omega'}\Big)^{g_0} G\Big(\frac{\omega'}{\lambda\rho}\Big)$$

If also

$$\lambda \le \frac{5}{6}\Big(\frac{\nu_0}{17}\Big)^{1/g_0},$$

Then we have

$$\begin{aligned} G\Big(\frac{\omega}{\rho}\Big) &\le \Big(\frac{\lambda\omega}{\omega'}\Big)^{g_0} G\Big(\frac{\omega'}{\lambda\rho}\Big) \\ &\le \Big(\frac{5\lambda}{6}\Big)^{g_0} G\Big(\frac{\omega'}{\lambda\rho}\Big) \\ &\le \frac{1}{9}\Big(\frac{5}{6}\Big)^2 \nu_0 G\Big(\frac{\omega'}{\lambda\rho}\Big) \end{aligned}$$

since $(5/6)^2/9 \ge 1/17$. It follows again that $Q^+_{\omega',\lambda\rho} \subset Q^+_{k,R}(\nu_0)$ and hence we obtain (3.8). Combining all these cases, we see that the result is true with

$$\lambda = \min\Big\{\frac{1}{9}, \frac{5}{6}\Big(\frac{\nu_0}{17}\Big)^{1/g_0}\Big\}$$

since $9^{-1/g_0} \ge 1/9$. □

From this lemma, we infer a continuity estimate near the initial surface. We recall from [18] that $B\Omega$ is the set of all $(x_0,t_0)\in\partial_P\Omega$ such that, for some positive numbers r and s, the cylinder

$$K_r^{x_0}\times(t_0,t_0+s)$$

is a subset of Ω. We also define the *initial surface* $B'\Omega$ of Ω as in [12] to be the set of all $(x_0,t_0)\in B\Omega$ such that $K_r\times\{t_0\}\subset\partial_P\Omega$ for some $r'>0$. As noted in [12], $B'\Omega$ need not be the same as $B\Omega$.

For $(x_0,t_0)\in B'\Omega$ and $\omega>0$, we write $\operatorname{dist}_B(x_0,t_0)$ for the supremum of the set of all numbers r such that $Q^{+;x_0,t_0}_{\omega,r}\subset\Omega$ and $K_r^{x_0,t_0}\times\{0\}\subset\partial_P\Omega$.

Theorem 3.5. *Let u be a bounded weak solution of (1.1) in Ω, and suppose $1<g_0\le g_1\le 2$. Suppose also that the restriction of u to $B'\Omega$ is continuous at some $(x_0,t_0)\in B'\Omega$. Then u is locally continuous up to (x_0,t_0). Specifically, if there is a continuous increasing function $\tilde\omega$ defined on $[0,\operatorname{dist}_B(x_0,t_0))$ with $\tilde\omega(0)=0$,*

$$\frac{5}{6}\tilde\omega(2r)\le\tilde\omega(r) \tag{3.11}$$

for all $r\in(0,\operatorname{dist}_B(x_0,t_0)/2)$, and with

$$|u(x_0,t_0)-u(x_1,t_0)|\le\tilde\omega(|x_0-x_1|)$$

for all x_1 with $|x_0-x_1|<\operatorname{dist}_B(x,t_0)$, then there exist constants γ and $\alpha\in(0,1)$ depending only upon the data such that, for any $(x,t)\in\Omega$ with $t\ge t_0$, we have

$$\begin{aligned}|u(x_0,t_0)-u(x,t)|\le{}&\gamma U\Big(\frac{|x_0-x|+|t_0-t|_G}{\operatorname{dist}_B(x_0,t_0)}\Big)^\alpha\\&+3\tilde\omega\left(\gamma|x_0-x|_\infty+\gamma\operatorname{dist}_B(x_0,t_0)^{1-\alpha}|t_0-t|_G^\alpha\right).\end{aligned}\tag{3.12}$$

Proof. We start by taking $\omega_0=U$ and $\rho_0=\operatorname{dist}_B(x_0,t_0)$. If $(x,t)\notin Q^{+,x_0,t_0}_{\omega_0,\rho_0}$, then the result is immediate for any α as long as $\gamma\ge1$. With λ as in Lemma 3.4, we set $\sigma=5/6$, and

$$\varepsilon=\min\{\lambda,\frac12\sigma^{(2-g_0)/g_0}\}.$$

If $(x,t)\in Q^{+,x_0,t_0}_{\omega_0,\rho_0}$, then we define $\rho_n=\lambda^n\rho_0$. We also define ω_n' for $n>0$ inductively as $\omega_{n+1}'=\max\{\frac56\omega_n',3\omega^*(\rho_n)\}$, and we set

$$Q_n=Q^{+,x_0,t_0}_{\omega_n',\rho_n}.$$

It follows from Lemma 3.4 that $\operatorname{ess\,osc}_{Q_n}u\le\omega_n'$, but this estimate must be improved. To this end, we set

$$\omega_n=\max\big\{\big(\frac56\big)^n\omega_0,3\tilde\omega(\rho_{n-1})\big\},$$

and infer from the proof of [12, Theorem 2.6] that $\omega_n'\le\omega_n$ for $n>0$. Hence

$$\operatorname{ess\,osc}_{Q_n}u\le\omega_n.$$

As before, we assume that $x\ne x_0$ and $t\ne t_0$, so there are nonnegative integers n and m such that

$$\rho_{n+1}\le|x_0-x|_\infty<\rho_n,$$

and

$$\omega_{m+1}^2G\Big(\frac{\omega_{m+1}}{\rho_{m+1}}\Big)^{-1}\le|t_0-t|<\omega_m^2G\big(\frac{\omega_m}{\rho_m}\big)^{-1}.$$

With $\alpha_1 = \log_{1/2}(5/6)$, it follows that

$$(\frac{5}{6})^n \le \Big(\frac{2|x_0 - x|_\infty}{\rho_0}\Big)^{\alpha_1}, \quad \tilde{\omega}(\rho_n) \le \tilde{\omega}(\frac{1}{\lambda}|x_0 - x|_\infty).$$

Moreover, if we set $\beta = \varepsilon\sigma^{(2-g_0)/g_0}$ and $\widehat{\omega}_m = \beta^m\omega_0$, it follows that $\widehat{\omega}_{m+1} \le \omega'_{m+1}$, so (as in the proof of Theorem 2.4)

$$|t_0 - t|_G \ge \beta^{m+1}\rho_0.$$

For $\alpha_2 = \log_\beta \sigma$, we infer again that

$$\widehat{\omega}_m \le \Big(\frac{|t_0 - t|_G}{\rho_0}\Big)^{\alpha_2}\omega_0.$$

In addition, for $\alpha_3 = \log_\beta \lambda$ (which is in the interval $(0,1]$), we infer that

$$\rho_m \le \Big(\frac{|t_0 - t|_G}{\beta\rho_0}\Big)^{\alpha_3}\rho_0.$$

Therefore,

$$\bar{\omega}(\rho_m) \le \bar{\omega}\Big(\rho_0^{1-\alpha_3}|t_0 - t|_G^{\alpha_3}\Big).$$

And the proof is complete by combining all these inequalities and taking $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$. $\square$

As in [12, Theorem 2.5], condition (3.11) involves no loss of generality in that any modulus of continuity for the restriction of u to $B'\Omega$ is controlled by one satisfying this condition.

4. Proof of the main lemma

Throughout this section, u is a bounded nonnegative weak solution of (1.1) with (1.2). The proof of Lemma 3.1 is composed of four steps under the assumption that u is large at least half of a cylinder $Q_{\omega,2R}$. First, Proposition 4.1 gives spatial cube at some fixed time level on which u is away from its minimum (zero value) on arbitrary fraction of the spatial cube. From the spatial cube, positive information spread in both later time and over the space variables with time limitations (Proposition 4.2 and Proposition 4.5). Controlling the positive quantity $\theta > 0$ in $T_{k,\rho}(\theta)$ is key to overcoming those time restrictions. Once we have a subcylinder centered at $(0,0)$ in $Q_{\omega,4R}$ with arbitrary fraction of the subcylinder, we finally apply modified De Giorgi iteration (Proposition 4.3) to obtain strictly positive infimum of u in a smaller cylinder around $(0,0)$.

4.1. **Basic results.** Our first proposition shows that if a nonnegative function is large on part of a cylinder, then it is large on part of a fixed cylinder. Except for some minor variation in notation, our result is [7, Lemma 7.1, Chapter III]; we refer the reader to [12, Proposition 3.1] for a proof using the present notation.

Proposition 4.1. *Let k, ρ, and T be positive constants. If u is a measurable nonnegative function defined on $Q = K_\rho \times (-T, 0)$ and if there is a constant $\nu_1 \in [0,1)$ such that*

$$|Q \cap \{u \le k\}| \le (1 - \nu_1)|Q|,$$

then there is a number

$$\tau_1 \in \Big(-T, -\frac{\nu_1}{2 - \nu_1}T\Big)$$

for which

$$|\{x \in K_\rho : u(x,\tau_1) \le k\}| \le \big(1 - \frac{\nu_1}{2}\big)|K_\rho|.$$

Our next proposition is similar to [7, Lemma IV.10.2].

Proposition 4.2. *Let ν, k, ρ, and θ be given positive constants with $\nu < 1$. If $g_1 \le 1$, then, for any $\epsilon \in (0,1)$, there exists a constant $\delta = \delta(\nu, \epsilon, \theta, data)$ such that, if u is a nonnegative supersolution of* (1.1) *in $K_\rho \times (-\tau, 0)$ with*

$$|\{x \in K_\rho : u(x,-\tau) < k\}| < (1-\nu)\,|K_\rho| \tag{4.1}$$

for some

$$\tau \le \theta(\delta k)^2 G\Big(\frac{\delta k}{\rho}\Big)^{-1}, \tag{4.2}$$

then

$$|\{x \in K_\rho : u(x,-t) < \delta k\}| < (1-(1-\epsilon)\nu)\,|K_\rho|$$

for any $-t \in (-\tau, 0]$.

Proof. The proof is almost identical to that of [12, Proposition 3.2]. With Ψ defined as

$$\Psi = \ln^+\Big(\frac{k}{(1+\delta)k - (u-k)_-}\Big),$$

we note that $\delta k|\Psi'| \le 1$. It follows that

$$\begin{aligned}|\Psi'|^2 G\big(\frac{|D\zeta|}{\Psi'}\big) &\le (\delta k|\Psi'|)^{2-g_1}\,(\delta k)^{-2}\,G\,(\delta k|D\zeta|)\\ &\le \sigma^{-g_1}\,(\delta k)^{-2}\,G\left(\frac{\delta k}{\rho}\right).\end{aligned}$$

Arguing as in the proof of [12, Proposition 3.2] (and noting that $2^{g_1} \ge 1$) then yields

$$\begin{aligned}&\int_{-\tau}^{-s}\int_{K_\rho} h(\Psi^2)|\Psi||\Psi'|^2 G\big(\frac{|D\zeta|}{|\Psi'|}\big)\,dx\,dt\\ &\le 2^{g_1}\theta h\left(j^2(\ln 2)^2\right)(j\ln 2)\sigma^{-g_1}|K_\rho|\\ &\le 2^{g_1}\theta\frac{H(j^2(\ln 2)^2)}{j\ln 2}\sigma^{-g_1}|K_\rho|\end{aligned}$$

for any $s \in (0,\tau)$. This inequality is the same as [12, (3.4)].

Since the remainder of the proof of [12, Proposition 3.2] is valid for the full range $1 < g_0 \le g_1$, we do not repeat it here. □

Our next step should be a proposition concerning the spread of positivity over space analogous to [12, Proposition 3.3]; however, because we need a much stronger result here, we defer its discussion to the next subsection. Instead, we present a modified DeGiorgi iteration with generalized structure conditions (1.2), which was proved as [12, Proposition 3.4]. Basically, our Proposition 4.3 is equivalent to [7, Lemmata III.4.1, III.9.1, IV.4.1]. We point out in particular that [7, Lemma IV.4.1], which is the same as [2, Lemma 3.1], follows from our proposition by taking $\theta = 1$, $k = \omega/2^m$ and $\rho = (2^{m+1}/\omega)^{(2-p)/p}R$.

Proposition 4.3. *For a given positive constant θ, there exists $\nu_0 = \nu_0(\theta, data) \in (0,1)$ such that, if u is a nonnegative supersolution of (1.1) in $Q_{k,2\rho}(\theta)$ with*
$$|\{(x,t) \in Q_{k,2\rho}(\theta) : u(x,t) < k\}| < \nu_0 |Q_{k,2\rho}(\theta)| \tag{4.3}$$
for some positive constants k and ρ, then
$$\operatorname{ess\,inf}_{Q_{k,\rho}(\theta)} u(x,t) \ge \frac{k}{2}.$$

We also recall [12, Proposition 3.5], which will be critical in our proof of initial regularity.

Proposition 4.4. *There exists $\nu_0 \in (0,1)$, determined only by the data, such that, if u is a nonnegative supersolution of (1.1) in $Q_{k,2\rho}(\theta)$ with*
$$|\{(x,t) \in Q_{k,2\rho}(\theta) : u(x,t) < k\}| < \frac{\nu_0}{\theta} |Q_{k,2\rho}(\theta)| \tag{4.4a}$$
for some positive constants k, ρ, and θ and if
$$u(x, -T_{k,2\rho}(\theta)) \ge k \tag{4.4b}$$
for all $x \in K_{2\rho}$, then
$$\operatorname{ess\,inf}_{K_\rho \times (-T_{k,2\rho}(\theta),0)} u \ge \frac{k}{2}.$$

4.2. **Expansion of positivity in space.** Throughout this subsection, ν, ν_0, ρ, and k are given positive constants with $\nu, \nu_0 < 1$. Also, to simplify notation, we set
$$T = \big(\frac{k}{2}\big)^2 G\big(\frac{k}{2\rho}\big)^{-1}.$$

We assume that u is a nonnegative supersolution of (1.1) in $K_{2\rho} \times (-T,0)$ such that
$$\big|\{x \in K_{2\rho} : u(x,t) \le \frac{k}{2}\}\big| \le (1-\nu_0)|K_{2\rho}| \tag{4.5}$$
for all $t \in (-T,0)$.

We wish to prove the following proposition, which is a generalization of [7, Lemma IV.5.1]. In fact, this lemma is not the complete first alternative as described in that source; we single it out as the crucial step in that alternative.

Proposition 4.5. *Let $\nu \in (0,1)$ and $\nu_0 \in (0,1]$ be constants. If $g_1 \le 2$ and if u is a nonnegative supersolution of (1.1) in $K_{2\rho} \times (-T,0)$ which satisfies (4.5), then there is a constant δ^* determined only by ν, ν_0, and the data such that*
$$\big|\{x \in K_\rho : u(x,t) \le \frac{\delta^* k}{2}\}\big| \le \nu |K_{2\rho}| \tag{4.6}$$
for all $t \in (-T_1,0)$, where
$$T_1 = \big(\frac{k}{2}\big)^2 G\big(\frac{k}{\rho}\big)^{-1}. \tag{4.7}$$

Our proof follows that of [7, Lemma IV.5.1] rather closely with a few modifications based on ideas from [17, Section 4]. In addition, our proof shows much more easily that the constants in [7, Chapter IV] are stable as $p \ne 2$.

Our first step is as in [7, Section IV.6]. We show that u satisfies an additional integral inequality, which is the basis of the proof of Proposition 4.5. Before stating our inequalities, we introduce some notation. For positive constants κ and δ with $\kappa \le k/2$ and $\delta < 1$, we define two functions Φ_κ and Ψ_κ as follows:

$$\Phi_\kappa(\sigma) = \int_0^{(\sigma-\kappa)_-} \frac{(1+\delta)\kappa - s}{G\big(\frac{(1+\delta)\kappa-s}{2\rho}\big)}\,ds, \tag{4.8a}$$

$$\Psi_\kappa(\sigma) = \ln\big[\frac{(1+\delta)\kappa}{(1+\delta)\kappa-(\sigma-\kappa)_-}\big]. \tag{4.8b}$$

We also note that there are two Lipschitz functions, ζ_1 defined on $K_{2\rho}$ and ζ_2 defined on $[-T,0]$, such that

$$\zeta_1 = 0 \text{ on the boundary of } K_{2\rho}, \tag{4.9a}$$

$$\zeta_1 = 1 \text{ in } K_\rho, \tag{4.9b}$$

$$|D\zeta_1| \le \frac{1}{\rho} \text{ in } K_{2\rho}, \tag{4.9c}$$

$$\{x \in K_{2\rho} : \zeta_1(x) > \varepsilon\} \text{ is convex for all } \varepsilon \in (0,1), \tag{4.9d}$$

$$\zeta_2(-T) = 0, \tag{4.9e}$$

$$\zeta_2 = 1 \text{ on } (-T_1, 0), \tag{4.9f}$$

$$0 \le \zeta_2' \le \big(\frac{2}{k}\big)^2 G\big(\frac{k}{\rho}\big) \text{ on } (-T,0). \tag{4.9g}$$

Let us note that it's easy to arrange that $\zeta_2' \ge 0$ and that

$$\frac{1}{\zeta_2'} \ge \big(\frac{k}{2}\big)^2 G\big(\frac{k}{2\rho}\big)^{-1} - \big(\frac{k}{2}\big)^2 G\big(\frac{k}{\rho}\big)^{-1}.$$

Since Lemma 2.1(b) implies that

$$G\big(\frac{k}{\rho}\big) \ge 2^{g_0} G\big(\frac{k}{2\rho}\big) \ge 2G\big(\frac{k}{2\rho}\big),$$

we infer the second inequality of (4.9g).

Also, we introduce the notation D^- to denote the derivative

$$D^- f(t) = \limsup_{h\to 0^+} \frac{f(t) - f(t-h)}{h}.$$

With these preliminaries, we can now state our integral inequality. Our proof of this inequality is essentially the same as that for [7, Lemma IV.6.1]; the new ingredient is a more careful estimate of the integral involving ζ_t (which we denote by I_4). In this way, we obtain an estimate which does not depend on $p-2$ being bounded away from zero, which was the case in [7, (6.9) Chapter IV].

Lemma 4.6. *If $g_1 \le 2$ and if u is a weak supersolution of* (1.1) *in $K_{2\rho} \times (-T,0)$ satisfying* (4.5)*, then there are positive constants γ and γ_0, determined only by ν, ν_0, and the data such that*

$$D^-\Big(\int_{K_{2\rho}} \Phi_\kappa(u(x,t))\zeta^q(x,t)\,dx\Big) + \gamma_0 \int_{K_{2\rho}} \Psi_\kappa^{g_0}(u(x,t))\zeta^q(x,t)\,dx \le \gamma|K_{2\rho}| \tag{4.10}$$

for all $t \in (-T,0)$, where

$$q = g_0/(g_0-1). \tag{4.11}$$

Proof. With
$$u^* = \frac{(1+\delta)\kappa - (u-\kappa)_-}{2\rho},$$
we use the test function
$$\frac{\zeta^q((1+\delta)\kappa - (u-\kappa)_-)}{G(u^*)}$$
in the weak form of the differential inequality satisfied by u to infer that, for all sufficiently small positive h, we have
$$I_1 + I_2 \le I_3 + I_4$$
with
$$\begin{aligned}
I_1 &= \int_{K_{2\rho}} \Phi_\kappa(u(x,t))\zeta^q(x,t)\,dx - \int_{K_{2\rho}} \Phi_\kappa(u(x,t-h))\zeta^q(x,t-h)\,dx,\\
I_2 &= \int_{t-h}^{h}\int_{K_{2\rho}} \zeta^q(x,\tau)D(u-\kappa)_-(x,\tau)A\frac{1}{G(u^*(x,\tau))}\\
&\quad\times\Big[1-\frac{u^*(x,\tau)g(u^*(x,\tau))}{G(u^*(x,\tau))}\Big]\,dx\,d\tau,\\
I_3 &= q\int_{t-h}^{t}\int_{K_{2\rho}} D\zeta(x,\tau)A\zeta^{q-1}(x,\tau)\frac{(1+\delta)\kappa-(u-\kappa)_-}{G\left(u^*(x,\tau)\right)}\,dx\,d\tau,\\
I_4 &= q\int_{t-h}^{t}\int_{K_{2\rho}} \Phi_\kappa(u(x,\tau))\zeta^{q-1}(x,\tau)\zeta_t(x,\tau)\,dx\,d\tau,
\end{aligned}$$
and A evaluated at $(x,\tau,u(x,\tau),Du(x,\tau))$ in I_2 and I_3. We now use (1.2a) and the first inequality in (1.3) to see that
$$I_2 \ge C_0(g_0-1)\int_{t-h}^{t}\int_{K_{2\rho}} \zeta^q(x,\tau)\frac{G(|D(u-\kappa)_-(x,\tau)|)}{G(u^*(x,\tau))}\,dx\,d\tau.$$
Also, (1.2b) and Lemma 2.1(e) (with $\sigma_1 = (qC_1/C_0)|D\zeta(x,\tau)|\rho u^*(x,\tau)$, $\sigma_2 = |D(u-\kappa)_-(x,\tau)|$, and $\epsilon = \zeta(x,\tau)(g_0-1)/(2g_1)$) imply that
$$qD\zeta(x,\tau)\cdot A(x,\tau,u,Du)\zeta^{q-1}(x,\tau)\frac{(1+\delta)\kappa-(u-\kappa)_-}{G\left(u^*(x,\tau)\right)} \le J_1 + J_2$$
with
$$\begin{aligned}
J_1 &= g_1^{g_1}(\frac{2}{g_0-1})^{g_1-1}\zeta^{q-g_1}\frac{G(q(C_1/C_0)|D\zeta|\rho u^*)}{G(u^*)},\\
J_2 &= \frac{1}{2}C_0(g_0-1)\zeta^q\frac{G(|D(u-\kappa)_-|)}{G(u^*)}.
\end{aligned}$$
From our conditions on ζ and because $q \ge 2 \ge g_1$, we conclude that there is a constant γ_1, determined only by data, such that $J_1 \le \gamma_1$, so
$$I_3 \le \gamma_1 h|K_{2\rho}| + \frac{1}{2}I_2.$$
Next, we estimate Φ_κ. Since $\kappa \le k/2$ and $\delta \in (0,1)$, it follows that, for all $s \in (0,(u-\kappa)_-)$, we have $(1+\delta)\kappa - s \le 2k$ and hence
$$G\Big(\frac{(1+\delta)\kappa-s}{2\rho}\Big) \ge \Big(\frac{(1+\delta)\kappa-s}{2k}\Big)^2 G\big(\frac{k}{\rho}\big).$$

It follows that

$$\Phi_\kappa(u) \le 4k^2 G\big(\frac{k}{\rho}\big)^{-1} \int_0^{(u-\kappa)_-} [(1+\delta)\kappa - s]^{-1}\, ds = 4k^2 G\big(\frac{k}{\rho}\big)^{-1} \Psi_\kappa(u),$$

and therefore

$$I_4 \le 16q \int_{t-h}^t \int_{K_{2\rho}} \Psi_\kappa(u(x,\tau))\zeta^{q-1}(x,\tau)\, dx\, d\tau.$$

Combining all these inequalities and setting

$$I_{21} = \int_{t-h}^t \int_{K_{2\rho}} \zeta^2(x,\tau) \frac{G(|D(u-\kappa)_-(x,\tau)|)}{G(u^*(x,\tau))}\, dx\, d\tau,$$

$$I_{41} = \int_{t-h}^t \int_{K_{2\rho}} \Psi_\kappa(u(x,\tau))\zeta^{q-1}(x,\tau)\, dx\, d\tau$$

yields

$$I_1 + \frac{1}{2} C_0(g_0 - 1) I_{21} \le \gamma_1 h |K_{2\rho}| + 16q I_{41}. \tag{4.12}$$

Our next step is to compare

$$I_{22} = \int_{t-h}^t \int_{K_{2\rho}} \zeta^q(x,\tau) \Psi_\kappa^{g_0}(u(x,\tau))\, dx\, d\tau$$

with I_{21}. To this end, we first use Lemma 5.3 with $\varphi = \zeta_1^q$, $v = (u-\kappa)_-$, and $p = g_0$ to conclude that there is a constant γ_2 determined only by the data and ν_0 such that, for almost all $\tau \in (t-h, t)$, we have

$$\int_{K_{2\rho}} \zeta^q(x,\tau) \Psi_\kappa^{g_0}(u(x,\tau))\, dx \le \gamma_2 \rho^{g_0} \int_{K_{2\rho}} \zeta^q(x,\tau) |D\Psi_\kappa(u(x,\tau))|^{g_0}\, dx. \tag{4.13}$$

(Of course, we have multiplied (5.4) by $\zeta_2^q(\tau)$ here.) Now we use the explicit expression for Ψ_κ to infer that

$$\rho |D\Psi_\kappa(u(x,\tau))| = \frac{|D(u-\kappa)_-(x,\tau)|}{2u^*(x,\tau)} \le \frac{|D(u-\kappa)_-(x,\tau)|}{u^*(x,\tau)}.$$

Whenever $|D(u-\kappa)_-(x,\tau)| \le u^*(x,\tau)$, we conclude that

$$\rho^{g_0} |D\Psi_\kappa(u(x,\tau))|^{g_0} \le 1$$

and, wherever $|D(u-\kappa)_-(x,\tau)| > u^*(x,\tau)$, we infer from Lemma 2.1 that

$$\rho^{g_0} |D\Psi_\kappa(u(x,\tau))|^{g_0} \le \frac{G(|D(u-\kappa)_-(x,\tau)|)}{G(u^*(x,\tau))}.$$

It follows that, for any (x,τ), we have

$$\rho^{g_0} |D\Psi_\kappa(u(x,\tau))|^{g_0} \le 1 + \frac{G(|D(u-\kappa)_-(x,\tau)|)}{G(u^*(x,\tau))}.$$

Inserting this inequality into (4.13) and integrating the resultant inequality with respect to τ yields

$$I_{22} \le \gamma_2 (I_{21} + h|K_{2\rho}|).$$

By invoking (4.12), we conclude that

$$I_1 + \frac{1}{2\gamma_2} C_0(g_0 - 1) I_{22} \le \Big(\gamma_1 + \frac{1}{2\gamma_2} C_0(g_0-1)\Big) h |K_{2\rho}| + 16q I_{41}.$$

We now note that

$$\Psi_\kappa(u)\zeta^{q-1} = \left(\Psi_\kappa^{g_0}(u)\zeta^q\right)^{1/g_0},$$

so Young's inequality shows that

$$\Psi_\kappa(u)\zeta^{q-1} \le \varepsilon\Psi_\kappa^{g_0}(u)\zeta^q + \varepsilon^{-q}$$

for any $\varepsilon \in (0,1)$. By choosing ε sufficiently small, we see that there are constants γ_0 and γ such that

$$I_1 + \gamma_0 I_2 \le \gamma h|K_{2\rho}|.$$

To complete the proof, we divide this inequality by h and take the limit superior as $h \to 0^+$. □

Our next step is to estimate the integral of ζ^q over suitable N-dimensional sets with q defined by (4.11). Specifically, for each positive integer n and a number $\delta \in (0,1)$ to be further specified, we define the set

$$K_{\rho,n}(t) = \{x \in K_{2\rho} : u(x,t) < \delta^n k\}$$

and we introduce the quantities

$$A_n(t) = \frac{1}{|K_{2\rho}|}\int_{K_{\rho,n}(t)} \zeta^q(x,t)\,dx, \quad Y_n = \sup_{-T<t<0} A_n(t)$$

We shall show that, for a suitable choice of δ (which will require at least that $\delta \le 1/2$) and n, we can make Y_n small. In fact, based on the discussion in [7, Section 7 Chapter IV], we shall find n_0 and δ so that $Y_{n_0} \le \nu$. In fact, our method is to estimate $A_{n+1}(t)$ in terms of Y_n for each n.

We first estimate $A_{n+1}(t)$ if

$$D^-\Big(\int_{K_{2\rho}} \zeta^q(x,t)\Phi_{\delta^n k}(u(x,t))\,dx\Big) \ge 0. \tag{4.14}$$

(This is the case [7, (7.5) Chapter IV].) Our estimate now takes the following form.

Lemma 4.7. *Let ν and ν_0 be constants in $(0,1)$. If (4.14) holds, then there is a constant δ_0, determined only by ν, ν_0, and the data, such that $\delta \le \delta_0$ implies that*

$$A_{n+1}(t) \le \nu. \tag{4.15}$$

Proof. On $K_{\rho,n+1}(t)$, we have

$$\begin{aligned}\Psi_{\delta^n k}(u) &= \ln\Big[\frac{(1+\delta)\delta^n k}{(1+\delta)\delta^n k - (u-\delta^n k)_-}\Big]\\ &\ge \ln\Big[\frac{(1+\delta)\delta^n k}{(1+\delta)\delta^n k - (\delta^{n+1}k-\delta^n k)_-}\Big]\\ &= \ln\frac{1+\delta}{2\delta}.\end{aligned}$$

It follows that

$$\Big(\ln\frac{1+\delta}{2\delta}\Big)^{g_0}\int_{K_{\rho,n+1}(t)} \zeta^q(x,t)\,dx \le \int_{K_{\rho,n+1}(t)} \zeta^q(x,t)\Psi_{\delta^n k}(u(x,t))\,dx.$$

By invoking (4.10) and (4.14), we conclude that

$$\int_{K_{\rho,n+1}(t)} \zeta^q(x,t)\,dx \le \frac{\gamma}{\gamma_0}\Big(\ln\frac{1+\delta}{2\delta}\Big)^{-g_0}|K_{2\rho}|.$$

By choosing δ_0 sufficiently small, we infer (4.15). □

Our estimate when (4.14) fails is more complicated, as shown for the power case in [7, Section IV.8].

Lemma 4.8. *Let ν and ν_0 be constants in $(0,1)$. There are positive constants δ_1 and $\sigma < 1$, determined only by ν, ν_0, and the data, such that if*

$$D^-\Big(\int_{K_{2\rho}} \zeta^q(x,t)\Phi_{\delta^n k}(u(x,t))\,dx\Big) < 0 \tag{4.16}$$

for some $\delta \in (0,\delta_1)$ and if $Y_n > \nu$, then

$$A_{n+1}(t) \le \sigma Y_n. \tag{4.17}$$

Proof. In this case, we define

$$t_* = \sup\Big\{\tau \in (-T,t) : D^-\Big(\int_{K_{2\rho}} \zeta^q(x,\tau)\Phi_{\delta^n k}(u(x,\tau))\,dx\Big) \ge 0\Big\}$$

(and note that this set is nonempty). From the definition of t_*, we have that

$$\int_{K_{2\rho}} \zeta^q(x,t)\Phi_{\delta^n k}u(x,t)\,dx \le \int_{K_{2\rho}} \zeta^q(x,t_*)\Phi_{\delta^n k}u(x,t_*)\,dx. \tag{4.18}$$

It follows from Lemma 4.6 and the definition of t_* that

$$\int_{K_{2\rho}} \zeta^q(x,t_*)\Psi^{g_0}_{\delta^n k}u(x,t_*)\,dx \le C|K_{2\rho}|,$$

with $C = \gamma/\gamma_0$. Now we set

$$K_*(s) = \{x \in K_{2\rho} : (u-\delta^n k)_-(x,t_*) > s\delta^n k\}$$

for $s \in (0,1)$, and

$$I_1(s) = \int_{K_*(s)} \zeta^q(x,t_*)\,dx.$$

As in the proof of Lemma 4.7, we have that

$$\Phi_{s\delta^n k}(u(x,t_*)) \ge \ln\frac{1+\delta}{1+\delta-s},$$

so

$$I_1(s) \le C\Big(\ln\frac{1+\delta}{1+\delta-s}\Big)^{-g_0}|K_{2\rho}|. \tag{4.19}$$

Moreover, if $x \in K_*(s)$, then

$$u(x,t_*) < (1-s)\delta^n k \le \delta^n k,$$

and hence $K_*(s) \subset K_{\rho,n}$, so

$$I_1(s) \le Y_n|K_{2\rho}|. \tag{4.20}$$

We now define

$$s_* = \Big[1 - \exp\Big(-\big(\frac{2C}{\nu}\big)^{1/g_0}\Big)\Big](1+\delta_*),$$

with $\delta_* \in (0,1)$ chosen so that $s_* < 1$. Since $Y_n > \nu$, a simple calculation shows that

$$C\Big(\ln\frac{1+\delta}{1+\delta-s}\Big)^{-g_0} \le \frac{1}{2}Y_n \tag{4.21}$$

for $s > s_*$ provided $\delta \le \delta_*$.

Next, we set

$$I_2 = \int_{K_{2\rho}} \zeta^q(x,t_*)\Phi_{\delta^n k}(u(x,t_*))\,dx$$

and use Fubini's theorem to conclude that

$$\begin{aligned} I_2 &= \int_{K_{2\rho}} \zeta^q(x,t_*)\Big(\int_0^{\delta^n k} \frac{\chi_{\{(\delta^n k-u)+>s\}}((1+\delta)\delta^n k - s)}{G\big(\frac{(1+\delta)\delta^n k-s}{2\rho}\big)}\,ds\Big)\,dx \\ &= \int_0^{\delta^n k} \frac{(1+\delta)\delta^n k - s}{G\big(\frac{(1+\delta)\delta^n k-s}{2\rho}\big)}\Big(\int_{K_{2\rho}} \zeta^q(x,t_*)\chi_{\{(\delta^n k-u)+>s\}}\,dx\Big)\,ds \end{aligned}$$

Using the change of variables $\tau = s/(\delta^n k)$, we see that

$$\begin{aligned} I_2 &= \int_0^1 \frac{(1+\delta)-\tau}{G\big(\frac{\delta^n k(1+\delta-\tau)}{2\rho}\big)}\Big(\int_{K_{2\rho}} \zeta^q(x,t_*)\chi_{\{(\delta^n k-u)+>\delta^n k\tau\}}\,dx\Big)\,d\tau \\ &= \int_0^1 \frac{(1+\delta)-\tau}{G\big(\frac{\delta^n k(1+\delta-\tau)}{2\rho}\big)} I_1(\tau)\,d\tau. \end{aligned}$$

Combining this equation with (4.19), (4.20), and (4.21) then yields

$$I_2 \le Y_n|K_{2\rho}|\Big[\int_0^{s_*} \frac{(1+\delta)-\tau}{G\big(\frac{\delta^n k(1+\delta-\tau)}{2\rho}\big)}\,d\tau + \frac{1}{2}\int_{s_*}^1 \frac{(1+\delta)-\tau}{G\big(\frac{\delta^n k(1+\delta-\tau)}{2\rho}\big)}\,d\tau\Big].$$

We now define the function

$$f(\tau) = \frac{\tau}{G\big(\frac{\delta^n k\tau}{2\rho}\big)}$$

and we set $\sigma_* = 1 - s_*$. Using the change of variables $s = 1-\tau$ then yields

$$I_2 \le Y_n|K_{2\rho}|\mathcal{K},$$

with

$$\mathcal{K} = \int_{\sigma_*}^1 f(\delta+s)\,ds + \frac{1}{2}\int_0^{\sigma_*} f(\delta+s)\,ds.$$

Since

$$\mathcal{K} = \int_0^1 f(\delta+s)\,ds - \frac{1}{2}\int_0^{\sigma_*} f(\delta+s)\,ds,$$

it follows from (2.3) that

$$\mathcal{K} \le \big(1 - \frac{\sigma_*}{2}\big)\int_0^1 f(\delta+s)\,ds,$$

and therefore

$$I_2 \le Y_n|K_{2\rho}|\Big(1-\frac{\sigma_*}{2}\Big)\int_0^1 f(\delta+s)\,ds. \tag{4.22}$$

Our next step is to obtain a lower bound for I_2. Taking into account (4.18), we have

$$I_2 \ge \int_{K_{\rho,n+1}(t)} \zeta^q(x,t)\Phi_{\delta^n k}(u(x,t))\,dx.$$

Next, for $z < \delta^{n+1}k$, we have

$$\begin{aligned} \Phi_{\delta^n k}(z) &= \int_0^{(z-\delta^n k)_-} \frac{(1+\delta)\delta^n k - s}{G\big(\frac{(1+\delta)\delta^n k-s}{2\rho}\big)}\,ds \\ &\ge \int_0^{\delta^n k(1-\delta)} \frac{(1+\delta)\delta^n k - s}{G\big(\frac{(1+\delta)\delta^n k-s}{2\rho}\big)}\,ds \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{1-\delta} f(\delta+s)\,ds \\
&\geq \Big(1-\frac{2}{2+\ln\delta}\Big)\int_0^1 f(\delta+s)\,ds
\end{aligned}
$$

by (2.1), so

$$
\Phi_{\delta^n k}(u(x,t)) \geq \Big(1-\frac{2}{2+\ln(1/\delta)}\Big)\int_0^\delta f(\delta+s)\,ds
$$

for all $x \in K_{\rho,n+1}(t)$ and hence

$$
I_2 \geq \Big(1-\frac{2}{2+\ln(1/\delta)}\Big)\Big(\int_{K_{\rho,n+1}(t)} \zeta^q(x,t)\,dx\Big)\Big(\int_0^\delta f(\delta+s)\,ds\Big).
$$

In conjunction with (4.22), this inequality implies that

$$
A_{n+1}(t) \leq \frac{1-(\sigma_*/2)}{1-(2/(2+\ln(1/\delta))}Y_n.
$$

By taking δ_2 sufficiently small, we can make sure that

$$
\sigma = \frac{1-(\sigma_*/2)}{1-(2/(2+\ln(1/\delta_2))}
$$

is in the interval $(0,1)$. If we take $\delta_1 = \min\{\delta_*, \delta_2\}$, we then infer (4.17) for $\delta \leq \delta_1$. $\square$

As shown in [7], if t_* and t are equal in this proof, we can infer (4.15) very simply. We are now ready to prove Proposition 4.5.

Proof of Proposition 4.5. Since $Y_{n+1} \leq Y_n$, it follows from Lemmata 4.7 and 4.8 that, for all positive integers n, we have

$$
A_{n+1}(t) \leq \max\{\nu, \sigma Y_n\}
$$

for all $t \in (-T,0)$ and hence $Y_{n+1} \leq \max\{\nu, \sigma Y_n\}$. Induction implies that

$$
Y_n \leq \max\{\nu, \sigma^{n-1} Y_1\}
$$

for all n. In addition $Y_1 \leq 1$, so there is a positive integer n_0, determined by ν, a_0, and the data such that $Y_{n_0} \leq \nu$.

Next, we recall that $\zeta = 1$ on $K_\rho \times (-T_1, 0)$, and hence, for all $t \in (-T_1, 0)$, we have

$$
\begin{aligned}
\big|\{x \in K_\rho : u(x,t) \leq \delta^{n_0} k\}\big| &= \int_{\{x\in K_\rho : u(x,t)\leq \delta^{n_0}k\}} \zeta^q(x,t)\,dx \\
&\leq \int_{\{x\in K_{2\rho} : u(x,t)\leq \delta^{n_0}k\}} \zeta^q(x,t)\,dx \leq Y_{n_0}.
\end{aligned}
$$

The proof is complete by using the inequality $Y_{n_0} \leq \nu$ and taking $\delta^* = \delta^{n_0}$. $\square$

4.3. **Proof of main lemma.**

Proof. With δ to be chosen, we use Proposition 4.1 with $k = \omega/2$, $\rho = 2R$, $\nu_1 = \frac{1}{2}$, and
$$T = 3\big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega}{2R}\big)^{-1}$$
to infer that there is a $\tau_1 \in (-T, -T/3)$ such that
$$\Big|\{x \in K_{2R} : u(x, \tau_1) \le \frac{\omega}{2}\}\Big| \le \frac{3}{4}|K_{2R}|.$$
Next, we set $\nu = \frac{1}{4}$, $\rho = 2R$, $k = \omega/2$, $\tau = \tau_1$, and $\theta = 3$. Since $\tau_1 \le T$ and
$$T = 3(\delta k)^2 G\big(\frac{2\delta k}{\rho}\big)^{-1} \le 3(\delta k)^2 G\big(\frac{\delta k}{\rho}\big)^{-1},$$
it follows that (4.1) is satisfied, so Proposition 4.2 implies that
$$\big|\{x \in K_{2R} : u(x, t) \le \frac{\delta\omega}{2}\}\big| \le \frac{7}{8}|K_{2R}| \tag{4.23}$$
for all $t \in (\tau_1, 0)$ provided we take δ to be the constant from that proposition. (In particular, δ is determined only by the data.) Since $\tau_1 \ge T/3$, it follows that
$$\tau_1 \ge \big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega}{2R}\big)^{-1},$$
and hence (4.23) holds for all
$$t \in \Big(-\big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega}{2R}\big)^{-1}, 0\Big).$$
Now we use Proposition 4.5, with $\omega = \delta\omega$ and ν to be chosen, to infer that there is a constant $\delta^* \in (0, 1)$, determined only by the data and ν such that
$$\big|\{x \in K_R : u(x, t) \le \frac{\delta^*\delta\omega}{2}\}\big| \le \nu|K_{2R}| \tag{4.24}$$
for all
$$t \in \Big(-\big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega}{R}\big)^{-1}, 0\Big).$$
Since $G\big(\frac{\delta\omega/2}{R}\big) \le G\big(\frac{\delta\omega}{R}\big)$ and $\delta^* \le 1$, it follows that
$$\big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega}{R}\big)^{-1} \ge \big(\frac{\delta\omega}{2}\big)^2 G\big(\frac{\delta\omega/2}{R}\big)^{-1} \ge \Big(\frac{\delta^*\delta\omega/2}{R}\Big)^2 G\Big(\frac{\delta^*\delta\omega/2}{R}\Big)^{-1}.$$
We now take ν_0 to be the constant corresponding to $\theta = 1$ in Proposition 4.3, and we set $\nu = 2^{-N}\nu_0$, which determines δ^*. Then (4.3) is satisfied for $k = \delta^*\delta\omega/2$ and $\rho = R/2$. Proposition 4.3 then yields (3.2) with $\mu = \delta^*\delta/4$. $\square$

5. Auxiliary theorems

We now present the basic results used in the previous sections of the paper. Since the results are either well-known or were proved in [12], we just state the results here.

5.1. **Local energy estimate.** The local energy estimate is a fundamental inequality playing an important role in the proofs of several results, especially Proposition 4.1, Proposition 4.2, and Proposition 4.5. We refer the reader to [12, Proposition 4.1] for a proof.

Proposition 5.1. *Let G satisfy structure conditions* (1.2) *in a cylinder $Q_\rho := K_\rho \times [t_0, t_1]$, and let ζ be a cutoff function on the cylinder Q_ρ, vanishing on the parabolic boundary of Q_ρ with $0 \le \zeta \le 1$. Define constants r, s, and q by*

$$r = 1 - \frac{1}{g_1}, \quad s = \frac{g_0}{g_1}, \quad \text{and} \quad q = 2g_1. \tag{5.1}$$

If u is a locally bounded weak supersolution of (1.1)*, then there exist constants c_0, c_1, and c_2 depending on data such that*

$$\begin{aligned}
&\int_{K_\rho \times \{t_1\}} G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s+2}\zeta^q \,dx \\
&+ c_0 \iint_{Q_\rho} G\left(|D(u-k)_-|\right) G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s}\zeta^q \,dx\,dt \\
&\le c_1 \iint_{Q_\rho} G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s+2}\zeta^{q-1}|\zeta_t| \,dx\,dt \\
&\quad + c_2 \iint_{Q_\rho} G\left(|D\zeta|\zeta(u-k)_-\right) G^{r-1}\Big(\frac{\zeta(u-k)_-}{\rho}\Big)(u-k)_-^{s} \,dx\,dt
\end{aligned} \tag{5.2}$$

for any constant k.

We refer the reader to [12, Proposition 4.1] for the corresponding result about nonpositive subsolutions.

5.2. **Logarithmic energy estimate.** With the functions h and H defined in Lemma 2.2, the logarithmic energy estimate was proved as [12, Proposition 4.2], which also contains the corresponding result for nonpositive weak subsolutions.

Proposition 5.2. *Assume that G satisfies* (1.2) *in a cylinder $K_R \times [t_0, t_1]$. Let $q \ge g_1$ and $\delta \in (0,1)$ be constants, and let ζ be a cut-off function which is independent of the time variable. Let u be a nonnegative weak supersolution of* (1.1)*and let k be a positive constant. Then*

$$\begin{aligned}
&\int_{K_R \times \{t_1\}} H(\Psi^2)\zeta^q \,dx + C_0(4g_0 - 2)\int_{t_0}^{t_1}\int_{K_R} G(|Du|)h(\Psi^2)(\Psi')^2\zeta^q \,dx\,dt \\
&\le \int_{K_R \times \{t_0\}} H(\Psi^2)\zeta^q \,dx + C^* \int_{t_0}^{t_1}\int_{K_R} h(\Psi^2)\Psi(\Psi')^2 G\Big(\frac{|D\zeta|}{|\Psi'|}\Big)\zeta^{q-g_1} \,dx\,dt
\end{aligned} \tag{5.3}$$

where

$$C^* = \frac{C_0}{g_1}\left(\frac{2qg_1C_1}{C_0}\right)^{g_1}, \quad \Psi(u) = \ln^+\left[\frac{k}{(1+\delta)k - (u-k)_-}\right].$$

5.3. **A Poincaré type inequality.** For our proofs, we shall need the following result which is [7, Proposition I.2.1].

Lemma 5.3. *Let Ω be a bounded convex subset of $\mathbb{R}^N$ and let φ be a nonnegative continuous function on $\overline{\Omega}$ such that $\varphi \le 1$ in Ω and such that the sets $\{x \in \Omega :$*

$\varphi(x) > k\}$ *are convex for all* $k \in (0,1)$. *Then, for any* $p \geq 1$, *there is a constant* C *determined only by* N *and* p *such that*

$$
\begin{aligned}
&\Big(\int_\Omega \varphi |v|^p\,dx\Big)^{1/p} \\
&\leq C \frac{(\operatorname{diam}\Omega)^N}{|\{x \in \Omega : v(x) = 0,\ \varphi(x) = 1\}|^{(N-1)/N}} \Big(\int_\Omega \varphi |Dv|^p\,dx\Big)^{1/p}
\end{aligned}
\tag{5.4}
$$

for all $v \in W^{1,p}$.

Note that if the set $\{x \in \Omega : v(x) = 0,\ \varphi(x) = 1\}$ has measure zero, then (5.4) is true because the right hand side is infinite.

5.4. **Embedding theorem.** Our next result is a variation on the Sobolev imbedding theorem, which is just [12, Theorem 4.4].

Theorem 5.4. *For a nonnegative function* $v \in W_0^{1,1}(Q)$ *where* $Q = K \times [t_0, t_1]$, $K \subset \mathbb{R}^N$, *we have*

$$
\begin{aligned}
\iint_Q v\,dx\,dt &\leq C(N)|Q \cap \{v > 0\}|^{\frac{1}{N+1}} \\
&\quad \times \Big[\operatorname{ess\,sup}_{t_0 \leq t \leq t_1} \int_K v\,dx\Big]^{\frac{1}{N+1}} \Big[\iint_Q |Dv|\,dx\,dt\Big]^{\frac{N}{N+1}}.
\end{aligned}
\tag{5.5}
$$

5.5. **Iteration.** Finally, we recall [7, Lemma I.4.1].

Lemma 5.5. *Let* $\{Y_n\}$, $n = 0, 1, 2, \dots$, *be a sequence of positive numbers, satisfying the recursive inequalities*

$$
Y_{n+1} \leq C b^n Y_n^{1+\alpha} \tag{5.6}
$$

where $C, b > 1$ *and* $\alpha > 0$ *are given numbers. If*

$$
Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},
$$

then $\{Y_n\}$ *converges to zero as* $n \to \infty$.

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Sukjung Hwang
Center for Mathematical Analysis and Computation, Yonsei University, Seoul 03722, Korea
E-mail address: `sukjung_hwang@yonsei.ac.kr`

Gary M. Lieberman Department of Mathematics, Iowa State University, Ames, IA 50011, USA
E-mail address: `lieb@iastate.edu`