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FAVARD SPACES AND ADMISSIBILITY FOR VOLTERRA SYSTEMS WITH SCALAR KERNEL

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Abstract. We introduce the Favard spaces for resolvent families, extending some well-known theorems for semigroups. Furthermore, we show the relationship between these Favard spaces and the L^p-admissibility of control operators for scalar Volterra linear systems in Banach spaces, extending some results in [22]. Assuming that the kernel $a(t)$ is a creep function which satisfies $a(0^+) > 0$, we prove an analogue version of the Weiss conjecture for scalar Volterra linear systems when $p = 1$. To this end, we also show that the finite-time and infinite-time (resp. finite-time and uniform finite-time) L^1-admissibility coincide for exponentially stable resolvent families (reps. for reflexive state space), extending well-known results for semigroups.

1. Introduction

Several authors have investigated the notion of the admissibility of control operator for semigroups [11, 12, 13, 23, 26, 27, 29]. The first studies on admissibility of control operator for Volterra scalar systems began with the paper of Jung [17]. Later, admissibility for linear Volterra scalar systems have been discussed by a number of authors in [10, 14, 15]. In [17], Jung links the notion of finite-time L^2-admissibility for Volterra scalar system with finite-time L^2-admissibility of the well-studied semigroups case for completely positive kernel. Likewise, in [14] the infinite-time L^2-admissibility for a Volterra scalar system is linked with the infinite-time L^2-admissibility for semigroups for a large class of kernel and the result subsumes that of [17]. In [15], the authors have given necessary and sufficient condition for finite-time L^2-admissibility of a linear integrodifferential Volterra scalar system when the underlying semigroup is equivalent to a contraction semigroup, which generalizes an analogous result known to hold for the standard Cauchy problem and it subsumes the result in [17]. Another result is related to the case where the generator of the underlying semigroup has a Riesz basis of eigenvectors in [10]. In Section 2, we give some preliminaries about the concept of resolvent family, and the relationship between linear integral equation of Volterra type with scalar kernel. It is well-known that for a Cauchy problem there are strong relations connecting its semigroup solution and its associated generator. Likewise, for a Volterra scalar

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problem, there are some results connecting its resolvent family and the domain of the associated generator; which will be reviewed in Section 3. There are many results available from semigroups theory concerning the Favard spaces (see. [3, 6]). In Section 4, we define the Favard spaces for scalar Volterra integral equations, and for these spaces we account for some results which are similar to those of semigroups. Especially, we account for a similar result to that in [5, Theorem 9] if the kernel is a creep function. In Section 5, we introduce the ideas of L^p-admissibility of resolvent families in the same spirit of semigroups and we describe the relationship between the L^p-admissibility to the Favard spaces, already introduced in Section 4. This extends some results obtained for the semigroups case in [22]. In particular, we are able to prove that for Volterra scalar systems with a creep kernel $a(t)$ such that $a(0^+) > 0$; the finite-time and the infinite-time L^1-admissibility are equivalent for exponentially stable resolvent family; and if the underlying state space is reflexive then the finite-time and the uniform finite-time L^1-admissibility are also equivalent; extending well-known results for semigroups for all $p \in [1, \infty[$. (See. [29, 8]).

2. Preliminaries

In this section we collect some elementary facts about scalar Volterra equations and resolvent family. These topics have been covered in detail in [25]. We refer to these works for reference to the literature and further information.

Let $(X, \|\cdot\|)$ be a Banach space, A be a linear closed densely defined operator in X and $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ is a scalar kernel. We consider the linear Volterra equation

$$\begin{gathered} x(t) = \int_0^t a(t-s)Ax(s)ds + f(t), \quad t \geq 0, \\ x(0) = x_0 \in X, \end{gathered} \tag{2.1}$$

where $f \in \mathcal{C}(\mathbb{R}^+, X)$.

Since A is a closed operator, we may consider $(X_1, \|x\|_1)$ the domain of A equipped with the graph-norm, i.e. $\|x\|_1 = \|x\| + \|Ax\|$. It is continuously embedded in X. If the resolvent set $\rho(A)$ of A is nonempty, $A_1 : D(A^2) \to X_1$, with $A_1 x = Ax$, is a closed operator in X_1 and $\rho(A) = \rho(A_1)$. On the other hand, we may consider X_{-1} the completion of X with respect to the norm

$$\|x\|_{-1} = \|(\mu_0 I - A)^{-1}x\| \quad \text{for some } \mu_0 \in \rho(A) \text{ and all } x \in X.$$

These spaces are independent of the choice of μ_0 and are related by the following continuous and dense injections

$$X_1 \overset{d}{\hookrightarrow} X \overset{d}{\hookrightarrow} X_{-1}.$$

Furthermore, the operator $A : D(A) \to X_{-1}$ is continuous and densely defined, its (unique) extension to X as domain makes it a closed operator in X_{-1}, and it is called A_{-1} and we have $\rho(A) = \rho(A_{-1})$ (see. e.g. [24]).

We define the convolution product of the scalar function a with the vector-valued function f by

$$(a * f)(t) := \int_0^t a(t-s)f(s)ds, \quad t \geq 0.$$

Definition 2.1. A function $x \in C(\mathbb{R}^+, X)$ is called:

(i) strong solution of (2.1) if $x \in C(\mathbb{R}^+, X_1)$ and (2.1) is satisfied.

(ii) mild solution of (2.1) if $a * x \in C(\mathbb{R}^+, X_1)$ and

$$x(t) = f(t) + A[a * x](t) \; t \geq 0. \tag{2.2}$$

Obviously, every strong solution of (2.1) is a mild solution. Conditions under which mild solutions are strong solutions are studied in [25].

Definition 2.2. Equation (2.1) is called well-posed if, for each $v \in D(A)$, there is a unique strong solution $x(t, v)$ on $\mathbb{R}^+$ of

$$x(t, v) = v + (a * Ax)(t) \quad t \geq 0, \tag{2.3}$$

and for a sequence $(v_n) \subset D(A)$, $v_n \to 0$ implies $x(t, v_n) \to 0$ in X, uniformly on compact intervals.

Definition 2.3. Let $a \in L^1_{\text{loc}}(\mathbb{R}^+)$. A strongly continuous family $(S(t))_{t\geq 0} \subset \mathcal{L}(X)$; (the space of bounded linear operators in X) is called resolvent family for equation (2.1), if the following three conditions are satisfied:

(S1) $S(0) = I$.
(S2) $S(t)$ commutes with A, which means $S(t)D(A) \subset D(A)$ for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.
(S3) For each $x \in D(A)$ and all $t \geq 0$ the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s)AS(s)x ds.$$

Note that the resolvent for (2.1) is uniquely determined. The proofs of these results and further information on resolvent can be found in the monograph by Prüss [25]. We also notice that the choice of the kernel a classifies different families of strongly continuous solution operators in $\mathcal{L}(X)$: For instance when $a(t) = 1$, then $S(t)$ corresponds to a $\mathcal{C}_0$-semigroup and when $a(t) = t$, then $S(t)$ corresponds to cosine operator function. In particular, when $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ with $0 < \alpha \leq 2$ and Γ denotes the Gamma function, they are the α-times resolvent families studied in [2] and corresponds to the solution families for fractional evolution equations, i.e. evolution equations where the integer derivative with respect to time is replaced by a derivative of fractional order.

The existence of a resolvent family allows one to find the solution for the equation (2.1). Several properties of resolvent families have been discussed in [1, 25].

The resolvent family is the central object to be studied in the theory of Volterra equations. The importance of the resolvent family $S(t)$ is that, if it exists, then the solution $x(t)$ of (2.1) is given by the following variation of parameters formula in [25]:

$$x(t) = \frac{d}{dt} \int_0^t S(t-s)f(s)ds, \tag{2.4}$$

for all $t \geq 0$, and

$$x(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)ds, \tag{2.5}$$

where $t \geq 0$ and $f \in W^{1,1}(\mathbb{R}^+, X)$, gives us a mild solution for (2.1).

The following well-known result [25, Proposition 1.1] establishes the relation between well-posedness and existence of a resolvent family. In what follows, $\mathcal{R}$ denotes the range of a given operator.

Theorem 2.4. *Equation* (2.1) *is well-posed if and only if* (2.1) *admits a resolvent family* $(S(t))_{t\geq 0}$. *If this is the case we have in addition* $\mathcal{R}(a*S(t)) \subset D(A)$, *for all* $t \geq 0$ *and*

$$S(t)x = x + A\int_0^t a(t-s)S(s)xds, \tag{2.6}$$

for each $x \in X$, $t \geq 0$.

From this we obtain that if $(S(t))_{t\geq 0}$ is a resolvent family of (2.1), we have $A(a*S)(.)$ is strongly continuous and the so-called mild solution $x(t) = S(t)x_0$ solves equation (2.1) for all $x_0 \in X$ with $f(t) = x$. A resolvent family $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and the pair (M, ω) is called type of $(S(t))_{t\geq 0}$. The growth bound of $(S(t))_{t\geq 0}$ is $\omega_0 = \inf\{\omega \in \mathbb{R}, \|S(t)\| \leq Me^{\omega t}, t \geq 0, M > 0\}$. The resolvent family is called exponentially stable if $\omega_0 < 0$.

Note that, contrary to the case of $\mathcal{C}_0$-semigroup, resolvent for (2.1) need not to be exponentially bounded: a counterexample can be found in [4, 25]. However, there is checkable condition guaranteeing that (2.1) possesses an exponentially bounded resolvent operator.

We will use the Laplace transform at times. Suppose $g : \mathbb{R}^+ \to X$ is measurable and there exist $M > 0$, $\omega \in \mathbb{R}$, such that $\|g(t)\| \leq Me^{\omega t}$ for almost $t \geq 0$. Then the Laplace transform

$$\widehat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)dt,$$

exists for all $\lambda \in \mathbb{C}$ with $Re\lambda > \omega$.

A function $a \in L^1_{\rm loc}(\mathbb{R}^+)$ is said to be ω (resp. ω^+)-exponentially bounded if $\int_0^\infty e^{-\omega s}|a(s)|ds < \infty$ for some $\omega \in \mathbb{R}$ (resp. $\omega > 0$).

The following proposition stated in [25], establishes the relation between resolvent family and Laplace transform.

Proposition 2.5. *Let* $a \in L^1_{\rm loc}(\mathbb{R}^+)$ *be* ω*-exponentially bounded. Then* (2.1) *admits a resolvent family* $(S(t))_{t\geq 0}$ *of type* (M, ω) *if and only if the following conditions hold:*

- (i) $\hat{a}(\lambda) \neq 0$ *and* $1/\hat{a}(\lambda) \in \rho(A)$, *for all* $\lambda > \omega$.
- (ii) $H(\lambda) := \frac{1}{\lambda \hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}$ *called the resolvent associated with* $(S(t))_{t\geq 0}$ *satisfies*

$$\|H^{(n)}(\lambda)\| \leq Mn!(\lambda - \omega)^{-(n+1)} \quad \text{for all } \lambda > \omega \text{ and } n \in \mathbb{N}.$$

Under these assumptions the Laplace-transform of $S(\cdot)$ is well-defined and it is given by $\widehat{S}(\lambda) = H(\lambda)$ for all $\lambda > \omega$.

3. Domains of A: A Review

Assuming the existence of a resolvent family $(S(t))_{t\geq 0}$ for (2.1), it is natural to ask how to characterize the domain $D(A)$ of the operator A in terms of the resolvent family. This is important, for instance in order to study the Favard class in perturbation theory (see. [16, 19]). For very special case, the answer to the above question is well-known. For instance, when $a(t) = 1$ or $a(t) = t$, A is the generator of a $\mathcal{C}_0$-semigroup $(\mathbb{T}(t))_{t\geq 0}$ or a cosine family $(C(t))_{t\geq 0}$ and we have:

$$D(A) = \big\{x \in X : \lim_{t\to 0^+} \frac{\mathbb{T}(t)x - x}{t} \text{ exists}\big\},$$

$$D(A)=\big\{x\in X:\lim_{t\to 0^+}\frac{C(t)x-x}{t^2}\text{ exists}\big\},$$

respectively (see. [25]).

A reasonable formula for the generator of resolvent families and k-regularized resolvent families introduced in [18, 21] have been established by assuming very mild conditions on the kernels $a(t)$ and $k(t)$. See. [16, Theorem 2.5] and [21, Theorem 2.1]. It was observed in [16] that $D(A)$ has the following characterization.

Proposition 3.1. *Let* (2.1) *admit a resolvent family with growth bound* ω *(such that the Laplace transform of the resolvent exists for* $\lambda>\omega$*) for* ω*-exponentially bounded* $a\in L^1_{\rm loc}(\mathbb{R}^+)$*. Set for* $0<\theta<\pi/2$ *and* $\epsilon>0$

$$\Omega_\theta^\epsilon:=\big\{\frac{1}{\widehat{a}(\lambda)}:Re\lambda>\omega+\epsilon,\ |\arg\lambda|\le\theta\big\}.$$

Then the following characterization of $D(A)$ *holds*

$$D(A)=\big\{x\in X:\lim_{|\mu|\to\infty,\,\mu\in\Omega_\theta^0}\mu A(\mu I-A)^{-1}x\ \ \textit{exists}\big\}.$$

Without loss of generality we may assume that $\int_0^t|a(s)|^pds\neq 0$ for all $t>0$ and some $1\le p<\infty$. Otherwise we would have for some $t_0>0$ and $p_0\ge 1$ that $a(t)=0$ for almost all $t\in[0,t_0]$, and thus by definition of resolvent family $S(t)=I$ for $t\in[0,t_0]$. This implies that A is bounded, which is the trivial case with $X=D(A)$.

In what follows, we will use in the forthcoming sections the following assumption on $a\in L^p_{\rm loc}(\mathbb{R}^+)$ with $1\le p<\infty$. It corresponds to [16, Assumption 2.3] when $p=1$.

(H1) There exist $\epsilon_a>0$ and $t_a>0$, such that for all $0<t\le t_a$,

$$|\int_0^t a(s)ds|\ge\epsilon_a\int_0^t|a(s)|^pds.$$

This is the case for functions a, which are positive (resp. $a(I)\subset]0,1]$) at some interval $I=[0,t_a[$ for $p=1$ (resp. $p>1$). For almost all reasonable functions in applications it is easy to see that they satisfy this assumption. There are nonetheless examples of functions that do not.

Now let us define the set $\widetilde{D}(A)$ as follows:

$$\widetilde{D}(A):=\big\{x\in X:\lim_{t\to 0^+}\frac{S(t)x-x}{(1*a)(t)}\text{ exists}\big\}$$

where $(S(t))_t\ge 0$ is a resolvent familly associated with (2.1).

It was proved in [16] that under (H1),

$$D(A)=\widetilde{D}(A)=\{x\in X:\lim_{t\to 0^+}\frac{S(t)x-x}{(1*a)(t)}=Ax\}. \tag{3.1}$$

From now and in view of this result we say that the pair (A,a) is a generator of a resolvent family $(S(t))_{t\ge 0}$.

Remark 3.2. When $a=1+1*k$, with $k\in L^1_{\rm loc}(\mathbb{R}^+)$, the Volterra system (2.1) with $f(t)=x_0$ is equivalent to the integrodifferential Volterra system

$$\dot{x}(t)=Ax(t)+\int_0^t k(t-s)Ax(s)ds,\quad t\ge 0. \tag{3.2}$$

Furthermore, if (3.2) admits a resolvent family $(S(t))_{t\geq 0}$, then it is easy to see that

$$\begin{aligned}\widetilde{D}(A) &= \{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{[1*(1+1*k)](t)} = Ax\},\\ &= \big\{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax\big\}.\end{aligned}$$

It is well-known that if $k \in BV_{\rm loc}(\mathbb{R}^+)$; (the space of functions locally of bounded variation), then the operator A becomes a generator of a $\mathcal{C}_0$-semigroup $(\mathbb{T}(t))_{t\geq 0}$, which is a necessary and sufficient condition for the existence of a resolvent family (see. [25]) . Whence $\widetilde{D}(A)$ is also characterized in term of $(\mathbb{T}(t))_{t\geq 0}$ and we have

$$\widetilde{D}(A) = \big\{x \in X : \lim_{t\to 0^+} \frac{\mathbb{T}(t)x - x}{t} = Ax\big\} = \big\{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax\big\}.$$

4. Favard spaces with kernel

In semigroup theory the Favard space sometimes called the generalized domain is defined for a given semigroup $(\mathbb{T}(t))_{t\geq 0}$ (with A as its generator) as

$$\widetilde{F}^\alpha(A) := \big\{x \in X : \sup_{t>0} \frac{\|\mathbb{T}(t)x - x\|}{t^\alpha} < \infty\big\}, \quad 0 < \alpha \leq 1,$$

with norm

$$\|x\|_{\widetilde{F}^\alpha(A)} := \|x\| + \sup_{t>0} \frac{\|\mathbb{T}(t)x - x\|}{t^\alpha},$$

which makes $\widetilde{F}^\alpha(A)$ a Banach space. $\mathbb{T}(t)$ is a bounded operator on $\widetilde{F}^\alpha(A)$ but is not necessary strongly continuous on it. X_1 is a closed subspace of $\widetilde{F}^\alpha(A)$ and both spaces coincide when $\alpha = 1$, and X is reflexive (see. e.g., [6]). It is natural to ask how to define in a similar way $\widetilde{F}^\alpha(A)$ of the operator A in terms of the resolvent family. In fact, these spaces can be defined for general solution families in a similar way. In fact, it can be defined for all A, for which there exists a sequence $(\lambda_n)_n$ with $\lambda_n \in \rho(A)$ and $|\lambda_n| \to \infty$ in a similar fashion, as was proved in [16] for resolvent family and in [19] for integral resolvent family and in [20] for (a,k)-resolvent family for the case $\alpha = 1$. Remark that both [16] and [19] have not considered the Favard class of order α. These spaces will be the topic of this section and will be useful for the notion of the admissibility considered in Section 5.

This leads to the following definition which corresponds to a natural extension, in our context, of the Favard classes frequently used in approximation theory for semigroups.

Definition 4.1. Let (2.1) admit a bounded resolvent family $(S(t))_{t\geq 0}$ on X, for ω^+-exponentially bounded $a \in L^1_{\rm loc}(\mathbb{R}^+)$. For $0 < \alpha \leq 1$, we define the "frequency" Favard space of order α associated with (A, a) as follows:

$$\begin{aligned}F^\alpha(A) &:= \big\{x \in X : \sup_{\lambda>\omega} \|\lambda^{\alpha-1} \frac{1}{\hat{a}(\lambda)} A\big(\frac{1}{\hat{a}(\lambda)} I - A\big)^{-1} x\| < \infty\big\},\\ &= \big\{x \in X : \sup_{\lambda>\omega} \|\lambda^\alpha A H(\lambda) x\| < \infty\big\}.\end{aligned}$$

Remark 4.2. (i) As for the semigroups it is natural to define the following space

$$\widetilde{F}^\alpha(A) := \Big\{x \in X : \sup_{t>0} \frac{\|S(t)x - x\|}{|(1*a)(t)|^\alpha} < \infty\Big\},$$

for (A, a) generator of a resolvent family $(S(t))_{t\geq 0}$ on X.

(ii) It is clear that $\widetilde{D}(A) \subset \widetilde{F}^1(A)$ and by virtue of Proposition 3.1 we have $D(A) \subset F^1(A)$. Moreover, if a satisfies (H1) then $D(A) \subset \widetilde{F}^1(A)$ due to the fact that $F^1(A) \subset \widetilde{F}^1(A)$ (see. [16]). In this way, for different functions $a(t)$ we obtain different Favard classes of order α which may be considered as extrapolation spaces between $D(A)$ and X.

(iii) When $a(t) = 1$, we recall that and $(S(t))_{t\geq 0}$ corresponds to a bounded $\mathcal{C}_0$-semigroup generated by A. In this situation we obtain

$$F^\alpha(A) = \big\{x \in X : \sup_{\lambda>0} \|\lambda^\alpha A(\lambda I - A)^{-1}x\| < \infty\big\}$$

and $F^\alpha(A) = \widetilde{F}^\alpha(A)$. This case is well known, (see. e.g. [6]).

(iv) The Favard class of A with kernel $a(t)$ can be alternatively defined as the subspace of X given by $\big\{x \in X : \limsup_{\lambda\to\infty} \|\lambda^{\alpha-1}\frac{1}{\hat{a}(\lambda)}A(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x\| < \infty\big\}$. As a consequence of $S(t)$ being bounded, the above space coincides with $F^\alpha(A)$ in Definition 4.1 and that $\widetilde{F}^\alpha(A) := \big\{x \in X : \sup_{0<t\leq 1} \frac{\|S(t)x-x\|}{|(1*a)(t)|^\alpha} < \infty\big\}$.

(v) Let $a = 1 + 1 * k$, with $k \in L^1_{\rm loc}(\mathbb{R}^+)$, and (A, a) be a generator of a bounded resolvent family $(S(t))_{t\geq 0}$ on X. Then, $\widetilde{F}^\alpha(A) = \big\{x \in X : \sup_{0<t\leq 1} \frac{\|S(t)x-x\|}{t^\alpha} < \infty\big\}$ (due to $\lim_{t\to 0^+} \frac{(1*a)(t)}{t} = 1$) and we have $S(t)F^\alpha(A) \subset F^\alpha(A)$ for all $\alpha \in]0, 1]$ and $t \geq 0$ thanks to [9, Theorem 7] ($(\mu I - A)^{-1}$commutes with $S(t)$ for all $\mu \in \rho(A)$).

The proof of the following proposition is immediate.

Proposition 4.3. *The Favard classes of order α of A with kernel $a(t)$, $F^\alpha(A)$ and $\widetilde{F}^\alpha(A)$ are Banach spaces with respect to the norms*

$$\|x\|_{F^\alpha(A)} := \|x\| + \sup_{\lambda>\omega} \|\lambda^{\alpha-1}\frac{1}{\hat{a}(\lambda)}A(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x\|,$$
$$\|x\|_{\widetilde{F}^\alpha(A)} := \|x\| + \sup_{0<t\leq 1} \frac{\|S(t)x - x\|}{|(1*a)(t)|^\alpha},$$

respectively.

As for the semigroups case, we obtain the natural inclusions between the Favard class for different exponents.

Proposition 4.4. *Let* (2.1) *admit a bounded resolvent family $(S(t))_{t\geq 0}$ on X for ω^+-exponentially bounded $a \in L^1_{\rm loc}(\mathbb{R}^+)$. For all $0 < \beta < \alpha \leq 1$, we have:*

(i) $D(A) \subset F^\alpha(A) \subset F^\beta(A)$.

(ii) $\widetilde{D}(A) \subset \widetilde{F}^\alpha(A) \subset \widetilde{F}^\beta(A)$.

Proof. (i) Let $x \in F^\alpha(A)$, then for all $\lambda > \omega$, we have

$$\begin{aligned}\|\lambda^{\beta-1}\frac{1}{\hat{a}(\lambda)}A(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x\| &= \|\lambda^{\beta-\alpha}\lambda^{\alpha-1}\frac{1}{\hat{a}(\lambda)}A(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x\| \\ &= \lambda^{\beta-\alpha}\|\lambda^{\alpha-1}\frac{1}{\hat{a}(\lambda)}A(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}x\|\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\lambda^{\alpha-\beta}} \sup_{\lambda>\omega} \|\lambda^{\alpha-1} \frac{1}{\hat{a}(\lambda)} A(\frac{1}{\hat{a}(\lambda)} I - A)^{-1} x\| \\
&\le \frac{1}{\omega^{\alpha-\beta}} \sup_{\lambda>\omega} \|\lambda^{\alpha-1} \frac{1}{\hat{a}(\lambda)} A(\frac{1}{\hat{a}(\lambda)} I - A)^{-1} x\|,
\end{aligned}
$$

which implies that $x \in F^{\beta}(A)$ and from Remark 4.2 (ii) we deduce that $D(A) \subset F^{\alpha}(A)$.

(ii) Let $x \in \widetilde{F}^{\alpha}(A)$, and $0 < t \le 1$. We have

$$
\begin{aligned}
\frac{\|S(t)x - x\|}{|(1*a)(t)|^{\beta}} &= \frac{1}{|\int_0^t a(s)ds|^{\beta-\alpha}} \frac{\|S(t)x - x\|}{|\int_0^t a(s)ds|^{\alpha}} \\
&\le \|a\|_{L^1[0,1]}^{\alpha-\beta} \sup_{0<t\le 1} \frac{\|S(t)x - x\|}{|\int_0^t a(s)ds|^{\alpha}}
\end{aligned}
$$

Hence $x \in \widetilde{F}^{\beta}(A)$ and that $\widetilde{D}(A) \subset \widetilde{F}^{\alpha}(A)$ due to Remark 4.2 (ii). $\square$

Note that under (H1) we have: (i) $F^1(A) \subset \widetilde{F}^1(A)$ (see. [16, Assumption 2.3]). Whereas the inclusion (ii) $\widetilde{F}^1(A) \subset F^1(A)$ was proved under a strong assumption in [16, Assumption 3.1]. Now we prove that (ii) holds for all non negative $a \in L^1_{\rm loc}(\mathbb{R}^+)$.

Proposition 4.5. *Let (2.1) admit a bounded resolvent family $(S(t))_{t\ge 0}$ on X, for ω^+-exponentially bounded non negative $a \in L^1_{\rm loc}(\mathbb{R}^+)$. Then, we have $F^1(A) = \widetilde{F}^1(A)$.*

Proof. Since $a(t)$ is a non negative, (H1) is satisfied and by [16] we have $F^1(A) \subset \widetilde{F}^1(A)$. Now let $x \in \widetilde{F}^1(A)$ and set $\sup_{0<t\le 1} \frac{\|S(t)x-x\|}{(1*a)(t)} := J_x < \infty$. We write

$$
\frac{1}{\hat{a}(\lambda)} A(\frac{1}{\hat{a}(\lambda)} I - A)^{-1} = \lambda A H(\lambda),
$$

for all $\lambda > \omega$. Using the integral representation of the resolvent (see. Proposition 2.5) we obtain:

$$
\begin{aligned}
\lambda A H(\lambda)x &= \frac{\lambda}{\widehat{a}(\lambda)} H(\lambda)x - \frac{1}{\widehat{a}(\lambda)} x \\
&= \frac{\lambda}{\widehat{a}(\lambda)} [H(\lambda)x - \frac{1}{\lambda} x] \\
&= \frac{\lambda}{\widehat{a}(\lambda)} \int_0^{\infty} e^{-\lambda s} (S(s)x - x) ds \\
&= \frac{\lambda}{\widehat{a}(\lambda)} \int_0^{\infty} e^{-\lambda s} (1*a)(s) \frac{S(s)x - x}{(1*a)(s)} ds.
\end{aligned}
$$

The resolvent family $(S(t))_{t\ge 0}$ being bounded; $\|S(t)\| \le M$ for some $M > 0$ and all $t \ge 0$. Then we obtain

$$
\begin{aligned}
\|\lambda A H(\lambda)x\| &\le \frac{\lambda}{\widehat{a}(\lambda)} \int_0^{\infty} e^{-\lambda s} (1*a)(s) ds. \sup_{t>0} \frac{\|S(t)x - x\|}{(1*a)(t)} \\
&\le \frac{\lambda}{\widehat{a}(\lambda)} \int_0^{\infty} e^{-\lambda s} (1*a)(s) ds. (L\|x\| + \sup_{0<t\le 1} \frac{\|S(t)x - x\|}{(1*a)(t)}) \\
&= \frac{\lambda}{\widehat{a}(\lambda)} \widehat{1*a}(\lambda). (L\|x\| + J_x) \\
&= L\|x\| + J_x,
\end{aligned}
$$

with $L = \frac{1+M}{(1*a)(1)}$. This implies that $\sup_{\lambda>\omega}\|\lambda AH(\lambda)x\| < \infty$, which completes the proof. $\square$

Note that in the semigroup case, i.e. $a(t) = 1$, we have the well-known result that $\widetilde{F}^\alpha(A){=}F^\alpha(A)$, (see. e.g. [6]). In what follows, we investigate conditions on the kernel a to prove that this is the case for the (A, a) generator of the resolvent families. Note that for all ω^+-exponentially bounded function a, it is clear that $(1 * a)^\alpha$ is also ω^+-exponentially bounded (due to $x^\alpha \leq 1 + x$ for all $x \geq 0$ and $\alpha \in]0, 1]$).

We consider the following assumption on $a \in L^1_{\rm loc}(\mathbb{R}^+)$ and $0 < \alpha \leq 1$.

(H2) a is ω^+-exponentially bounded and there exists $\epsilon_{a,\alpha} > 0$, such that for all $\lambda > \omega$
$$|\hat{a}(\lambda)| \geq \epsilon_{a,\alpha}\lambda^\alpha \int_0^\infty e^{-\lambda t}|(1 * a)(t)|^\alpha dt.$$

Note that conditions (H2) and $\lambda\hat{a}(\lambda)$ being bounded, are independent (see. e.g. Example 4.6 (ii)).

Example 4.6. (i) The famous case $a(t) = 1$ satisfies the condition (H2) for all $\alpha \geq 0$ due to
$$\frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}((1 * 1)(t))^\alpha dt = \Gamma(\alpha + 1) \quad \text{for all } \lambda > 0,$$
which corresponds to the semigroup case.

(ii) Consider the standard kernel $a(t) = t^{\beta-1}/\Gamma(\beta)$ for $\beta \in [0, 1[$. a is non negative and for all $\lambda > 0$
$$\begin{aligned}\frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}((1 * a)(t))^\alpha dt &= \frac{\lambda^{\alpha+\beta-\alpha\beta-1}}{\beta^\alpha(\Gamma(\beta))^\beta\Gamma(\alpha\beta + 1)},\\ &= \frac{\lambda^{(\alpha-1)(1-\beta)}}{\beta^\alpha(\Gamma(\beta))^\beta\Gamma(\alpha\beta + 1)}.\end{aligned}$$
Thus a satisfies (H2) and $\lambda\hat{a}(\lambda) = \lambda^{1-\beta}$ is not bounded for $\beta \in [0, 1[$.

(iii) Let $a(t) = \mu + \nu t^\beta$, $0 < \beta < 1$, $\mu > 0$, $\nu > 0$. Then we have $\widehat{a}(\lambda) = \frac{\mu}{\lambda} + \frac{\nu}{\lambda^{\beta+1}}\Gamma(\beta + 1)$ for $\lambda > 0$ and $(1 * a)(t) = \mu t + \nu\frac{t^{\beta+1}}{\beta+1}$. Further, for $\alpha \in]0, 1]$ we have
$$\begin{aligned}&\frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}((1 * a)(t))^\alpha dt\\ &= \frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}(\mu t + \nu\frac{t^{\beta+1}}{\beta + 1})^\alpha dt,\\ &= \frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^1 e^{-\lambda t}(\mu t + \nu\frac{t^{\beta+1}}{\beta + 1})^\alpha dt + \frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_1^\infty e^{-\lambda t}(\mu t + \nu\frac{t^{\beta+1}}{\beta + 1})^\alpha dt,\\ &\leq (\mu + \frac{\nu}{\beta + 1})^\alpha\frac{\Gamma(\alpha + 1)}{\mu} + (\mu + \frac{\nu}{\beta + 1})^\alpha\frac{\Gamma(\alpha\beta + \alpha + 1)}{\mu}\lambda^{-\alpha\beta}.\end{aligned}$$
Then (H2) is satisfied. Note that for $\beta = 1$, $a(t) = \mu + \nu t$, Equqation (2.1) corresponds to the model of a solid of Kelvin-Voigt (see. [25]).

(iv) Let $a = 1 + 1 * k$ with $k(t) = e^{-t}$. We have $\widehat{a}(\lambda) = \frac{\lambda+2}{\lambda(\lambda+1)}$ for all $\lambda > 0$ and $(1 * a)(t) = 2t + e^{-t} - 1 \le 2t$ for all $t \ge 0$. Hence

$$\frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}((1*a)(t))^\alpha dt \le \frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}(2t)^\alpha dt, = \frac{\lambda+1}{\lambda+2} \cdot 2^\alpha \Gamma(\alpha+1).$$

Then a satisfies (H2).

(v) Let $a = 1 + 1 * k$ with $k(t) = -e^{-t}$. We have $\widehat{a}(\lambda) = \frac{1}{\lambda+1}$ for all $\lambda > 0$ and that $(1 * a)(t) = 1 - e^{-t} \le t$ for all $t \ge 0$. Hence

$$\frac{\lambda^\alpha}{\widehat{a}(\lambda)} \int_0^\infty e^{-\lambda t}((1*a)(t))^\alpha dt \le \lambda^\alpha(\lambda+1) \int_0^\infty e^{-\lambda t} t^\alpha dt = \frac{\lambda+1}{\lambda}\Gamma(\alpha).$$

Then a satisfies (H2).

The following result establishes the relation between the spaces $\widetilde{F}^\alpha(A)$ and $F^\alpha(A)$ and therefore generalizes [6, Proposition 5.12].

Proposition 4.7. *Let* (2.1) *admit a bounded resolvent family* $(S(t))_{t\ge0}$ *on* X, *for* ω^+*-exponentially bounded* $a \in L^1_{\rm loc}(\mathbb{R}^+)$ *and* $0 < \alpha \le 1$.

(i) *If a satisfies (H1) and $\lambda\widehat{a}(\lambda)$ is bounded for $\lambda > \omega$, then $F^\alpha(A) \subset \widetilde{F}^\alpha(A)$.*
(ii) *If a is non negative satisfying (H2), then $\widetilde{F}^\alpha(A) \subset F^\alpha(A)$.*

Proof. (i) Let $x \in F^\alpha(A)$ and $0 < t \le 1$. Then $\sup_{\lambda>\omega}\|\lambda^\alpha AH(\lambda)x\| =: K_x < \infty$. Using the integral representation of the resolvent (see. Proposition 2.5), we obtain

$$x = \lambda H(\lambda)x - \lambda\widehat{a}(\lambda)AH(\lambda)x \text{ for } \lambda > \omega, =: x_\lambda - y_\lambda.$$

Since $x_\lambda \in D(A)$ and using (S2)-(S3) we have

$$\begin{aligned}
\|S(t)x_\lambda - x_\lambda\| &= \|\int_0^t a(t-s)S(s)Ax_\lambda ds\| \\
&\le \int_0^t |a(t-s)| \cdot \|S(s)\| \cdot \|Ax_\lambda\| ds \\
&\le M\|Ax_\lambda\| \int_0^t |a(s)| ds \\
&= M\|\lambda^\alpha AH(\lambda)x\|\lambda^{1-\alpha}(1*|a|)(t) \\
&\le MK_x\lambda^{1-\alpha}(1*|a|)(t).
\end{aligned}$$

On the other hand, $(S(t))_{t\ge0}$ is bounded by M and we have

$$\begin{aligned}
\|S(t)y_\lambda - y_\lambda\| &\le \|S(t)y_\lambda\| + \|y_\lambda\| \\
&\le \|S(t)\|\,\|y_\lambda\| + \|y_\lambda\| \\
&\le (M+1)\|y_\lambda\| \\
&= (M+1)\|\lambda\widehat{a}(\lambda)AH(\lambda)x\| \\
&= (M+1)|\widehat{a}(\lambda)|\,\|\lambda^\alpha AH(\lambda)x\|\lambda^{1-\alpha} \\
&\le (M+1)K_x|\widehat{a}(\lambda)|\lambda^{1-\alpha}.
\end{aligned}$$

This implies

$$\frac{\|S(t)x - x\|}{|(1*a)(t)|^\alpha}$$

$$
\begin{aligned}
&\le \frac{MK_x\lambda^{1-\alpha}(1*|a|)(t)}{|(1*a)(t)|^\alpha} + \frac{(M+1)K_x.|\widehat{a}(\lambda)|\lambda^{1-\alpha}}{|(1*a)(t)|^\alpha} \\
&\le \frac{MK_x}{\epsilon_a^\alpha}\lambda^{1-\alpha}((1*|a|)(t))^{1-\alpha} + \frac{(M+1)K_x}{\epsilon_a^\alpha}|\lambda\widehat{a}(\lambda)|.\lambda^{-\alpha}((1*|a|)(t))^{-\alpha} \\
&\le \frac{MK_x}{\epsilon_a^\alpha}\lambda^{1-\alpha}((1*|a|)(t))^{1-\alpha} + \frac{(M+1)K_xK'}{\epsilon_a^\alpha}\lambda^{-\alpha}((1*|a|)(t))^{-\alpha}.
\end{aligned}
$$

The third inequality is realized under (H1): $|(1*a)(t)| \ge \epsilon_a(1*|a|)(t)$ and that $|\lambda\widehat{a}(\lambda)| \le K'$ for some $K' > 0$ and for λ large enough. Substituting $\lambda_t = \frac{N_\omega}{(1*|a|)(t)} > \omega$ for $t \in]0,1]$ ($\lambda_t \to \infty$ as $t \to 0$) with $N_\omega = 1 + \omega(1*|a|)(1)$, we obtain

$$
\frac{\|S(t)x - x\|}{|(1*a)(t)|^\alpha} \le \frac{MK_xN_\omega^{1-\alpha}}{\epsilon_a^\alpha} + \frac{(M+1)K_xK'N_\omega^{-\alpha}}{\epsilon_a^\alpha},
$$

for all $0 < t \le 1$. Thus $\sup_{0<t\le 1}\frac{\|S(t)x-x\|}{|(1*a)(t)|^\alpha} < \infty$, and hence $x \in \widetilde{F}^\alpha(A)$.

(ii) Let $x \in \widetilde{F}^\alpha(A)$ be given, then $\sup_{0<t\le 1}\frac{\|S(t)x-x\|}{|(1*a)(t)|^\alpha} := J_x < \infty$. For $\lambda > \omega$ we write $\lambda H(\lambda)x - x = \lambda\widehat{a}(\lambda)AH(\lambda)x$ then

$$
\lambda AH(\lambda)x = \frac{\lambda}{\widehat{a}(\lambda)}(H(\lambda)x - \frac{1}{\lambda}x) = \frac{\lambda}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda t}(S(t)x - x)dt,
$$

and

$$
\lambda^\alpha AH(\lambda)x = \frac{\lambda^\alpha}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda t}(1*a)^\alpha(t)(\frac{S(t)x - x}{(1*a)^\alpha(t)})dt,
$$

The fact that a is non negative and satisfies (H2), implies

$$
\|\lambda^\alpha AH(\lambda)x\| \le \frac{(L_\alpha\|x\| + J_x)}{\epsilon_{a,\alpha}} \quad \text{with } L_\alpha = \frac{1+M}{(1*a)^\alpha(1)}.
$$

Therefore, $\sup_{\lambda>\omega}\|\lambda^\alpha AH(\lambda)x\| < \infty$ which completes the proof. □

Remark 4.8. Let $\alpha \in]0,1]$.

(i) $a(t) = 1$. Then $\lambda\widehat{a}(\lambda)$ is bounded for all $\lambda > 0$ and a satisfies (H1). Furthermore a satisfies (H2) (see. Example 4.6 (i)) and by virtue of Proposition 4.7, we obtain $F^\alpha(A) = \widetilde{F}^\alpha(A)$. Hence we recover a result for $\mathcal{C}_0$-semigroups case which corresponds to [6, Proposition 5.12].

(ii) $a(t) = t$ satisfies (H1) and we have $\lambda\widehat{a}(\lambda) = \frac{1}{\lambda}$ is bounded for all $\lambda > \omega > 0$. By virtue of Proposition 4.7 (i) we obtain $F^\alpha(A) \subset \widetilde{F}^\alpha(A)$.

(iii) Let a be a completely positive function. Then (see. [25]) a is non negative and

$$
\lambda\widehat{a}(\lambda) = \frac{1}{k_0 + \frac{1}{k_\infty} + \widehat{k_1}(\lambda)},
$$

for all $\lambda > 0$ where $k_0 \ge 0$, $k_\infty \ge 0$ and k_1 is non negative decreasing function tending to 0 as $t \to \infty$. That is $\lambda\widehat{a}(\lambda)$ is bounded and by Proposition 4.7 (i) we obtain $F^\alpha(A) \subset \widetilde{F}^\alpha(A)$.

(iv) Consider the standard kernel $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, with $\beta \in [1,2[$. Then a satisfies (H1) and that $\lambda\widehat{a}(\lambda) = \lambda\cdot\lambda^{-\beta} = \lambda^{1-\beta}$, for all $\lambda > \omega > 0$ is bounded, thus from Proposition 4.7 (i) $F^\alpha(A) \subset \widetilde{F}^\alpha(A)$.

(v) Let $a(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, with $\beta \in [0,1[$. Then a is non negative and we have

$$\begin{aligned}\frac{\lambda^\alpha}{\widehat{a}(\lambda)}\int_0^\infty e^{-\lambda t}((1*a)(t))^\alpha dt &= \frac{\lambda^{\alpha+\beta-\alpha\beta-1}}{\beta^\alpha(\Gamma(\beta))^\beta\Gamma(\alpha\beta+1)}\\ &= \frac{\lambda^{(\alpha-1)(1-\beta)}}{\beta^\alpha(\Gamma(\beta))^\beta\Gamma(\alpha\beta+1)}\end{aligned}$$

which is bounded, for all $\lambda > \omega > 0$ due to $\beta \in [0,1[$. This implies that a satisfies (H2) and according to Proposition 4.7 (ii) we can conclude that $\widetilde{F}^\alpha(A) \subset F^\alpha(A)$.

(vi) Let $a(t) = \mu + \nu t^\beta$, $0 < \beta < 1$, $\mu > 0$, $\nu > 0$. By Proposition 4.7 we have $\widetilde{F}^\alpha(A) = F^\alpha(A)$ according to the Example 4.6 (iii).

(vii) Let $a = 1 + 1 * k$, with $k(t) = \pm e^{-t}$. Proposition 4.7 yields $\widetilde{F}^\alpha(A) = F^\alpha(A)$ according to the Example 4.6 (iv)-(v). In general, for $k \in L^1_{\rm loc}(\mathbb{R}^+)$, ω^+-exponentially bounded, we have $\lambda\hat{a}(\lambda) = 1 + \widehat{k}(\lambda)$ which is is bounded for all $\lambda > 0$, according to the Riemann-Lebesgue Lemma. If in addition a satisfies (H1), Proposition 4.7 (i) asserts that $F^\alpha(A) \subset \widetilde{F}^\alpha(A)$. Now, if $k(t)$ is negative with $\widehat{k}(0) \geq -1$ then we obtain a non negative kernel a satisfying $0 \leq (1*a)(t) \leq t$. Hence, both (H1) and (H2) are satisfied (see. Example 4.6 (iv)) and using Proposition 4.7 we obtain $\widetilde{F}^\alpha(A) = F^\alpha(A)$.

Definition 4.9. A scalar function $a :]0,\infty[\to \mathbb{R}$ is called creep if it is continuous, non-negative, non-decreasing and concave.

According to [25, Definition 4.4], a creep function has the standard form

$$a(t) = a_0 + a_\infty t + \int_0^t a_1(\tau)d\tau,$$

where $a_0 = a(0^+) \geq 0$, $a_\infty = \lim_{t\to\infty}\frac{a(t)}{t}$ and $a_1(t) = \dot{a}(t) - a_\infty$ is non negative, non increasing and $\lim_{t\to\infty} a_1(t) = 0$.

The concept of creep function is well known in viscoelasticity theory and corresponds to a class of functions which are normally verified in practical situations. We refer to the monograph of Prüss [25] for further information.

We finish this section by proving an analogue version to a well-known result for semigroups in [5, Theorem 9] for resolvent family. We remark that a similar result was proved for integral resolvent families in [19]. For the sake of completeness we give here the details of the proof.

Lemma 4.10. *Assume that (A,a) generates a bounded resolvent family $(S(t))_{t\geq 0}$, and a is a creep function with $a(0^+) > 0$. Then for all $\xi \in L^1_{\rm loc}(\mathbb{R}^+, \widetilde{F}^1(A))$ and $t > 0$, we have $\int_0^t S(t-s)\xi(s)ds \in D(A)$, and there exists $N > 0$ such that*

$$\|A\int_0^t S(t-s)\xi(s)ds\|_X \leq N\int_0^t \|\xi(s)\|_{\widetilde{F}^1(A)}ds \quad \text{for all } t > 0.$$

Proof. We give the proof in three steps. Let $t > 0$ and $\xi \in L^1([0,t], \widetilde{F}^1(A))$, there exists $\xi_n \in \mathcal{C}^2([0,t], \widetilde{F}^1(A))$, such that $\xi_n \to \xi$ in $L^1([0,t], \widetilde{F}^1(A))$ as $n \to \infty$.

Step 1. For all $t \geq 0$, $\int_0^t S(t-s)\xi_n(s)ds \in D(A)$. In fact, let $\varphi_n(s) = \xi_n(s) - \xi_n(0) - s\xi_n'(s)$. Then $\varphi_n(0) = 0$, and $\varphi_n'(0) = 0$. Define $i(s) = s$ and observe that $\xi_n(s) = (i * \varphi_n'')(s) + \xi_n'(0)i(s) + \xi_n(0)$, for all $s \in [0,t]$. Since a is a creep function, there exists a scalar function b such that $a*b=i$, see. [25, Proposition 4.4]. Hence, we obtain

$$\begin{aligned}\int_0^t S(t-s)\xi_n(s)ds &= (S*\xi_n)(t)\\ &= (a*S*b*\varphi_n^{''})(t)+(a*S*b)(t)\xi_n'(0)+\int_0^t S(s)\xi_n(0)ds.\end{aligned}$$

Since a is a creep function with $a(0^+)>0$, it is easy to see that the resolvent family $(S(t))_{t\geq 0}$ is a solution of an integrodifferential Volterra equation of the form (3.2). Thus $\int_0^t S(s)\xi_n(0)ds\in D(A)$, (see [9, Lemma 1] and [7]) and that $\mathcal{R}((a*S)(t))\subset D(A)$, we obtain $\int_0^t S(t-s)\xi_n(s)ds\in D(A)$ for all $t\geq 0$.

Step 2. $\int_0^t S(t-s)\xi_n(s)ds\to\int_0^t S(t-s)\xi(s)ds$ as $n\to\infty$ for all $t\geq 0$. In fact, by hypothesis there exists $M>0$ such that $\|S(t)\|\leq M$ for all $t\geq 0$, hence

$$\begin{aligned}\|\int_0^t S(t-s)[\xi_n(s)-\xi(s)]ds\| &\leq \int_0^t \|S(t-s)\|\,\|\xi_n(s)-\xi(s)\|ds\\ &\leq M\int_0^t \|\xi_n(s)-\xi(s)\|_{\widetilde{F}^1(A)}ds,\end{aligned}$$

which tends to zero as $n\to\infty$.

Step 3. $\int_0^t S(t-s)\xi(s)ds\in D(A)$ for all $t\geq 0$. In fact, let $\epsilon>0$ be given. Since $\int_0^t S(t-s)\xi_n(s)ds\in D(A)$ (see. step1) and a is non negative, (3.1) implies that there exists $\delta>0$, such that for all $0\leq h\leq\delta$ we have

$$\|\frac{S(h)-I}{(1*a)(h)}\int_0^t S(t-s)\xi_n(s)ds - A\int_0^t S(t-s)\xi_n(s)ds\|<\epsilon,$$

equivalently,

$$\|\int_0^t S(t-s)\frac{S(h)-I}{(1*a)(h)}\xi_n(s)ds - A\int_0^t S(t-s)\xi_n(s)ds\|<\epsilon.$$

Using that $\xi_n(\cdot)\in\widetilde{F}^1(A)$ and the boundedness of $(S(t))_{t\geq 0}$ we obtain

$$\begin{aligned}\|A\int_0^t S(t-s)\xi_n(s)ds\| &\leq \|\frac{S(h)-I}{(1*a)(h)}\int_0^t S(t-s)\xi_n(s)ds - A\int_0^t S(t-s)\xi_n(s)ds\|\\ &\quad+\int_0^t \|S(t-s)\|\sup_{0<h\leq 1}\|\frac{S(h)-I}{(1*a)(h)}\xi_n(s)\|ds\\ &\leq \epsilon + M\int_0^t \|\xi_n(s)\|_{\widetilde{F}^1(A)}ds,\end{aligned}$$

for all $\epsilon>0$, which implies that

$$\|A\int_0^t S(t-s)\xi_n(s)ds\|\leq M\int_0^t \|\xi_n(s)\|_{\widetilde{F}^1(A)}ds. \tag{4.1}$$

Now let $x_n:=\int_0^t S(t-s)\xi_n(s)ds$, then by step 1 we have $x_n\in D(A)$ and by step 2, $x_n\to x:=\int_0^t S(t-s)\xi(s)ds$ as $n\to\infty$ for all $t>0$. Moreover by (4.1) we have

$$\|Ax_m - Ax_n\| = \|A\int_0^t S(t-s)[\xi_m(s)-\xi_n(s)]ds\|$$

$$\le M \int_0^t \|\xi_m(s) - \xi_n(s)\|_{\widetilde{F}^1(A)} ds \to 0,$$

as $m, n \to \infty$. This proves that the sequence $(Ax_n)_n$ is Cauchy, and hence $(Ax_n)_n$ converges in X, say $Ax_n \to y \in X$.

Since A is closed, we conclude that $x \in D(A)$ proving the step 3. Moreover, from (4.1) we deduce that

$$\|A \int_0^t S(t-s)\xi(s)ds\| \le M \int_0^t \|\xi(s)\|_{\widetilde{F}^1(A)} ds$$

for all $t > 0$ which completes the proof. □

5. Sufficient and necessary conditions for admissibility

In this section we go back to the admissibility. We give sufficient and necessary conditions in terms of the Favard classes introduced in the above section for the L^p-admissibility of control operators for Volterra systems of the form

$$\begin{gathered} x(t) = x_0 + \int_0^t a(t-s)Ax(s)ds + \int_0^t Bu(s)ds, \quad t \ge 0 \\ x(0) = x_0 \in X \end{gathered} \tag{5.1}$$

Here A is a closed densely defined operator on a Banach space and U is another Banach space. It is further assumed that the uncontrolled system (i.e. (5.1) with $B = 0$)

$$x(t) = x_0 + \int_0^t a(t-s)Ax(s)ds, \quad t \ge 0, \tag{5.2}$$

admits a resolvent family $(S(t))_{t\ge0}$.

Since the resolvent of (5.2) commutes with the operator A, then it can be easily seen that the restriction $(S_1(t))_{t\ge0}$ to X_1 of $(S(t))_{t\ge0}$, the solution of (5.2), is strongly continuous. Moreover, if $\rho(A) \ne \emptyset$ (in particular if $(S(t))_{t\ge0}$ is exponentially bounded; see. Proposition 2.5) $(S_1(t))_{t\ge0}$ solves (5.2) for each $x_0 \in X$ and A_1 replacing A . Likewise, $S(t)$ has a unique bounded extension to X_{-1} for each $t \ge 0$ and $t \longmapsto S_{-1}(t)$ is also strongly continuous, and it solves (5.2) in X_{-1} with A_{-1} replacing A.

If $\rho(A) \ne \emptyset$ and $B \in \mathcal{L}(U, X_{-1})$, then the mild solution of (5.1) is formally given by the variation of constant formula

$$x(t) = S(t)x_0 + \int_0^t S_{-1}(t-s)Bu(s)ds, \tag{5.3}$$

which is actually the classical solution if $B \in \mathcal{L}(U, X)$ and $x_0 \in D(A)$ and u sufficiently smooth. In general however, B is not a bounded operator from U into X and so an additional assumption on B will be needed to ensure that $x(t) \in X$ for every $x_0 \in X$ and every $u \in L^p([0,\infty[; U)$ or $L^p_{\rm loc}([0,\infty[; U)$.

In the same spirit of semigroups, the following are the most natural definitions of the L^p-admissibility for resolvent families.

Definition 5.1. Let $p \in [1, \infty[$ and $B \in \mathcal{L}(U, X_{-1})$ and assume that $(S(t))_{t\ge0}$ is exponentially bounded .

(i) B is called infinite-time L^p-admissible operator for $(S(t))_{t\ge0}$, if there exists a constant $M > 0$ such that

$$\|S_{-1} * Bu(t)\| \le M\|u\|_{L^p([0,\infty[,U)} \quad \text{for all } u \in L^p([0,\infty[, U) \text{ and } t > 0.$$

(ii) B is called finite-time L^p-admissible operator for $(S(t))_{t\geq 0}$ if there exists $t_0>0$ and a constant $M(t_0)>0$ such that:

$$\|S_{-1}*Bu(t_0)\| \leq M(t_0)\|u\|_{L^p([0,t_0],U)} \quad \text{for all } u\in L^p([0,t_0],U).$$

(iii) B is called uniformly finite-time L^p-admissible operator for $(S(t))_{t\geq 0}$ if for all $t>0$, there exists a constant $M(t)>0$ such that

$$\|S_{-1}*Bu(t)\| \leq M(t)\|u\|_{L^p([0,t],U)}$$

for all $u\in L^p([0,t],U)$ with $\limsup_{t\to 0^+} M(t)<\infty$.

For $p\in[1,\infty[$, we denote by $\mathcal{A}^p_\infty(U,X)$, $\mathcal{A}^p(U,X)$ and $\mathcal{A}^p_u(U,X)$, the space of the infinite-time, the finite-time and the uniformly finite-time L^p-admissible operators for $(S(t))_{t\geq 0}$, respectively.

Recall that the condition $\limsup_{t\to 0^+} M(t)<\infty$ is always satisfied for semigroups (see. [27]). We prove in Proposition 5.6 (i), that this the case; in particular if X is reflexive. Note that the definition of infinite-time L^p-admissible control operator for $(S(t))_{t\geq 0}$ was introduced in [14] when $p=2$ and implies the finite-time L^2-admissibility condition considered by [17]. Our definitions of finite-time and uniformly finite-time L^p-admissible control operator for $(S(t))_{t\geq 0}$ correspond to that of the semigroups, also imply that of [17] when $p=2$. Furthermore, it is well-known that:

(P1) the finite-time L^p-admissibility and the uniform finite-time L^p-admissibility are equivalent for semigroups and
(P2) the finite-time L^p-admissibility and the infinite-time L^p-admissibility are equivalent for exponentially stable semigroups (i.e. $a(t)=1$) for all $p\in[1,\infty[$.

One question that remains open to our knowledge, is whether for Volterra systems, these problems (i.e. (P1)–(P2)) are still true for resolvent families. In the end of this section we give a partial response to these problems when $p=1$.

Claim 5.2. *Let* (5.2) *admit an exponentially stable resolvent family* $(S(t))_{t\geq 0}$ *and* $p\in[1,\infty[$. *The following is a necessary condition for infinite-time L^p-admissibility of control operator* $B\in\mathcal{L}(U,X_{-1})$*: there exists* $L_p>0$ *such that*

$$\|\frac{1}{\lambda\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}I-A_{-1})^{-1}B\|_{\mathcal{L}(U,X)} < \frac{L_p}{(Re\lambda)^{1/p}}, \tag{5.4}$$

for all $Re\lambda>0$.

Proof. Thanks to Proposition 2.5, the Laplace-transform of $S_{-1}(\cdot)$ is well-defined; similarly it is given by

$$\widehat{S_{-1}}(\lambda)=\frac{1}{\lambda\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}I-A_{-1})^{-1}=: H_{-1}(\lambda)$$

for all $\mathrm{Re}(\lambda)>0$. Let $B\in\mathcal{A}^p_\infty(U,X)$ and take $v\in U$ and $\lambda\in\mathbb{C}$, such that $\mathrm{Re}(\lambda)>0$. The infinite-time L^p-admissibility of B guarantees (see. [10, Remark 2.2]) that the operator $\mathcal{B}_\infty: L^p_c(\mathbb{R}^+,U)\to X$ given by $\mathcal{B}_\infty u:=\int_0^\infty S_{-1}(t)Bu(t)dt$ possesses a unique extension to a linear bounded operator from $L^p(\mathbb{R}^+,U)$ to X where $L^p_c(\mathbb{R}^+,U)$ denotes the space of functions in $L^p(\mathbb{R}^+,U)$ with compact support. Since $(S(t))_{t\geq 0}$ is exponentially stable, then $\mathcal{B}_\infty u=\int_0^\infty S_{-1}(t)Bu(t)dt$ for every

$u \in L^p(\mathbb{R}^+, U)$. Substituting $u(t) = ve^{-\lambda t}$ where $v \in U$, we deduce that there exists $M > 0$ such that

$$\begin{aligned}\|H_{-1}(\lambda)Bv\| &= \|\int_0^\infty S_{-1}(t)Bve^{-\lambda t}dt\| \\ &\leq M\|ve^{-\lambda \cdot}\|_{L^p([0,\infty[,U)} = \frac{M\|v\|_U}{p^{1/p}(Re\lambda)^{1/p}}\,.\end{aligned}$$

Hence

$$\|H_{-1}(\lambda)B\|_{\mathcal{L}(U,X)} \leq \frac{L_p}{(\operatorname{Re}\lambda)^{1/p}}\,,$$

for some constant $L_p > 0$ depending only on p. □

In a similar way we define the extrapolated Favard spaces of A_{-1} denoted by $F^\alpha(A_{-1})$ and $\widetilde{F}^1(A_{-1})$. The following results give an extension of [22, Proposition 15] (i.e. $a(t) = 1$).

Theorem 5.3. *Let* (5.2) *admit a bounded resolvent family* $(S(t))_{t\geq 0}$ *on* X *for* ω^+-*exponentially bounded* $a \in L^p_{\rm loc}(\mathbb{R}^+)$ *with* $p \geq 1$. *Then, we have the following assertions.*

- (i) *If* $\omega_0(S) < 0$ *then* $\mathcal{A}^p_\infty(U,X) \subset \mathcal{L}(U, F^{1/p}(A_{-1}))$.
- (ii) *If a is non negative satisfying* (H1), *then* $\mathcal{A}^p_u(U,X) \subset \mathcal{L}(U, \widetilde{F}^{1/p}(A_{-1}))$.
- (iii) *If a is a creep function with* $a(0^+) > 0$, *then* $\mathcal{L}(U, \widetilde{F}^1(A_{-1})) \subset \mathcal{A}^1_\infty(U,X) \subset \mathcal{A}^p_u(U,X)$.

Proof. Without loss of generality, we may assume that $0 \in \rho(A)$. See [7].

(i) Let $B \in \mathcal{A}^p_\infty(U,X)$ and let $u_0 \in U$ fixed. Thanks to the Claim 5.2, there exists $L_p > 0$ such that

$$\|\frac{1}{\lambda\widehat{a}(\lambda)}(\frac{1}{\widehat{a}(\lambda)}I - A_{-1})^{-1}Bu_0\| \leq \frac{L_p\|u_0\|}{\lambda^{1/p}} \quad \lambda > 0.$$

Equivalently,

$$\|\lambda^{\frac{1}{p}-1}\frac{1}{\widehat{a}(\lambda)}A_{-1}(\frac{1}{\widehat{a}(\lambda)}I - A_{-1})^{-1}Bu_0\|_{-1} \leq L_p\|u_0\|,$$

for all $\lambda > 0$ and for some $L_p > 0$. Hence

$$\sup_{\lambda>\omega}\|\lambda^{\frac{1}{p}-1}\frac{1}{\widehat{a}(\lambda)}A_{-1}(\frac{1}{\widehat{a}(\lambda)}I - A_{-1})^{-1}Bu_0\|_{-1} < L_p\|u_0\|,$$

which implies that $Bu_0 \in F^{1/p}(A_{-1})$, and by closed graph theorem we deduce that $B \in \mathcal{L}(U, F^{1/p}(A_{-1}))$.

(ii) Let $B \in \mathcal{A}^p_u(U,X)$ and $u_0 \in U$, then $b := Bu_0$ is uniformly finite-time L^p-admissible vector for $(S(t))_{t\geq 0}$. By (H1) and (S3) for $(S_{-1}(t))_{t\geq 0}$ for all $0 < t \leq 1$, we have

$$\frac{\|S_{-1}(t)b - b\|_{-1}}{((1*a)(t))^{1/p}} = \frac{\|A_{-1}\int_0^t a(t-s)S_{-1}(s)bds\|_{-1}}{((1*a)(t))^{1/p}} = \frac{\|\int_0^t S_{-1}(t-s)ba(s)ds\|}{(\int_0^t a(s)ds)^{1/p}}.$$

With $u(t) := a(t)$, the uniform finite-time L^p-admissibility of B and (H1) imply that

$$\frac{\|S_{-1}(t)b - b\|_{-1}}{((1*a)(t))^{1/p}} \leq \frac{M(t)\|a\|_{L^p([0,t])}}{(\int_0^t a(s)ds)^{1/p}} \leq \frac{M(t)}{\epsilon_a^{1/p}} \leq K$$

for some constant $K > 0$ and all $0 < t \le t_a$ due to $\limsup_{t\to 0^+} M(t) < \infty$, which implies that $b \in \widetilde{F}^{1/p}(A_{-1})$ if $t_a \ge 1$. Now, if $t_a < 1$ we obtain once again that $b \in \widetilde{F}^{1/p}(A_{-1})$ due to

$$\sup_{t_a \le t \le 1} \frac{\|a\|_{L^p([0,t])}}{(1*a)(t)} \le \frac{\|a\|_{L^p([0,1])}}{(1*a)(t_a)}.$$

Hence $B \in \mathcal{L}(U, \widetilde{F}^{1/p}(A_{-1}))$ thanks to the closed graph theorem.

(iii) Let $B \in \mathcal{L}(U, \widetilde{F}^1(A_{-1}))$, then $Bu(\cdot) \in \mathcal{L}([0,\infty[, \widetilde{F}^1(A_{-1}))$ for all $u \in L^1([0,\infty[, U)$. Since a is creep with $a(0^+) > 0$ and (A_{-1}, a) is a generator of the resolvent family $(S_{-1}(t))_{t\ge 0}$, Lemma 4.10 implies that there exists $N > 0$ such that $\int_0^t S_{-1}(t-s)Bu(s)ds \in D(A_{-1}) = X$ for all $t > 0$ and we have

$$\|A_{-1}\int_0^t S_{-1}(t-s)Bu(s)ds\|_{-1} \le N\|Bu\|_{L^1([0,t],\widetilde{F}^1(A_{-1}))}.$$

Whence,

$$\|\int_0^t S_{-1}(t-s)Bu(s)ds\| \le N\|B\|_{\mathcal{L}(U,\widetilde{F}^1(A_{-1}))}\|u\|_{L^1([0,t],U)} \le L\|u\|_{L^1([0,\infty[,U)},$$

for some $L > 0$ and all $u \in L^1([0,\infty[, U)$ which implies that $B \in \mathcal{A}^1_\infty(U,X)$. The proof of the inclusion $\mathcal{A}^1_\infty(U,X) \subset \mathcal{A}^p_u(U,X)$ is immediate. $\square$

Theorem 5.3 together with Proposition 4.5 give us the following corollary that is well-known for semigroups (see. [22] for (i) when $p = 1$ and [29, 26] for (ii) when $p \ge 1$).

Corollary 5.4. *Let* (5.2) *admit a bounded resolvent family for ω^+-exponentially creep function a with $a(0^+) > 0$. Then, we have the following assertions.*

(i) $\mathcal{A}^1_u(U,X) = \mathcal{L}(U, F^1(A_{-1})) = \mathcal{L}(U, \widetilde{F}^1(A_{-1}))$.
(ii) *If $\omega_0(S) < 0$, then $\mathcal{A}^1_u(U,X) = \mathcal{A}^1_\infty(U,X)$.*

Remark 5.5. Let (5.2) admit an exponentially bounded resolvent family $(S(t))_{t\ge 0}$ for ω-exponentially function $a \in L^1_{\rm loc}(\mathbb{R}^+)$ satisfying (H1). Let us consider the following "adjoint" Volterra equation

$$z(t) = z_0 + \int_0^t a(t-s)A^*z(s)ds, \quad z_0 \in X^*. \tag{5.5}$$

where A^* is the adjoint operator of A . Then (5.5) admits a resolvent family, denoted by $(\widetilde{S}(t))_{t\ge 0}$ if and only if $D(A^*)$ is densely defined. If this is the case we have in addition $\widetilde{S}(t) = S^*(t)$ for all $t \ge 0$ where $S^*(t)$ is the adjoint of $S(t)$.

Proof. Since $(S(t))_{t\ge 0}$ is exponentially bounded resolvent family, there exist $M > 0$ and $\omega \in \mathbb{R}^+$ such that $\|S(t)\| \le Me^{\omega t}$, $t \ge 0$. Then thanks to Proposition 2.5, we have:

(i) $\hat{a}(\lambda) \neq 0$ and $\frac{1}{\hat{a}(\lambda)} \in \rho(A)$ for all $\lambda > \omega$;
(ii) $H(\lambda) := \frac{1}{\lambda\hat{a}(\lambda)}(\frac{1}{\hat{a}(\lambda)}I - A)^{-1}$, the resolvent associated with $(S(t))_{t\ge 0}$ satisfies

$$\|H^{(n)}(\lambda)\| \le Mn!(\lambda-\omega)^{-(n+1)} \quad \text{for all } \lambda > \omega \text{ and } n \in \mathbb{N}.$$

This implies

$$\|(H^*)^{(n)}(\lambda)\| = \|H^{(n)}(\lambda)\| \leq (\lambda-\omega)^{-(n+1)} \quad \text{for all } \lambda > \omega \text{ and } n \in \mathbb{N}.$$

If (5.5) admits a resolvent family $(\widetilde{S}(t))_{t\geq 0}$, then by [16, Proposition 2.6], $D(A^*)$ is densely defined. Using Proposition 2.5 once again, $(\widetilde{S}(t))_{t\geq 0}$ becomes exponentially bounded and we have

$$\widehat{\widetilde{S}}(\lambda) = H^*(\lambda) = \widehat{S^*}(\lambda) \quad \text{for all } \lambda > \omega.$$

Since the Laplace transform is one-to-one, and that $\widetilde{S}(t)$ and $S^*(t)$ are continuous, we obtain $\widetilde{S}(t) = S^*(t)$ for every $t \geq 0$. Conversely, assume that $D(A^*)$ is densely defined then the above argument implies that $(S^*(t))_{t\geq 0}$ is the resolvent family of (5.5) which completes the proof. □

Note that a partial result was obtained in [14, Theorem 3.1] when A generates a $\mathcal{C}_0$-semigroup on reflexive Banach space X.

It has been proved in [24, p. 46] (resp. Remark 5.5), that if both A and A^* are densely defined (e.g. if A is densely defined and X is reflexive), then

$$(X_1)^* = (X^*)_{-1}.$$

(resp. $(S^*(t))_{t\geq 0}$ is exactly the resolvent family associated with the adjoint Volterra equation (5.5)). As a first consequence, if $C \in \mathcal{L}(X_1, Y)$; where Y is another Banach space (of observation), is an observation operator for the Volterra equation (5.2), then its adjoint $C^* \in \mathcal{L}(Y^*, (X^*)_{-1})$ becomes a control operator for Volterra equation (5.5). Likewise, it has been proved in [24, p. 50] that if in addition A is densely defined generalized Hille-Yosida operator then

$$(X_{-1})^* = (X^*)_1,$$

As a second consequence, if $B \in \mathcal{L}(U, X_{-1})$ is a control operator for the Volterra equation (5.2), then its adjoint $B^* \in \mathcal{L}((X^*)_1, U^*)$ becomes an observation operator for Volterra equation (5.5). Finally, as for the semigroups case (see. [28, Theorem 6.9]); if both A and A^* are densely defined and A is a generalized Hille-Yosida operator, then it is easy to see that there is a natural duality theorem between admissibility of the control operators and admissibility of observation operators. That is B is a finite-time L^p-admissible (with $p \in [1, \infty[$) control operator for $(S(t))_{t\geq 0}$ if and only if B^* is a finite-time $L^{\overline{p}}$-admissible observation operator for $(S^*(t))_{t\geq 0}$ with $\frac{1}{p} + \frac{1}{\overline{p}} = 1$, i.e. there exists $t_0 > 0$ such that

$$\int_0^{t_0} \|B^* S^*(s) z\|_{U^*}^{\overline{p}} ds \leq N(t_0) \|z\|_{X^*}^{\overline{p}}, \quad z \in D(A^*),$$

and $N(t_0) > 0$. This duality has already been considered in [14, Section 4], when $p = 2$ and A generates a $\mathcal{C}_0$-semigroup on reflexive Banach space X. In this case, it is well-known that both A and A^* are densely defined and A is a generalized Hille-Yosida operator.

We now have the following interesting results that are well-known for semigroups.

Proposition 5.6. *Let* (5.2) *admit a bounded resolvent family* $(S(t))_{t\geq 0}$ *for* ω^+-*exponentially creep function* a *with* $a(0^+) > 0$. *If* (5.5) *admits a resolvent family (equivalently* $\overline{D(A^*)} = X^*$*), then we have the following assertions.*

(i) $\mathcal{A}_u^1(U, X) = \mathcal{A}^1(U, X)$.
(ii) *If* $\omega_0(S) < 0$, *then* $\mathcal{A}_u^1(U, X) = \mathcal{A}^1(U, X) = \mathcal{A}_\infty^1(U, X)$.

Proof. (i) $\mathcal{A}^1_u(U,X) \subset \mathcal{A}^1(U,X)$ is immediate and it remains only to show that $\mathcal{A}^1(U,X) \subset \mathcal{A}^1_u(U,X)$. Let $B \in \mathcal{A}^1(U,X)$, then there exists $t_0 > 0$ and $M(t_0) > 0$ such that

$$\| \int_0^{t_0} S_{-1}(t_0 - s)Bu(s)ds\| \le M(t_0)\|u\|_{L^1([0,t_0],U)}. \tag{5.6}$$

Since (5.2) has an exponentially bounded resolvent family, [25, Corollary 1.6] implies that A is a generalized Hille-Yosida operator. By Remark 5.5, $(S^*(t))_{t\ge 0}$ is the unique resolvent family for (5.5). Hence, by duality, (5.6) is equivalent to

$$\sup_{0<t\le t_0} \|B^*S^*(t)z\|_{U^*} \le M(t_0)\|z\|_{X^*}$$

(i.e. $\overline{p} = \infty$), which in turns implies that

$$\sup_{0<t\le \tau} \|B^*S^*(t)z\|_{U^*} \le N(\tau)\|z\|_{X^*}, \tag{5.7}$$

for all $z \in D(A^*)$ and $0 < \tau \le t_0$ with $N(\tau) \le M(t_0)$. Using once again the duality argument, we deduce that (5.7) yields

$$\| \int_0^{\tau} S_{-1}(\tau - s)Bu(s)ds\| \le M(t_0)\|u\|_{L^1([0,\tau],U)} \quad \text{for all } 0 < \tau \le t_0.$$

Without loss of generality, we assume that $0 \in \rho(A)$. Thanks to (S3) for $(S_{-1}(t))_{t\ge 0}$ and for all $b \in X_{-1}$, we have

$$\frac{\|S_{-1}(\tau)b - b\|_{-1}}{(1*a)(\tau)} = \frac{\|A_{-1}\int_0^\tau a(\tau - s)S_{-1}(s)bds\|_{-1}}{(1*a)(\tau)} = \frac{\|\int_0^\tau S_{-1}(\tau - s)ba(s)ds\|}{\int_0^\tau a(s)ds}.$$

Since a is non negative; (H1) is satisfied with $t_a = \infty$. Substituting $u(\tau) := a(\tau)$, the finite-time L^1- admissibility of B implies that

$$\frac{\|S_{-1}(\tau)b - b\|_{-1}}{(1*a)(\tau)} \le M(t_0),$$

for all $0 < \tau \le t_0$ which implies that $b \in \widetilde{F}^1(A_{-1})$ if $t_0 \ge 1$. Now, if $t_0 < 1$ we obtain once again that $b \in \widetilde{F}^1(A_{-1})$ due to

$$\sup_{t_0\le\tau\le 1} \frac{\|S_{-1}(t)b - b\|_{-1}}{(1*a)(t)} \le \frac{(M+1)\|b\|_{-1}}{(1*a)(t_0)},$$

where M is the bound of $(S(t))_{t\ge 0}$. Thus $B \in \mathcal{L}(U, \widetilde{F}^1(A_{-1}))$ according to the closed graph theorem. By virtue of Corollary 5.4 (i) we obtain $B \in \mathcal{A}^1_u(U,X)$. Assertion (ii) is directly obtained from (i) and Corollary 5.4 (i) and this ends the proof. □

Remark 5.7. We remark that Corollary 5.4 (i) was proved for the $\mathcal{C}_0$-semigroups (i.e. $a(t) = 1$) in [22, Corollary 17], and also implies that the analogue Weiss conjecture is true for this class of Volterra integral systems. Note that a partial answer to Weiss conjecture for $p = 2$ and for some class of Volterra systems was given in [15] when $a = 1 + 1 * k$ with $k \in W^{1,2}(\mathbb{R}^+)$ and that the semigroup generated by A is equivalent to a contraction semigroup on a Hilbert space X and U is finite-dimensional. Now, we can see that Corollary 5.4 (ii) and Proposition 5.6 (i) give an affirmative answer to (P2) and (P1) respectively for some Volterra systems when $p = 1$.

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