Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 169, pp. 1–26.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

HOMOGENIZATION OF SOME EVOLUTION PROBLEMS IN DOMAINS WITH SMALL HOLES

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Abstract. This article concerns the asymptotic behavior of the wave and heat equations in periodically perforated domains with small holes and Dirichlet conditions on the boundary of the holes. In the first part we extend to time-dependent functions the periodic unfolding method for domains with small holes introduced in [6]. Therein, the method was applied to the study of elliptic problems with oscillating coefficients in domains with small holes, recovering the homogenization result with a "strange term" originally obtained in [11] for the Laplacian. In the second part we obtain some homogenization results for the wave and heat equations with oscillating coefficients in domains with small holes. The results concerning the wave equation extend those obtained in [12] for the case where the elliptic part of the operator is the Laplacian.

1. Introduction

The aim of this work is the study of the asymptotic behavior as $\varepsilon \to 0$ of the wave and heat equations in a perforated domain with holes distributed periodically with period ε, and with a Dirichlet condition on the boundary of the holes. We consider here "small" holes, that is to say with size of the order of $\varepsilon\delta$ ($\varepsilon \to 0$, $\delta \to 0$). The case $\delta = 1$ corresponds to the classical case of homogenization where the size of the holes and of the period is of the same order. We will use for the proofs an adaptation to the case of time dependent equations of the periodic unfolding method for small holes from Cioranescu, Damlamian, Griso and Onofrei [7].

The periodic unfolding method for the classical homogenization was introduced in Cioranescu, Damlamian and Griso [4] for fixed domains (see [5] for detailed proofs) and extended to perforated domains in [9] (see Cioranescu, Damlamian, Donato, Griso and Zaki [7] for more general situations). The method was applied in particular, for the classical homogenization of the wave and heat equations in periodically perforated domains by Gaveau [17] and more recently, by Donato and Yang [15] and [16].

The asymptotic behavior of the homogeneous Dirichlet problem for the Poisson equation in perforated domains with small holes of size ε^{α}, $\alpha > 0$, was studied by Cioranescu and Murat in [11]. They showed that for each dimension N of the space, the size $\varepsilon^{N/N-2}$ is "critical" in the sense that in the limit problem appears an additional zero order term (called in [11] "strange term") which is related to the

2010 *Mathematics Subject Classification.* 35B27, 35L20, 35K20.
Key words and phrases. Periodic unfolding method; homogenization in perforated domains; small holes; wave equation; heat equation.

Submitted March 2, 2016. Published July 4, 2016.

capacity of the set of holes as $\varepsilon \to 0$. There were afterward many works treating the same geometrical framework with various conditions on the boundary of the holes. Let us list a few of them. The case of Stokes equations was studied by Allaire in [1], the Poisson equation with non homogeneous Neumann conditions was treated by Conca and Donato [14] where it was shown that the contribution of the holes of size of order of $\varepsilon^{N/N-1}$, is reflected by an extra term in the right hand side of the limit equation. The case of mixed boundary conditions was studied by Cardone, D'Apice and De Maio in [3]. As concerning the parabolic case, we refer to Gontcharenko [18] where the homogenization result is obtained via the convergence of some cost-functionals. Homogenization and corrector results for the wave equation have been proved by Cioranescu, et al. [12].

In all these papers, the elliptic part of the operator is the Laplacian. For the asymptotic study, standard variational homogenization methods, as for instance Tartar's oscillating test functions method ([25]), are used (see also [2, 13, 23]). They need to introduce extension operators (since the domains are changing with ε) and to construct test functions, specific for each situation.

As mentioned before, in the paper we present here, we will use the periodic unfolding method. On one hand, we take the advantage of the simplicity of this method when applied to perforated domains as can be seen in [9] or [7]. Indeed, the periodic unfolding, being a fixed-domain method, no extension operator is needed. On the other hand, the method does not use any construction of special test functions and so, one can treat general second order operators with highly oscillating (in ε) coefficients, which was not the case in the papers cited above.

For the case of small holes for the Laplace equation and homogeneous Dirichlet boundary condition, first applications of the unfolding method have been done in Cioranescu, et al. [6], Onofrei [20], and Zaki [26]. Then the same operator was used in the framework of [14], with small holes of size $\varepsilon^{N/N-1}$ and non homogeneous Neumann conditions, in Ould Hammouda [21] and in Cioranescu and Ould Hammouda [10] for mixed boundary conditions.

In this work we first extend the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ introduced in [6] to time-dependent functions and study in details its related properties. In the second part, we apply the periodic unfolding method to obtain some homogenization results for the wave and heat equations with oscillating coefficients in domains with small holes.

We present here the proofs for the wave equation while for the heat equation we only state the problem together with the main convergence results. We skip the proofs for this case, since they follow step by step the outlines of those for the wave equation.

This paper is organized as follows: Sections 2-4 recalls the geometric framework for the perforated domain as well as some definitions and properties of the unfolding operators for fixed and perforated domains with small holes. In Section 5 we extend the operator $\mathcal{T}_{\varepsilon,\delta}$ given in [6] to time-dependent functions with detailed proofs of its properties. One can also find in this section the extension of the local average operator to time-dependent functions together with the related properties needed in this work. Section 6 is devoted to the main homogenization results for the wave and heat equations while Section 7 contains the proofs for the wave equation.

2. Notation and definitions

We recall here some notation and definitions as given in [4] for fixed domains. Let Ω be a bounded open set in $\mathbb{R}^n$, such that $|\partial\Omega| = 0$ and

$$Y = \Big] -\frac{\ell_i}{2}, \frac{\ell_i}{2}\Big[^N, \quad 0 < \ell_i,\ \ell_i \in \mathbb{R}^+ \text{ for } i = 1, \dots, N,$$

be the reference periodicity cell. Let us now introduce the sets

$$\begin{gathered} \widehat{\Omega}_\varepsilon = \text{interior}\big\{ \cup_{\xi\in\Xi_\varepsilon} \varepsilon\big(\xi + \overline{Y}\big)\big\}, \quad \Xi_\varepsilon = \big\{\xi \in \mathbb{Z}^n : \varepsilon(\xi + Y) \subset \Omega\big\}, \\ \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon. \end{gathered} \tag{2.1}$$

By construction, $\widehat{\Omega}_\varepsilon$ is the interior of the largest union of $\varepsilon(\xi + \overline{Y})$ cells fully contained in Ω, while Λ_ε is the subset of Ω containing the parts from the $\varepsilon(\xi + Y)$ cells intersecting the boundary $\partial\Omega$ (see Figure 1).

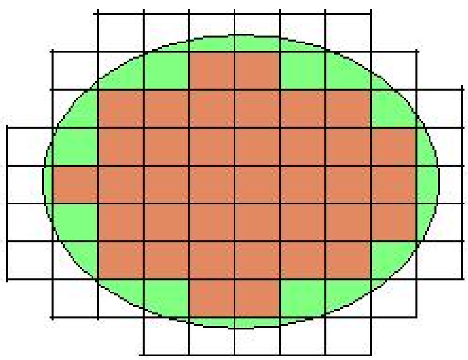

Figure 1. Sets $\widehat{\Omega}_\varepsilon$ (brown) and Λ_ε (light green)

As in [4], for every z in $\mathbb{R}^N$, we denote by $[z]_Y$ the unique integer combination of periods such that

$$\{z\}_Y = z - [z]_Y \in Y \tag{2.2}$$

which is depicted in Figure 2. Then, because of the periodicity and recalling (2.2), each $x \in \mathbb{R}^N$ can be uniquely written as

$$x = \varepsilon\big(\{\frac{x}{\varepsilon}\}_Y + [\frac{x}{\varepsilon}]_Y\big). \tag{2.3}$$

3. Time-dependent unfolding operator in fixed domains

Throughout this paper, T will be a given positive number. This section recalls the time-dependent unfolding operator for fixed domains as introduced in [17].

Definition 3.1 ([17]). Let $\varphi \in L^q(0,T; L^p(\Omega))$ where $p \in [1,+\infty[$ and $q \in [1,+\infty]$. The unfolding operator $\mathcal{T}_\varepsilon : L^q(0,T;L^p(\Omega)) \mapsto L^q(0,T;L^p(\Omega\times Y))$ is defined as

$$\mathcal{T}_\varepsilon(\varphi)(x,y,t) = \begin{cases} \varphi(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y, t) & \text{a.e. for } (x,y,t) \in \widehat{\Omega}_\varepsilon \times Y \times]0,T[, \\ 0 & \text{a.e. for } (x,y,t) \in \Lambda_\varepsilon \times Y \times]0,T[. \end{cases}$$

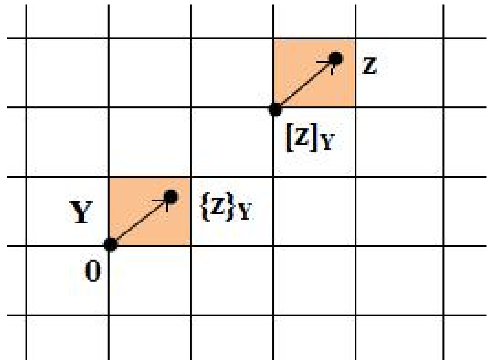

FIGURE 2. $\{z\}_Y$ and $[z]_Y$.

Some of the properties of this operator which were stated in [17] are listed below. For perforated domains with holes of the same size as the period and for detailed proofs (in Definition 3.1 obviously true for fixed domains), we refer to [15].

Remark 3.2. Notice that if in Definition 3.1 we take φ in $L^p(\Omega)$ independent of time, we recover the definition of the unfolding operator for fixed domains from [4].

Proposition 3.3 ([15, 17])**.** *Let $p \in [1, +\infty[$ and $q \in [1, +\infty]$. Suppose that u and v are functions in $L^q(0, T; L^p(\Omega))$. Then:*

(1) *$\mathcal{T}_\varepsilon$ is linear and continuous from $L^q(0, T; L^p(\Omega))$ to $L^q(0, T; L^p(\Omega \times Y))$;*
(2) *$\mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v)$;*
(3) *if $u \in L^q(0, T; W^{1,p}(\Omega))$ then $\mathcal{T}_\varepsilon(u) \in L^q(0, T; L^p(\Omega; W^{1,p}(Y)))$ and*
$$\nabla_y(\mathcal{T}_\varepsilon(u)) = \varepsilon \mathcal{T}_\varepsilon(\nabla u) \quad \text{in } \Omega \times Y \times]0, T[\,;$$
(4) *for almost every $t \in]0, T[$,*
$$\begin{aligned} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(u)(x, y, t)\, dx\, dy\, dt &= \int_\Omega u(x, t)\, dx\, dt - \int_{\Lambda_\varepsilon} u(x, t)\, dx\, dt \\ &= \int_{\widehat{\Omega}_\varepsilon} u(x, t)\, dx\, dt. \end{aligned}$$

Proposition 3.4 ([15, 17])**.** *Let $p, q \in [1, +\infty[$. Suppose that $\phi \in L^q(0, T; L^p(\Omega))$ and $\{\phi_\varepsilon\}$ is a sequence in $L^q(0, T; L^p(\Omega))$.*

(1) *$\mathcal{T}_\varepsilon(\phi) \to \phi$ strongly in $L^q(0, T; L^p(\Omega \times Y))$.*
(2) *If $\phi_\varepsilon \to \phi$ strongly in $L^q(0, T; L^p(\Omega))$, then $\mathcal{T}_\varepsilon(\phi_\varepsilon) \to \phi$ strongly in the space $L^q(0, T; L^p(\Omega \times Y))$.*

Proposition 3.5 ([15, 17])**.** *Let $p \in]1, +\infty[$ and $\{\varphi_\varepsilon\}$ be a sequence in the space $L^\infty(0, T; W_0^{1,p}(\Omega))$ such that*
$$\|\nabla \varphi_\varepsilon\|_{L^\infty(0,T;L^p(\Omega))} \le C.$$
Then there exist $\varphi \in L^\infty(0, T; W_0^{1,p}(\Omega))$ and $\widehat{\varphi} \in L^\infty(0, T; L^p(\Omega; W^{1,p}_{\rm per}(Y)))$ such that up to a subsequence,

(i) *$T_\varepsilon(\varphi_\varepsilon) \rightharpoonup \varphi$ weakly* in $L^\infty(0, T; L^p(\Omega; W^{1,p}(Y)))$,*
(ii) *$T_\varepsilon(\nabla \varphi_\varepsilon) \rightharpoonup \nabla_x \varphi + \nabla_y \widehat{\varphi}$ weakly* in $L^\infty(0, T; L^p(\Omega \times Y))$.*

We end this section by recalling the definition of the mean value operator $\mathcal{M}_Y$ and that of the local average operator $\mathcal{M}_Y^\varepsilon$ and give some of their properties that will be useful in the sequel.

Definition 3.6. Let $p \in [1, +\infty[$ and $q \in [1, +\infty]$. The mean value operator $\mathcal{M}_Y : L^q(0, T; L^p(\Omega \times Y)) \longmapsto L^q(0, T; L^p(\Omega))$ is defined by

$$\mathcal{M}_Y(u)(x, t) = \frac{1}{|Y|} \int_Y u(x, y, t)\, dy,$$

for every $u \in L^q(0, T; L^p(\Omega \times Y))$.

Definition 3.7. Let $p \in [1, +\infty[$ and $q \in [1, +\infty]$. The local average operator $\mathcal{M}_Y^\varepsilon : L^q(0, T; L^p(\Omega)) \longmapsto L^q(0, T; L^p(\Omega))$ is defined by

$$\mathcal{M}_Y^\varepsilon(\varphi)(x, t) = \frac{1}{|Y|} \int_Y \mathcal{T}_\varepsilon(\varphi)(x, y, t)\, dy,$$

for any $\varphi \in L^q(0, T; L^p(\Omega))$.

Remark 3.8. In connection with Remark 3.2, some of the properties of $\mathcal{T}_\varepsilon$ (in the case of dependence on time) can be derived directly for those of the unfolding operator for fixed domains from [4] with the time t as a mere parameter.

As a consequence, we have the following result.

Proposition 3.9. *Let $p \in [1, \infty[$ and $q \in [1, \infty]$.*

(1) *For $\varphi \in L^q(0, T; L^p(\Omega))$, one has*
$$\mathcal{T}_\varepsilon(\mathcal{M}_Y^\varepsilon(\varphi))(x, y, t) = \mathcal{M}_Y(\mathcal{T}_\varepsilon(\varphi))(x, t) = \mathcal{M}_Y^\varepsilon(\varphi)(x, t) \quad \text{in } \Omega \times]0, T[.$$

(2) *Let $\{w_\varepsilon\}$ be a sequence in $L^q(0, T; L^p(\Omega))$ such that*
$$w_\varepsilon \to w \quad \text{strongly in } L^q(0, T; L^p(\Omega)).$$
Then
$$\mathcal{M}_Y^\varepsilon(w_\varepsilon) \to \mathcal{M}_Y(w) = w \quad \text{strongly in } L^q(0, T; L^p(\Omega)).$$

(3) *For any $\varphi \in L^q(0, T; L^p(\Omega))$,*
$$\|\mathcal{M}_Y^\varepsilon(\varphi)\|_{L^q(0,T;L^p(\Omega))} \leq |Y|^{\frac{1-p}{p}} \|\varphi\|_{L^q(0,T;L^p(\Omega))}.$$

Proof. Property 1 corresponds to [4, Remarks 2.23 and 2.24]. For the reader's convenience, let us sketch the proof. One has successively, by using Definitions 3.1, 3.6 and 3.7,

$$\begin{aligned}\mathcal{T}_\varepsilon(\mathcal{M}_Y^\varepsilon(\varphi))(x, y, t) &= \mathcal{M}_Y^\varepsilon(\varphi)\Big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y, t\Big) = \frac{1}{|Y|} \int_Y \mathcal{T}_\varepsilon(\varphi)(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y, y, t)\, dy \\ &= \frac{1}{|Y|} \int_Y \mathcal{T}_\varepsilon(\varphi)(x, y, t)\, dy = \mathcal{M}_Y(\mathcal{T}_\varepsilon(\varphi))(x, t) = \mathcal{M}_Y^\varepsilon(\varphi)(x, t),\end{aligned}$$

for a.e. (x, t) in $\Omega \times (0, T)$.

Property 2 (corresponding to [4, Proposition 2.25 (iii)]) follows immediately from Proposition 3.4(2) and Definition 3.6.

Property 3 is a consequence of [4, Proposition 2.25(iii)] which shows that for all $w \in L^p(\Omega)$,

$$\|\mathcal{T}_\varepsilon(w)\|_{L^p(\Omega \times Y)} \leq |\ Y\ |^{1/p} \|w\|_{L^p(\Omega)}.$$

Then the result is straightforward by taking into account Remark 3.8 and Definition 3.7. $\square$

4. Unfolding operator in domains depending on two parameters

In this section we recall the definition and some of its properties of the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending on two mall parameters ε and δ, as introduced in [6].

Definition 4.1 ([6])**.** Let $p \in [1, +\infty[$. For $\phi \in L^p(\Omega)$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ is the function $\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbb{R}^N)$ defined by

$$T_{\varepsilon,\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(\phi)(x, \delta z) & \text{if } (x,z) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta} Y, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{T}_\varepsilon$ is the operator for fixed domains as introduced in [4] (see Remark 3.2).

To go further, let us introduce what is called a *perforated domain with small holes*, denoted here $\Omega^*_{\varepsilon,\delta}$. Let $B \subset\subset Y$ and denote $Y^*_\delta = Y \backslash \delta \overline{B}$. Then $\Omega^*_{\varepsilon,\delta}$ is defined as

$$\Omega^*_{\varepsilon,\delta} = \{x \in \Omega \text{ such that } \{\frac{x}{\varepsilon}\}_Y \in Y^*_\delta\},$$

where $\delta \to 0$ with ε. This definition means that $\Omega^*_{\varepsilon,\delta}$ is a domain ε-periodically perforated by holes $\varepsilon\delta B$, see Figure 3.

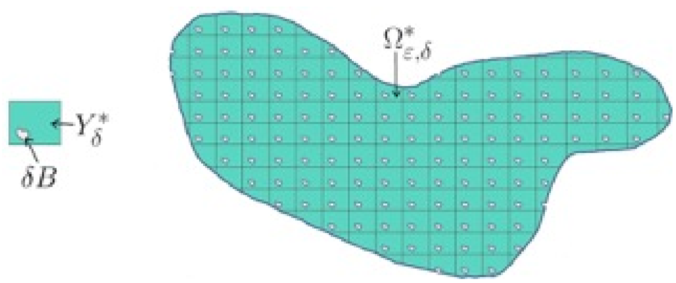

Figure 3. Perforated domain with small holes $\Omega^*_{\varepsilon,\delta}$.

Remark 4.2. As shown in [6], it turns out that the operator $\mathcal{T}_{\varepsilon,\delta}$ is well-adapted for domains with small holes when dealing with functions which vanish on the boundary of $\Omega^*_{\varepsilon,\delta}$. It is precisely the case we treat in this work. We will deal with functions belonging in particular, to $H^1_0(\Omega^*_{\varepsilon,\delta})$. The extensions of these functions by zero to the whole of Ω, belong to $H^1_0(\Omega)$. Consequently in the sequel, we will not distinguish the elements of $H^1_0(\Omega^*_{\varepsilon,\delta})$ and their extensions from $H^1_0(\Omega)$.

Proposition 4.3. [6]

(1) *For any* $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$.

(2) *For any* $u \in L^1(\Omega)$,

$$\delta^N \int_{\Omega \times \mathbb{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)| \, dx \, dz \le \int_\Omega |u| \, dx.$$

(3) *For any* $u \in L^2(\Omega)$,

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega \times \mathbb{R}^N)} \le \frac{1}{\delta^N} \|u\|^2_{L^2(\Omega)}.$$

(4) *For any* $u \in L^1(\Omega)$,
$$\Big|\int_\Omega u\,dx - \delta^N \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(u)\,dx\,dz\Big| \le \int_{\Lambda_\varepsilon} |u|\,dx.$$

(5) *Let* $u \in H^1(\Omega)$. *Then*
$$\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u)), \quad \text{in } \Omega\times\frac{1}{\delta}Y.$$

(6) *Suppose* $N \ge 3$ *and let* $\omega \subset \mathbb{R}^N$ *be open and bounded. The following estimates hold:*
$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))\|^2_{L^2(\Omega\times\frac{1}{\delta}Y)} \le \frac{\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$
$$\|\mathcal{T}_{\varepsilon,\delta}(u - M^\varepsilon_Y(u))\|^2_{L^2(\Omega;L^{2^*}(\mathbb{R}^N))} \le \frac{C\varepsilon^2}{\delta^{N-2}}\|\nabla u\|^2_{L^2(\Omega)},$$
$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^2_{L^2(\Omega\times\omega)} \le \frac{2C\varepsilon^2}{\delta^{N-2}}|\omega|^{2/N}\|\nabla u\|^2_{L^2(\Omega)} + 2|\omega|\|u\|^2_{L^2(\Omega)},$$
where C *is the Sobolev-Poincaré-Wirtinger constant for* $H^1(Y)$.

(7) *Suppose* $N \ge 3$ *and let* $\{w_{\varepsilon,\delta}\}$ *be a sequence in* $H^1(\Omega)$ *which is uniformly bounded as both* ε *and* δ *approach* 0. *Then there exists* W *in* $L^2(\Omega;L^{2^*}(\mathbb{R}^N))$ *with* $\nabla_z W$ *in* $L^2(\Omega\times\mathbb{R}^N)$ *such that, up to a subsequence,*
$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\Big(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - M^\varepsilon_Y(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}\Big) \rightharpoonup W \quad w\text{-}L^2(\Omega;L^{2^*}(\mathbb{R}^N)),$$
and
$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}))1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in } L^2(\Omega\times\mathbb{R}^N).$$
Furthermore, if
$$\limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty,$$
then one can choose the subsequence above and some $U \in L^2(\Omega;L^2_{\rm loc}(\mathbb{R}^N))$ *such that*
$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2(\Omega;L^2_{\rm loc}(\mathbb{R}^N)).$$

Definition 4.4. A sequence $\{v_{\varepsilon,\delta}\}$ in $L^1(\Omega)$ satisfies the unfolding criterion for integrals (u.c.i.) if
$$\int_\Omega v_{\varepsilon,\delta}\,dx - \delta^N\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})\,dx\,dz \to 0,$$
for every sequence $(\varepsilon,\delta)\to(0^+,0^+)$. This property is denoted
$$\int_\Omega v_{\varepsilon,\delta}\,dx \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \delta^N\int_{\Omega\times\mathbb{R}^N}\mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})\,dx\,dz.$$

Proposition 4.5 ([6](u.c.i.)). *If* $\{v_\varepsilon\}$ *is a sequence in* $L^1(\Omega)$ *satisfying*
$$\int_{\Lambda_\varepsilon}|u_\varepsilon|\,dx \to 0,$$
then it satisfies u.c.i..

Corollary 4.6 ([6]). *Let $\{u_\varepsilon\}$ be bounded in $L^2(\Omega)$ and $\{v_\varepsilon\}$ be bounded in $L^p(\Omega)$ with $p > 2$. Then $\{u_\varepsilon v_\varepsilon\}$ satisfies u.c.i..*

Remark 4.7. As observed in [6], for any $\psi \in \mathcal{D}(\Omega)$, one has
$$\|\mathcal{T}_{\varepsilon,\delta}(\psi) - \psi\|_{L^\infty(\hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y)} \to 0.$$

5. Time-dependent unfolding operator in domains with two parameters

In this section, we extend the operator $\mathcal{T}_{\varepsilon,\delta}$ defined in the previous section to time-dependent functions by adapting what is done in [15]. We start by defining the unfolding operator for time-dependent functions in the domain $\Omega^*_{\varepsilon,\delta} \times]0,T[$, depending on ε and δ.

In what follows, we have $(\varepsilon, \delta) \to (0,0)$ through any sequence and subsequence.

Definition 5.1. Let $p \in [1,+\infty[$ and $q \in [1,+\infty]$. Let $\varphi \in L^q(0,T;L^p(\Omega))$. The unfolding operator $\mathcal{T}_{\varepsilon,\delta} : L^q(0,T;L^p(\Omega)) \to L^q(0,T;L^p(\Omega \times \mathbb{R}^N))$ is defined as
$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t) = \begin{cases} \mathcal{T}_\varepsilon(\varphi)(x,\delta z,t) & \text{if } (x,z,t) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y \times]0,T[, \\ 0 & \text{otherwise.} \end{cases}$$
that is,
$$\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t) = \begin{cases} \varphi\big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon\delta z, t\big) & \text{if } (x,z,t) \in \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y \times]0,T[, \\ 0 & \text{otherwise.} \end{cases}$$

As mentioned above, for $\delta = 1$ we are in presence of the unfolding operator for fixed domains introduced in [4].

Remark 5.2. From now on, if a function does not depend on t, by $\mathcal{T}_{\varepsilon,\delta}(\varphi)$ we simply mean the operator introduced in Definition 4.1.

Being defined by means of the operator $\mathcal{T}_\varepsilon$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ inherits most of the general properties of it. In particular, the following proposition is straightforward:

Proposition 5.3. *Let $p \in [1,+\infty[$ and $q \in [1,+\infty]$.*

1. *$\mathcal{T}_{\varepsilon,\delta}$ is linear and continuous from $L^q(0,T;L^p(\Omega))$ to $L^q(0,T;L^p(\Omega \times \mathbb{R}^N))$.*
2. *$\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$ for every $v,w \in L^q(0,T;L^p(\Omega))$.*
3. *$\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi)) = \varepsilon\delta\mathcal{T}_{\varepsilon,\delta}(\nabla\varphi)$ in $\Omega \times \frac{1}{\delta}Y \times]0,T[$ for all $\varphi \in L^q(0,T;H^1(\Omega))$.*

Theorem 5.4. *Let $p \in [1,+\infty[$ and $q \in [1,+\infty]$.*

- *Let $\varphi \in L^q(0,T;L^p(\Omega))$.*
  (1)
  $$\begin{aligned} \frac{\delta^N}{|Y|}\int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)\,dx\,dz &= \int_{\widehat{\Omega}_\varepsilon} \varphi(x,t)\,dx \\ &= \int_\Omega \varphi(x,t)\,dx - \int_{\Lambda_\varepsilon} \varphi(x,t)\,dx \end{aligned}$$
  for a.e. $t \in]0,T[$.

(2) *The continuity of the operator $\mathcal{T}_{\varepsilon,\delta}$ from Proposition 5.3 reads as follows:*

$$\|\mathcal{T}_{\varepsilon,\delta}(\varphi)\|_{L^q(0,T;L^p(\Omega))} \leq \Big(\frac{|Y|}{\delta^N}\Big)^{1/p} \|\varphi\|_{L^q(0,T;L^p(\Omega))}. \tag{5.1}$$

- *Let $\varphi \in L^q(0,T;H^1(\Omega))$ and $N \geq 3$. Then, for a.e. $t \in]0,T[$,*

(3)

$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi))\|_{L^p(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon|Y|^{1/p}}{\delta^{\frac{N}{p}-1}}\|\nabla\varphi\|_{L^p(\Omega)}.$$

Proof. As a rule, all the properties above are proved by using the change of variable $z = (1/\delta)y$ and the fact that the integral $\int_{\widehat{\Omega}_\varepsilon}$ can be written as a sum on the cells $\varepsilon\xi + \varepsilon Y$ for $\xi \in \Xi_\varepsilon$ (see (2.1) for the definition of Ξ_ε).

(1) With this rule in mind, for every $\varphi \in L^q(0,T;L^p(\Omega))$ and recalling Definition 5.1, one has

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)\,dx\,dz &= \int_{\widehat{\Omega}_\varepsilon\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)\,dx\,dz \\
&= \sum_{\xi\in\Xi_\varepsilon}\int_{(\varepsilon\xi+\varepsilon Y)\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)\,dx\,dz \\
&= \sum_{\xi\in\Xi_\varepsilon}\int_{(\varepsilon\xi+\varepsilon Y)\times\frac{1}{\delta}Y} \varphi(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon\delta z, t)\,dx\,dz
\end{aligned} \tag{5.2}$$

for almost every $t \in]0,T[$. For each element of the last sum, we have successively,

$$\begin{aligned}
&\delta^N \int_{(\varepsilon\xi+\varepsilon Y)\times\frac{1}{\delta}Y} \varphi\big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon\delta z, t\big)\,dx\,dz \\
&= \delta^N|\varepsilon\xi+\varepsilon Y| \int_{\frac{1}{\delta}Y} \varphi\big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon\delta z, t\big)\,dz \\
&= \varepsilon^N|Y|\int_Y \varphi\big(\varepsilon[\frac{x}{\varepsilon}]_Y + \varepsilon y, t\big)\,dy = |Y|\int_{(\varepsilon\xi+\varepsilon Y)}\varphi(x,t)\,dx.
\end{aligned} \tag{5.3}$$

Using (2.1), the first property follows by summing up with respect to ξ in Ξ_ε.

(2) For the second property we proceed in the same way as for (5.3), to obtain

$$\int_{(\varepsilon\xi+\varepsilon Y)\times\mathbb{R}^N} \big|\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)\big|^p\,dxdz = \frac{|Y|}{\delta^N}\int_{(\varepsilon\xi+\varepsilon Y)} |\varphi(x,t)|^p\,dx.$$

Summing as above yields

$$\int_{\Omega\times\mathbb{R}^N} \big|\mathcal{T}_{\varepsilon,\delta}(\varphi)(x,z,t)\big|^p\,dxdz = \frac{|Y|}{\delta^N}\int_{\widehat{\Omega}_\varepsilon}|\varphi(x,t)|^p\,dx \leq \frac{|Y|}{\delta^N}\int_\Omega |\varphi(x,t)|^p\,dx.$$

Hence

$$\|\mathcal{T}_{\varepsilon,\delta}(\varphi)\|_{L^p(\Omega\times\mathbb{R}^N)} \leq \Big(\frac{|Y|}{\delta^N}\Big)^{1/p}\|\varphi\|_{L^p(\Omega)}, \tag{5.4}$$

which when integrated with respect to time gives (5.1).

(3) For $\varphi \in L^q(0,T;L^p(\Omega))$, from property 3 of Proposition 5.3 and (5.4),

$$\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(\varphi))\|_{L^p(\Omega\times\frac{1}{\delta}Y)} = \|\varepsilon\delta\mathcal{T}_{\varepsilon,\delta}(\nabla\varphi)\|_{L^p(\Omega\times\frac{1}{\delta}Y)} \leq \varepsilon\delta\Big(\frac{|Y|}{\delta^N}\Big)^{1/p}\|\nabla\varphi\|_{L^p(\Omega)},$$

for a.e. $t \in]0,T[$, which gives the desired result. $\square$

Regarding the integral formulas, one still has an unfolding criterion for integrals, which is very useful in homogenization problems.

Proposition 5.5. *Let $q \in [1, +\infty]$ and $\varphi_\varepsilon \in L^q(0,T;L^1(\Omega))$ satisfying*

$$\int_0^T \int_{\Lambda_\varepsilon} \varphi_\varepsilon \, dx \, dt \to 0, \tag{5.5}$$

then

$$\int_0^T \int_\Omega \varphi_\varepsilon \, dx \, dt \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega \times \mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon) \, dx \, dz \, dt.$$

The proof of the following proposition is essentially the same as that of [15, Proposition 2.6].

Proposition 5.6. *Let $p, q \in]1, +\infty]$. Let $\{\varphi_\varepsilon\}$ be a sequence in $L^q(0,T;L^p(\Omega))$ and $\{\psi_\varepsilon\}$ be a sequence in $L^{q'}(0,t;L^{p_0}(\Omega))$, such that*

$$\|\varphi_\varepsilon\|_{L^q(0,T;L^p(\Omega))} \leq C \quad \text{and} \quad \|\psi_\varepsilon\|_{L^{q'}(0,T;L^{p_0}(\Omega))} \leq C,$$

where $\frac{1}{p} + \frac{1}{p_0} < 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then,

$$\int_0^T \int_{\widehat{\Omega}_\varepsilon} \varphi_\varepsilon \psi_\varepsilon \, dx \, dt \overset{\mathcal{T}_{\varepsilon,\delta}}{\cong} \frac{\delta^N}{|Y|} \int_0^T \int_{\Omega \times \frac{1}{\delta} Y} \mathcal{T}_{\varepsilon,\delta}(\varphi_\varepsilon \psi_\varepsilon) \, dx \, dz dt.$$

The next two propositions extend to time-dependent functions some properties given in [6, Theorem 2.11].

Proposition 5.7. *Let $u \in L^q(0,T;H^1(\Omega))$. For $q \in [1, +\infty[$, one has the estimates*

$$\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}_Y^\varepsilon(u))\|_{L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N)))} \leq \frac{C\varepsilon |Y|^{1/p}}{\delta^{\frac{N}{p}-1}} \|\nabla u\|_{L^q(0,T;L^p(\Omega))}, \tag{5.6}$$

and for ω an open and bounded subset of $\mathbb{R}^N$,

$$\begin{aligned} &\|\mathcal{T}_{\varepsilon,\delta}(u)\|_{L^q(0,T;L^p(\Omega\times\omega))} \\ &\leq \frac{2C\varepsilon |Y|^{1/p}}{\delta^{\frac{N}{p}-1}} \|\nabla u\|_{L^q(0,T;L^p(\Omega))} + 2|\omega||Y|^{\frac{1-p}{p}} \|u\|_{L^q(0,T;L^p(\Omega))}, \end{aligned} \tag{5.7}$$

where C is the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$.

Proof. Let $u \in L^q(0,T;H^1(\Omega))$.

Step 1. Let us prove (5.6). By a change of variable, the linearity of the unfolding operator and using Proposition 3.9(1), we have for almost every $x \in \Omega$ and $t \in]0,T[$,

$$\begin{aligned} &\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}_Y^\varepsilon(u))(x,\cdot,t)\|_{L^{p^*}(\frac{1}{\delta}Y)} \\ &= \Big(\int_{\frac{1}{\delta}Y} |\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}_Y^\varepsilon(u))(x,z,t)|^{p^*} \, dz \Big)^{1/p^*} \\ &= \Big(\int_{\frac{1}{\delta}Y} |\mathcal{T}_{\varepsilon}(u - \mathcal{M}_Y^\varepsilon(u))(x,\delta z,t)|^{p^*} \, dz \Big)^{1/p^*} \\ &= \Big(\frac{1}{\delta^N} \int_{Y} |\mathcal{T}_{\varepsilon}(u - \mathcal{M}_Y^\varepsilon(u))(x,y,t)|^{p^*} \, dy \Big)^{1/p^*} \\ &= \frac{1}{\delta^{N/p^*}} \Big(\int_{Y} |(\mathcal{T}_{\varepsilon}(u) - \mathcal{M}_Y(\mathcal{T}_\varepsilon(u)))(x,y,t)|^{p^*} \, dy \Big)^{1/p^*} \end{aligned}$$

$$= \frac{1}{\delta^{N/p^*}} \|(\mathcal{T}_\varepsilon(u) - \mathcal{M}_Y(\mathcal{T}_\varepsilon(u)))(x,\cdot,t)\|_{L^{p^*}(Y)}.$$

On the other hand, using the Sobolev-Poincaré-Wirtinger inequality in $H^1(Y)$, Proposition 3.3(3), Proposition 5.3(3) and a change of variable, we obtain

$$\begin{aligned}
&\frac{1}{\delta^{N/p^*}} \|(\mathcal{T}_\varepsilon(u) - \mathcal{M}_Y(\mathcal{T}_\varepsilon(u)))(x,\cdot,t)\|_{L^{p^*}(Y)} \\
&\le \frac{C}{\delta^{N/p^*}} \|\nabla_y(\mathcal{T}_\varepsilon(u)(x,\cdot,t))\|_{L^p(Y)} \\
&= \frac{C}{\delta^{N/p^*}} \|\varepsilon \mathcal{T}_\varepsilon(\nabla(u))(x,\cdot,t)\|_{L^p(Y)} \\
&= \frac{C\varepsilon}{\delta^{N/p^*}} \Big(\int_Y |\mathcal{T}_\varepsilon(\nabla(u))(x,y,t)|^p\,dy\Big)^{1/p} \\
&= \frac{C\varepsilon}{\delta^{N/p^*}} \Big(\int_{\frac{1}{\delta}Y} |\mathcal{T}_\varepsilon(\nabla(u))(x,\delta z,t)|^p\,\delta^N dz\Big)^{1/p} \\
&= \frac{C\varepsilon}{\delta^{N/p^*}} \Big(\int_{\frac{1}{\delta}Y} |\mathcal{T}_{\varepsilon,\delta}(\nabla(u))(x,z,t)|^p\,\delta^N dz\Big)^{1/p} \\
&= \frac{C\varepsilon}{\delta^{N/p^*}} \Big(\int_{\frac{1}{\delta}Y} \big|\frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u)(x,z,t))\big|^p\,\delta^N dz\Big)^{1/p} \\
&= C\delta^{\frac{N}{p}-\frac{N}{p^*}-1} \|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u)(x,\cdot,t))\|_{L^p(\frac{1}{\delta}Y)} \\
&= C\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u)(x,\cdot,t))\|_{L^p(\frac{1}{\delta}Y)},
\end{aligned}$$

since $\frac{N}{p} - \frac{N}{p^*} - 1 = 0$, and where C is the Sobolev-Poincaré-Wirtinger constant for $H^1(Y)$. Thus,

$$\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))(x,\cdot,t)\|_{L^{p^*}(\frac{1}{\delta}Y)} \le C\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u)(x,\cdot,t))\|_{L^p(\frac{1}{\delta}Y)},$$

which implies

$$\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))(\cdot,\cdot,t)\|_{L^p(\Omega;L^{p^*}(\frac{1}{\delta}Y))} \le C\|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u)(\cdot,\cdot,t))\|_{L^p(\Omega\times\frac{1}{\delta}Y)},$$

for almost every $t \in]0,T[$. Taking the L^q-norm over $]0,T[$ gives

$$\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))\|_{L^q(0,T;L^p(\Omega;L^{p^*}(\frac{1}{\delta}Y)))} \le \|\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u))\|_{L^q(0,T;L^p(\Omega\times\frac{1}{\delta}Y))}.$$

This, together with Definition 5.1 and Theorem 5.4(5) yields (5.6) for a.e. $t \in]0,T[$. smallskip

Step 2. For estimate (5.7), we use Proposition 3.9(3) and note that

$$\begin{aligned}
|\mathcal{T}_{\varepsilon,\delta}(u)|^p &= |\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u)) + \mathcal{T}_{\varepsilon,\delta}(\mathcal{M}^\varepsilon_Y(u))|^p \\
&\le 2^p(|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))|^p + |\mathcal{T}_{\varepsilon,\delta}(\mathcal{M}^\varepsilon_Y(u))|^p) \\
&= 2^p(|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))|^p + |\mathcal{M}^\varepsilon_Y(u)|^p).
\end{aligned}$$

Thus, one has

$$\begin{aligned}
\|\mathcal{T}_{\varepsilon,\delta}(u)\|_{L^p(\Omega\times\omega)} &\le 2(\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))\|_{L^p(\Omega\times\omega)} + \|\mathcal{M}^\varepsilon_Y(u)\|_{L^p(\Omega\times\omega)}) \\
&= 2(\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))\|_{L^p(\Omega\times\omega)} + |\omega|\|\mathcal{M}^\varepsilon_Y(u)\|_{L^p(\Omega)}) \\
&\le 2(\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))\|_{L^p(\Omega;L^{p^*}(\omega))} + |\omega|\|\mathcal{M}^\varepsilon_Y(u)\|_{L^p(\Omega)}) \\
&\le 2(\|\mathcal{T}_{\varepsilon,\delta}(u - \mathcal{M}^\varepsilon_Y(u))\|_{L^p(\Omega;L^{p^*}(\mathbb{R}^N))} + |\omega|\|\mathcal{M}^\varepsilon_Y(u)\|_{L^p(\Omega)}).
\end{aligned}$$

In view of Proposition 3.9(3) and (5.6), taking the L^q-norm over $]0,T[$ yields inequality (5.7). ∎

Theorem 5.8. *Let $p \in [1,+\infty[$, $q \in [1,+\infty]$, $N \geq 3$, $\{w_{\varepsilon,\delta}\}$ be a sequence in $L^q(0,T;H^1(\Omega))$ which is uniformly bounded with respect to ε and δ as $(\varepsilon,\delta) \to (0,0)$. Then up to a subsequence, there exists W in $L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N)))$ with $\nabla_z W$ in $L^q(0,T;L^p(\Omega \times \mathbb{R}^N))$ such that*

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) - \mathcal{M}^\varepsilon_Y(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}) \rightharpoonup W \quad \text{weakly in } L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N))), \tag{5.8}$$

and

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}))1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z W \quad \text{weakly in }, L^q(0,T;L^p(\Omega \times \mathbb{R}^N)). \tag{5.9}$$

Furthermore, if

$$k^* = \limsup_{(\varepsilon,\delta)\to(0^+,0^+)} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon} < +\infty, \tag{5.10}$$

then one can choose the subsequence above and some $U \in L^q(0,T;L^p(\Omega;L^p_{\rm loc}(\mathbb{R}^N)))$ with

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^q(0,T;L^p(\Omega;L^p_{\rm loc}(\mathbb{R}^N))). \tag{5.11}$$

Proof. We follow the arguments from [6] and [20]. The existence of W in the space $L^q(0,T;L^p(\Omega;L^{p^*}(\mathbb{R}^N)))$ in (5.8) is a consequence of estimate (5.6).

Let us prove (5.9). From Theorem 5.4(5), we have

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\|\nabla_z\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\|_{L^q(0,T;L^p(\Omega\times\frac{1}{\delta}Y))} \leq |Y|^{\frac{1}{p}}\|\nabla w_{\varepsilon,\delta}\|_{L^q(0,T;L^p(\Omega))},$$

and thus, there exists $U \in L^q(0,T;L^p(\Omega \times \mathbb{R}^N))$ such that

$$\frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \rightharpoonup U, \quad \text{weakly in } L^q(0,T;L^p(\Omega \times \mathbb{R}^N)). \tag{5.12}$$

Let us show that $U = \nabla_z W$.

For $\varphi \in \mathcal{D}(\Omega \times \mathbb{R}^N \times]0,T[)$, in view of Definition 3.7 one has

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\mathbb{R}^N} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})\varphi\,dx\,dz\,dt \\
&= \int_0^T \int_{\Omega\times\mathbb{R}^N} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta} - \mathcal{M}^\varepsilon_Y(w_{\varepsilon,\delta})))\varphi\,dx\,dz\,dt \\
&= -\int_0^T \int_{\Omega\times\mathbb{R}^N} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta} - \mathcal{M}^\varepsilon_Y(w_{\varepsilon,\delta}))\nabla_z\varphi\,dx\,dz\,dt.
\end{aligned}$$

Thus, passing to the limit for any subsequences such that $(\varepsilon,\delta) \to (0,0)$ using (5.8) and (5.12) in this equation yields

$$\begin{aligned}
\int_0^T \int_{\Omega\times\mathbb{R}^N} U\varphi\,dx\,dz\,dt &= -\int_0^T \int_{\Omega\times\mathbb{R}^N} W\nabla_z\varphi\,dx\,dz\,dt \\
&= \int_0^T \int_{\Omega\times\mathbb{R}^N} \nabla_z W\varphi\,dx\,dz\,dt.
\end{aligned}$$

Therefore, $U = \nabla_z W$ and from (5.12), we have (5.9).

Finally, by using (5.7), convergence (5.11) follows from (5.10). $\square$

6. Statement of the main homogenization results

In this section, we suppose that $N \geq 3$ and that ε and $\delta = \delta(\varepsilon)$ are such that (5.10) holds, that is, there exists the following limit and is finite:

$$k^* \doteq \lim_{\varepsilon \to 0} \frac{\delta^{\frac{N}{2}-1}}{\varepsilon} < +\infty. \tag{6.1}$$

Remark 6.1. Often in the literature (see for instance [11, 14, 18, 24]), the size of the reference hole is denoted a_ε. Then (6.1) is equivalent to

$$(k^*)^{\frac{2}{N}} = \lim_{\varepsilon \to 0} \frac{a_\varepsilon^{\frac{N-2}{N}}}{\varepsilon}.$$

The case $k^* > 0$ concerns the situation where the reference hole has a critical size, giving rise to the "strange term" ([11]), in the homogenized problem. The noncritical case $k^* = 0$ does not present this phenomenon.

If one assumes that $\delta = a_0 \varepsilon^\alpha$, for some a_0 a positive constant, then, in order for (6.1) to be satisfied, a simple computation shows that necessarily, $\alpha = \frac{2}{N-2}$. This implies that the size a_ε of the holes in $\Omega^*_{\varepsilon,\delta}$ and k^* are

$$a_\varepsilon = a_0 \varepsilon^{\frac{N}{N-2}}, \quad k^* = a_0^{\frac{N-2}{N}}.$$

These are precisely the values from [11] leading to the presence of the "strange term" in the limit equation.

We also denote by $M(\alpha, \beta, \Omega)$ the set of $N \times N$ matrices $A = (a_{ij})_{1 \leq i,j \leq N}$ in $(L^\infty(\Omega))^{N\times N}$ such that

(i) $(A(x)\lambda, \lambda) \geq \alpha |\lambda|^2$,

(ii) $|A(x)\lambda| \leq \beta |\lambda|$,

for any $\lambda \in \mathbb{R}^N$ and almost everywhere on Ω, where $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$.

6.1. **Wave equation.**

We want to study the asymptotic behavior as $\varepsilon \to 0$, of the problem

$$\begin{gathered}
u''_{\varepsilon,\delta}(x,t) - \operatorname{div}(A^\varepsilon(x) \nabla u_{\varepsilon,\delta}(x,t)) = f_{\varepsilon,\delta}(x,t) \quad \text{in } \Omega^*_{\varepsilon,\delta} \times]0,T[, \\
u_{\varepsilon,\delta}(x,t) = 0 \quad \text{on } \partial\Omega^*_{\varepsilon,\delta} \times]0,T[, \\
u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x), \quad u'_{\varepsilon,\delta}(x,0) = u^1_{\varepsilon,\delta}(x) \quad \text{in } \Omega^*_{\varepsilon,\delta}.
\end{gathered} \tag{6.2}$$

We suppose that the data satisfy the following assumptions:

$$\begin{aligned}
&(i) \quad A^\varepsilon \in \mathcal{M}(\alpha, \beta, \Omega), \ A^\varepsilon \text{ symmetric}, \\
&(ii) \quad f_{\varepsilon,\delta} \in L^2(0,T; L^2(\Omega^*_{\varepsilon,\delta})), \\
&(iii) \quad u^0_{\varepsilon,\delta} \in H^1_0(\Omega^*_{\varepsilon,\delta}), \\
&(iv) \quad u^1_{\varepsilon,\delta} \in L^2(\Omega).
\end{aligned} \tag{6.3}$$

Moreover, we assume that

$$\begin{aligned}
&(i) \quad u^0_{\varepsilon,\delta} \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega), \\
&(ii) \quad u^1_{\varepsilon,\delta} \rightharpoonup u^1 \quad \text{weakly in } L^2(\Omega), \\
&(iii) \quad f_{\varepsilon,\delta} \rightharpoonup f \quad \text{weakly in } L^2(0,T; L^2(\Omega)).
\end{aligned} \tag{6.4}$$

The set

$$\mathcal{W}_{\varepsilon,\delta} = \{ v_{\varepsilon,\delta} \in L^2(0,T; H^1_0(\Omega^*_{\varepsilon,\delta})) : v'_{\varepsilon,\delta} \in L^2(0,T; L^2(\Omega^*_{\varepsilon,\delta})) \},$$

is equipped with the norm

$$\|v_{\varepsilon,\delta}\|_{\mathcal{W}_{\varepsilon,\delta}} = \|v_{\varepsilon,\delta}\|_{L^2(0,T;H_0^1(\Omega^*_{\varepsilon,\delta}))} + \|v'_{\varepsilon,\delta}\|_{L^2(0,T;L^2(\Omega^*_{\varepsilon,\delta}))}.$$

The variational formulation of problem (6.2) is: Find $u_{\varepsilon,\delta} \in \mathcal{W}_{\varepsilon,\delta}$ such that for all $v \in H_0^1(\Omega^*_{\varepsilon,\delta})$,

$$\begin{gathered}
\langle u''_{\varepsilon,\delta}(x,t), v(x)\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))',H_0^1(\Omega^*_{\varepsilon,\delta})} + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v(x)\,dx \\
= \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \\
u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x), \quad u'_{\varepsilon,\delta}(x,0) = u^1_{\varepsilon,\delta}(x) \quad \text{in } \Omega^*_{\varepsilon,\delta}.
\end{gathered} \tag{6.5}$$

Classical results [19, 8] provide for every fixed ε and δ the existence and uniqueness of a solution of problem (6.5) such that

$$u_{\varepsilon,\delta} \in \mathcal{C}^0([0,T]; H_0^1(\Omega^*_{\varepsilon,\delta})) \cap \mathcal{C}^1([0,T]; L^2(\Omega^*_{\varepsilon,\delta})),$$

and satisfies the estimate

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;H_0^1(\Omega^*_{\varepsilon,\delta}))} + \|u'_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,\delta}))} \le C, \tag{6.6}$$

where C is independent of ε and δ.

Remark 6.2. In the following, we identify functions in $H_0^1(\Omega^*_{\varepsilon,\delta})$ with their zero extension to $H_0^1(\Omega)$ so that we can write (6.6) as

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;H_0^1(\Omega))} + \|u'_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{6.7}$$

where C is independent of ε and δ.

We adapt here for the evolution problem some arguments introduced in [6]. Let us introduce the functional space

$$K_B = \{\Phi \in L^2(0,T; L^{2^*}(\mathbb{R}^N)) : \nabla\Phi \in L^2(0,T; L^2(\mathbb{R}^N)),\ \Phi \text{ is constant on } B\}. \tag{6.8}$$

We also need the following lemmas from [6] in order to pass to the limit in equation (6.5).

Lemma 6.3 ([6]). *Let $N \ge 3$. Then, for every $\delta_0 > 0$, the set*

$$\cup_{0<\delta<\delta_0}\{\phi \in H^1_{\rm per}(Y) : \phi = 0 \ \text{ on } \delta B\},$$

is dense in $H^1_{\rm per}(Y)$.

Lemma 6.4 ([6]). *Let $v \in \mathcal{D}(\mathbb{R}^N) \cap K_B$ (i.e., $v = v(B)$ is constant on B) and set*

$$w_{\varepsilon,\delta}(x) = v(B) - v\Big(\frac{1}{\delta}\{\frac{x}{\varepsilon}\}_Y\Big) \quad \text{for } x \in \mathbb{R}^N.$$

Then

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \quad \text{weakly in } H^1(\Omega). \tag{6.9}$$

Remark 6.5. (1) From the definition of $w_{\varepsilon,\delta}$ above, one has

$$\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})(x,z) = v(B) - v(z) \quad \text{in } \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y,$$

and consequently (see [6]),

$$\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = \frac{1}{\varepsilon\delta}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(w_{\varepsilon,\delta})) = -\frac{1}{\varepsilon\delta}\nabla_z v \quad \text{in } \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y. \tag{6.10}$$

(2) Let $\{w_{\varepsilon,\delta}\}$ be a sequence satisfying (6.9). We have,

$$\mathcal{T}_\varepsilon(w_{\varepsilon,\delta}) \to v(B) \quad \text{strongly in } L^2(\Omega \times Y). \tag{6.11}$$

Indeed, it was shown in [6] that $\{w_{\varepsilon,\delta}\}$ is bounded in $H^1(\Omega)$ so that together with (6.9) and Rellich compactness theorem, one has $w_{\varepsilon,\delta} \to v(B)$ strongly in $L^2(\Omega)$; that is,

$$\|w_{\varepsilon,\delta} - v(B)\|_{L^2(\Omega)} \to 0.$$

(see [6]) This, together with Proposition 3.4(2) gives (6.11).

We state now a homogenization theorem for system (6.2):

Theorem 6.6. *Under assumptions* (6.3) *and* (6.4)*, suppose that as* $\varepsilon \to 0$*, there is a matrix field* A *such that*

$$\mathcal{T}_\varepsilon(A^\varepsilon)(x,y) \to A(x,y) \quad \text{a.e. in } \Omega \times Y, \tag{6.12}$$

and as both $\varepsilon, \delta \to 0$*, there exists a matrix field* A^0 *such that*

$$\mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(x,z) \to A^0(x,z) \quad \text{a.e. in } \Omega \times (\mathbb{R}^N \setminus B). \tag{6.13}$$

Let $u_{\varepsilon,\delta}$ *be the solution of* (6.5)*. Then there exists* u *in* $L^\infty(0,T;H_0^1(\Omega))$ *and* $\hat{u}$ *in* $L^\infty(0,T;L^2(\Omega;H^1_{\rm per}(Y)))$ *such that*

$$\begin{aligned}
&(i) \quad u_{\varepsilon,\delta} \rightharpoonup u \quad \text{weakly* in } L^\infty(0,T;H_0^1(\Omega)),\\
&(ii) \quad u'_{\varepsilon,\delta} \rightharpoonup u' \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega)),\\
&(iii) \quad \mathcal{T}_\varepsilon(u_{\varepsilon,\delta}) \rightharpoonup u \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega;H^1(Y))),\\
&(iv) \quad \mathcal{T}_\varepsilon(u'_{\varepsilon,\delta}) \rightharpoonup u' \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega\times Y)).\\
&(v) \quad \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y \widehat{u} \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega\times Y)).
\end{aligned} \tag{6.14}$$

and $U \in L^2(0,T;L^2(\Omega;L^2_{\rm loc}(\mathbb{R}^N)))$ *such that*

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^2(0,T;L^2(\Omega;L^2_{\rm loc}(\mathbb{R}^N))), \tag{6.15}$$

with U *vanishing on* $\Omega \times B \times]0,T[$ *and* $U - k^*u \in L^2(0,T;L^2(\Omega;K_B))$ *(*K_B *being defined by* (6.8)*).*

The couple $(u,\hat{u})$ *satisfies the limit equation*

$$\int_Y A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\nabla_y \phi(y)\,dy = 0, \tag{6.16}$$

for a.e. $x \in \Omega$*, a.e.* $t \in]0,T[$ *and for* $\phi \in H^1_{\rm per}(Y)$*. While the function* U *obeys*

$$\int_{\mathbb{R}^N\setminus B} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\,dz = 0, \tag{6.17}$$

for a.e. $x \in \Omega$*, a.e.* $t \in]0,T[$ *and for all* $v \in K_B$*, with* $v_B = 0$*.*

The ordered triplet $(u, \hat{u}, U)$ satisfies the limit equation

$$
\begin{aligned}
&\langle u''(\cdot,t), \psi\rangle_{(H_0^1(\Omega))', H_0^1(\Omega)} \\
&+ \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \widehat{u}(x,y,t))\nabla\psi(x)\,dx\,dy \\
&- k^* \int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B \psi(x)\,dx\,d\sigma_z \qquad (6.18)\\
&= \int_\Omega f(x,t)\psi(x)\,dx, \quad \text{for a.e. } t\in]0,T[\text{ and for all } \psi\in H_0^1(\Omega), \\
&\qquad u(x,0) = u^0, \quad u'(x,0) = u^1 \quad \text{in } \Omega,
\end{aligned}
$$

where ν_B is the inward normal to ∂B and $d\sigma_z$ its surface measure.

In what follows, we will use the notation $m_Y(\cdot)$ for the average over Y defined as

$$
m_Y(v) = \frac{1}{|Y|}\int_Y v(y)\,dy, \quad \forall v\in L^1(Y).
$$

The result below describes now the homogenized problem in the variable (x,t) in $\Omega\times]0,T[$. To this aim, let us consider the correctors $\hat{\chi}_j$, $j = 1,\dots,N$ solutions of the cell problem; they are the same for domains without holes (see [2, 8]).

$$
\begin{gathered}
\hat{\chi}_j \in L^\infty(\Omega; H^1_{\rm per}(Y)), \\
\int_Y A\nabla(\hat{\chi}_j - y_j)\nabla\varphi\,dy = 0 \quad \text{a.e. } x\in\Omega,\ \forall\varphi\in H^1_{\rm per}(Y) \\
m_Y(\hat{\chi}_j) = 0,
\end{gathered} \tag{6.19}
$$

where A is given by (6.12).

We consider also the cell problem corresponding to the holes B defining the corrector θ for small holes, introduced in [6],

$$
\begin{gathered}
\theta\in L^\infty(\Omega; K_B), \quad \theta(x,B)\equiv 1, \\
\int_{\mathbb{R}^N\setminus B} {}^t\!A^0(x,z)\ \nabla_z\theta(x,z)\ \nabla_z\Psi(z)\ dz = 0 \\
\text{a.e. for } x\in\Omega,\ \forall\Psi\in K_B \text{ with } \Psi(B) = 0.
\end{gathered} \tag{6.20}
$$

Corollary 6.7. *Under assumptions* (6.3) *and* (6.4), *$u\in H_0^1(\Omega)$ is the unique solution of the limit problem*

$$
\begin{gathered}
u'' - \operatorname{div}(\mathcal{A}^{\rm hom}\nabla u) + (k^*)^2\Theta u = f \quad \text{in } \Omega\times]0,T[, \\
u = 0 \quad \text{in } \partial\Omega\times]0,T[, \\
u(x,0) = u^0, \quad u'(x,0) = u^1 \quad \text{in } \Omega,
\end{gathered} \tag{6.21}
$$

where the homogenized matrix field is

$$
\mathcal{A}^{\rm hom} = m_Y\Big(a_{ij} + \sum_{k=1}^N a_{ik}\frac{\partial\hat{\chi}_j}{\partial y_k}\Big), \tag{6.22}
$$

and

$$
\Theta = \int_{\partial B} {}^t\!A^0\nabla_z\theta\nu_B\,d\sigma_z. \tag{6.23}
$$

Remark 6.8. As shown in [6], Θ can be interpreted as the local capacity of B. (See also [11, 12].) Moreover, from (6.20) it is easily seen that Θ is non-negative, i.e.,

$$\boldsymbol{\Theta}(x) = \int_{\mathbb{R}^N\setminus B} A_0(x,z)\,\nabla_z\theta(x,z)\,\nabla_z\theta(x,z)\,dz \geq 0,$$

that is essential for the existence of the solution of the homogenized system (6.21).

Theorem 6.6 is proved in the next section together with Corollary 6.7.

6.2. **Heat equation.** We want to study now the asymptotic behavior as $\varepsilon \to 0$ of the problem

$$\begin{gathered} u'_{\varepsilon,\delta}(x,t) - \operatorname{div}(A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)) = f_{\varepsilon,\delta}(x,t) \quad \text{in } \Omega^*_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,t) = 0 \quad \text{on } \partial\Omega^*_{\varepsilon,\delta}\times]0,T[, \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x), \quad \text{in } \Omega^*_{\varepsilon,\delta}. \end{gathered} \tag{6.24}$$

We suppose that the data satisfy the assumptions:

$$\begin{gathered} (i) A^\varepsilon \in \mathcal{M}(\alpha,\beta,\Omega), \\ (ii) f_{\varepsilon,\delta} \in L^2(0,T;L^2(\Omega)), \\ (iii) u^0_{\varepsilon,\delta} \in L^2(\Omega). \end{gathered} \tag{6.25}$$

Moreover, we assume that

$$\begin{gathered} (i) u^0_{\varepsilon,\delta} \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega), \\ (iii) f_{\varepsilon,\delta} \rightharpoonup f \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \end{gathered} \tag{6.26}$$

Set

$$W_{\varepsilon,\delta} = \{v_{\varepsilon,\delta} \in L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta})) : v'_{\varepsilon,\delta} \in L^2(0,T;H^{-1}(\Omega^*_{\varepsilon,\delta}))\},$$

equipped with the norm

$$\|v_{\varepsilon,\delta}\|_{W_{\varepsilon,\delta}} = \|v_{\varepsilon,\delta}\|_{L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta}))} + \|v'_{\varepsilon,\delta}\|_{L^2(0,T;H^{-1}(\Omega^*_{\varepsilon,\delta}))}.$$

The variational formulation of problem (6.24) is: Find $u_{\varepsilon,\delta} \in W_{\varepsilon,\delta}$ such that, for all $v \in H^1_0(\Omega^*_{\varepsilon,\delta})$,

$$\begin{gathered} \langle u'_{\varepsilon,\delta}(x,t), v(x)\rangle_{(H^1_0(\Omega^*_{\varepsilon,\delta}))', H^1_0(\Omega^*_{\varepsilon,\delta})} + \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v(x)\,dx \\ = \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v(x)\,dx \quad \text{in } \mathcal{D}'(0,T), \\ u_{\varepsilon,\delta}(x,0) = u^0_{\varepsilon,\delta}(x), \quad \text{in } \Omega^*_{\varepsilon,\delta}. \end{gathered} \tag{6.27}$$

For this problem, classical results [8, 19] provide for every fixed ε and δ the existence and uniqueness of a solution of problem (6.27) such that

$$u_{\varepsilon,\delta} \in L^2(0,T;H^1_0(\Omega^*_{\varepsilon,\delta})) \cap \mathcal{C}^0([0,T];L^2(\Omega^*_{\varepsilon,\delta}))$$

and, according to Remark 6.2, satisfies the estimate

$$\|u_{\varepsilon,\delta}\|_{L^\infty(0,T;L^2(\Omega))} + \|u'_{\varepsilon,\delta}\|_{L^2(0,T;H^1_0(\Omega))} \leq C, \tag{6.28}$$

where C is independent of ε and δ. We have the following homogenization result for problem (6.24).

Theorem 6.9. *Under assumptions* (6.25)*,* (6.26)*,* (6.12) *and* (6.13)*, let* $u_{\varepsilon,\delta}$ *be the solution of problem* (6.27)*. Then there exist* u *in* $L^\infty(0,T;H_0^1(\Omega))$ *and* $\hat{u}$ *in* $L^\infty(0,T;L^2(\Omega;H^1_{\rm per}Y)))$*, such that*

$$\begin{gathered}
(i) u_{\varepsilon,\delta} \rightharpoonup u \quad \textit{weakly}^* \textit{ in } L^\infty(0,T;H_0^1(\Omega)),\\
(ii) \mathcal{T}_\varepsilon(u_{\varepsilon,\delta}) \rightharpoonup u \quad \textit{weakly}^* \textit{ in } L^\infty(0,T;L^2(\Omega;H^1(Y))),\\
(iii) \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u + \nabla_y \widehat{u} \quad \textit{weakly}^* \textit{ in } L^\infty(0,T;L^2(\Omega\times Y)).
\end{gathered} \tag{6.29}$$

Moreover, there exists $U \in L^2(0,T;L^2(\Omega;L^2_{\rm loc}(\mathbb{R}^N)))$ *such that* (6.15) *holds.*

The couple $(u,\hat{u})$ *still satisfies the limit equation* (6.16) *while the function* U *still obeys* (6.17)*.*

The ordered triplet $(u,\hat{u},U)$ *satisfies the limit equation*

$$\begin{gathered}
\langle u'(\cdot,t),\psi\rangle_{(H_0^1(\Omega))',H_0^1(\Omega)} - k^*\int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B\psi(x)\,dx\,d\sigma_z\\
+\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\,dx\,dy\\
=\int_\Omega f(x,t)\psi(x)\,dx, \quad \textit{for a.e. } t\in]0,T[\textit{ and for all } \psi\in H_0^1(\Omega),\\
u(x,0)=u^0 \quad \textit{in } \Omega.
\end{gathered} \tag{6.30}$$

On the other hand, the homogenized problem in the variable $(x,t)\in\Omega\times]0,T[$ is given below.

Corollary 6.10. *Under assumptions* (6.3) *and* (6.4)*,* $u\in H_0^1(\Omega)$ *is the unique solution of the limit problem*

$$\begin{gathered}
u' - \operatorname{div}(\mathcal{A}^{\rm hom}\nabla u) + (k^*)^2\Theta u = f \quad \textit{in } \Omega\times]0,T[,\\
u = 0 \quad \textit{in } \partial\Omega\times]0,T[,\\
u(x,0)=u^0, \quad \textit{in } \Omega,
\end{gathered}$$

where the homogenized matrix field $\mathcal{A}^{\rm hom}$ *and the function* Θ *are given by* (6.22) *and* (6.23)*, respectively.*

The proofs of Theorem 6.9 and Corollary 6.10 follow step by step the outlines of those of the corresponding results for the wave equation, hence we omit here their proofs.

7. Proof of main results

Let us now present the proofs of the homogenization results stated in the previous section. We adapt here some ideas in [6, 15].

7.1. **Proof of Theorem 6.6.** We prove the results in several steps.

Step 1. The existence of $u\in L^\infty(0,T;H_0^1(\Omega))$ such that up to subsequences, convergences (6.14)(i)-(ii) hold, follows from estimate (6.6) while the existence of $\hat{u}\in L^\infty(0,T;L^2(\Omega;H^1_{\rm per}(Y)))$ and such that convergences (6.14)(iii)-(v) hold, follows from Proposition 3.5 (see also Remark 6.2).

On the other hand, from (6.7) and Theorem 5.8 there exists a function W in $L^2(0,T;L^2(\Omega;L^{2^*}(\mathbb{R}^N)))$ with $\nabla_z W\in L^2(0,T;L^2(\Omega\times\mathbb{R}^N))$ such that (up to a subsequence)

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})-\mathcal{M}^{\varepsilon}_{Y}(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y}) \rightharpoonup W \quad \text{weakly in } L^2(0,T;L^2(\Omega;L^{2^*}(\mathbb{R}^N))). \tag{7.1}$$

Moreover, in view of (5.10), again by Theorem 5.8 there exists U such that (up to a subsequence) (6.15) holds.

Step 2. Let us check the properties of the function U. From (i) and (ii) of (6.14) we have by compactness,

$$u_{\varepsilon,\delta} \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{7.2}$$

so that from Proposition 3.9(2) and (5.10),

$$\frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\mathcal{M}^{\varepsilon}_{Y}(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to k^*u \quad \text{strongly in } L^2(0,T;L^2(\Omega;L^2_{\rm loc}(\mathbb{R}^N))). \tag{7.3}$$

Thus, from (6.15), (7.1) and (7.3) we conclude that

$$U = W + k^*u \quad \text{and} \quad \nabla_z U = \nabla_z W.$$

Moreover, by using (5.9) of Theorem 5.8, we have

$$\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) = \frac{\delta^{\frac{N}{2}-1}}{\varepsilon}\nabla_z(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad \text{w-}L^2(0,T;L^2(\Omega\times\mathbb{R}^N)). \tag{7.4}$$

Also, from Definition 5.1,

$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \quad \text{in } \Omega\times B\times]0,T[,$$

and thus from (6.15), Definition 3.7 and (7.3),

$$U = u = 0 \quad \text{in } \Omega\times B\times]0,T[. \tag{7.5}$$

This means that

$$W = U - k^*u \in L^2(0,T;L^2(\Omega;K_B)).$$

Step 3. Let us prove the first limit equation. Let $\psi \in \mathcal{D}(\Omega)$ and $\phi \in C^1_{\rm per}(Y)$ vanishing in a neighborhood of $y=0$, and set $v_\varepsilon(x) = \varepsilon\psi(x)\phi^\varepsilon(x)$ with $\phi^\varepsilon(x) = \phi(\frac{x}{\varepsilon})$. By Proposition 3.3,

$$\mathcal{T}_\varepsilon(\nabla v_\varepsilon) \to \psi\nabla_y\phi \quad \text{strongly in } L^2(\Omega\times Y). \tag{7.6}$$

Taking v_ε as a test function in (6.5), multiplying by $\varphi \in \mathcal{D}(0,T)$, and integrating over $]0,T[$, we obtain

$$\begin{aligned}
&\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi''(t)\,dx\,dt\\
&+\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt\\
&=\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt.
\end{aligned} \tag{7.7}$$

Note that this equation can be rewritten as

$$\begin{aligned} &\varepsilon \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t)\psi(x)\phi^\varepsilon(x)\varphi''(t)\,dx\,dt \\ &+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt \\ &= \varepsilon \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)\psi(x)\phi^\varepsilon(x)\varphi(t)\,dx\,dt. \end{aligned} \tag{7.8}$$

We first use the unfolding operator $\mathcal{T}_\varepsilon$ to pass to the limit in the second term of the left-hand side of this equation. Using Proposition 3.3(2) and Proposition 3.5(ii) together with (6.12) and (7.6), we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt \\ &= \lim_{\varepsilon\to 0} \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} \mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla v_\varepsilon)\varphi(t)\,dx\,dy\,dt \\ &= \frac{1}{|Y|} \int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \widehat{u}(x,y,t))\psi(x)\nabla_y\phi(y)\,\varphi(t)\,dx\,dy\,dt. \end{aligned}$$

On the other hand, the first term on the left-hand side of (7.8) as well as the term on the right-hand side goes to zero as $\varepsilon \to 0$, which implies

$$\lim_{\varepsilon\to 0} \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt = 0,$$

so that

$$\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \hat{u}(x,y,t))\psi(x)\nabla_y\phi(y)\,\varphi(t)\,dx\,dy\,dt = 0.$$

By Lemma 6.3, we obtain (6.16) which describes the asymptotic behavior of the problem based on the oscillations in the coefficients of (6.5).

Now, to take into account the effect of the perforations, let us use $w_{\varepsilon,\delta}\psi$ as a test function in (6.5), where $w_{\varepsilon,\delta}$ is the function defined in Lemma 6.4 and for $\psi \in \mathcal{D}(\Omega)$. Thus, we have

$$\begin{aligned} &\langle u''_{\varepsilon,\delta}(x,t), w_{\varepsilon,\delta}(x)\psi(x)\rangle_{(H^1_0(\Omega^*_{\varepsilon,\delta}))', H^1_0(\Omega^*_{\varepsilon,\delta})} \\ &+ \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\,dx \\ &+ \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\nabla\psi(x)\,dx \\ &= \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\,dx. \end{aligned}$$

Let $\varphi \in \mathcal{D}(0,T)$ and multiply the integrands in this equation and integrate over $]0,T[$,

$$\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x)\psi(x)\varphi''(t)\,dx\,dt \\
&+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt \\
&+ \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)\,dx\,dt \\
&= \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt.
\end{aligned} \tag{7.9}$$

For the first term on the left-hand side of this equation, we apply the operator $\mathcal{T}_\varepsilon$. Thus, from Proposition 3.3(2)(4), Proposition 3.4(1), Definition 5.1 together with Remark 6.5(2) and (6.14)(iii), we obtain,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) w_{\varepsilon,\delta}(x)\psi(x)\varphi''(t)\,dx\,dt \\
&= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T \int_{\Omega\times Y} \mathcal{T}_\varepsilon(u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\psi)\varphi''(t)\,dx\,dy\,dt \\
&= \frac{v(B)}{|Y|}\int_0^T \int_{\Omega\times Y} u(x,t)\psi(x)\varphi''(t)\,dx,dy\,dt.
\end{aligned} \tag{7.10}$$

For the second term on the left-hand side of equation (7.9), we use the operator $\mathcal{T}_{\varepsilon,\delta}$. Then, Remark 4.7, together with (6.1), (6.13), (7.4), (7.5), Proposition 5.3(2), Proposition 5.5 and Remark 6.5(1), yield

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt \\
&= \lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_0^T \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\mathcal{T}_{\varepsilon,\delta}(\psi(x))\varphi(t)\,dx\,dzdt \\
&= \lim_{\varepsilon\to 0}\frac{\delta^N}{|Y|}\int_0^T \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})(-\frac{1}{\varepsilon\delta}\nabla_z v)\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi(t)\,dx\,dz\,dt \\
&= \lim_{\varepsilon\to 0}\Big(-\frac{\delta^{\frac{N}{2}-1}}{\varepsilon|Y|}\int_0^T \int_{\Omega\times\mathbb{R}^N} \mathcal{T}_{\varepsilon,\delta}(A^\varepsilon)(\delta^{\frac{N}{2}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}))\nabla_z v\mathcal{T}_{\varepsilon,\delta}(\psi)\varphi(t)\,dx\,dz\,dt\Big) \\
&= -\frac{k^*}{|Y|}\int_0^T \int_{\Omega\times\mathbb{R}^N} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dzdt \\
&= -\frac{k^*}{|Y|}\int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt,
\end{aligned}$$

so that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt \\
&= -\frac{k^*}{|Y|}\int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt.
\end{aligned} \tag{7.11}$$

For the third term on the left-hand side of (7.9), we use $\mathcal{T}_\varepsilon$. From Proposition 3.3(2)(4), Proposition 3.4(1), Definition 5.1 together with Remark 6.5(2), (6.12), Proposition 3.5(ii), passing to the limit gives

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\nabla\psi(x)\varphi(t)\,dx\,dt\\
&=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(A^\varepsilon)\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\nabla\psi)\varphi(t)\,dx\,dy\,dt \qquad (7.12)\\
&=\frac{v(B)}{|Y|}\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt.
\end{aligned}$$

For the term on the right-hand side of equation (7.9), we also apply $\mathcal{T}_\varepsilon$, Definition 5.1, Remark 6.5(2), Proposition 3.3(2) and (6.4)(iii) and passing to the limit, yields

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)w_{\varepsilon,\delta}(x)\psi(x)\varphi(t)\,dx\,dt\\
&=\lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_0^T\int_{\Omega\times Y}\mathcal{T}_\varepsilon(f_{\varepsilon,\delta})\mathcal{T}_\varepsilon(w_{\varepsilon,\delta})\mathcal{T}_\varepsilon(\psi)\varphi(t)\,dx\,dy\,dt \qquad (7.13)\\
&=\frac{v(B)}{|Y|}\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt.
\end{aligned}$$

Thus, combining (7.10)-(7.13), the limit equation of (7.9) is

$$\begin{aligned}
&v(B)\int_0^T\int_{\Omega\times Y} u(x,t)\psi(x)\varphi''(t)\,dx\,dy\,dt\\
&-k^*\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt\\
&\quad+v(B)\int_0^T\int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t)+\nabla_y\widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt \qquad (7.14)\\
&=v(B)\int_0^T\int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt,
\end{aligned}$$

which is true for all $\varphi\in\mathcal{D}(0,T)$, $\psi\in H_0^1(\Omega)$ and $v\in K_B$. So, we obtain (6.17) for $v\in K_B$ such that $v(B)=0$.

If $v(B)\neq 0$, by applying Stoke's formula and (6.17), we have

$$\begin{aligned}
&\int_0^T\int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dzdt\\
&=v(B)\int_0^T\int_{\Omega\times\partial B} A^0(x,z)\nabla_z U(x,z,t)\nu_B\psi(x)\varphi(t)\,dx\,d\sigma_z\,dt,
\end{aligned}$$

which used in (7.14) gives the first equation of problem (6.18).

Step 4. It remains now to check the limit initial conditions. Let $v_\varepsilon = w_{\varepsilon,\delta}\psi$ where $w_{\varepsilon,\delta}$ is given by Lemma 6.4 and $\psi\in\mathcal{D}(\Omega)$. Let $\varphi\in C^\infty([0,T])$ with $\varphi(0)=1$ and $\varphi(T)=0$. Take $v_\varepsilon\varphi$ as a test function in (6.5). Using the initial condition in (6.5) and by integration by parts, we have

$$\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t)v_\varepsilon(x)\varphi(t)\,dx\,dt-\int_0^T\int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x)\varphi(t)\,dx\,dt$$

$$
\begin{aligned}
&= \int_0^T \langle u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x)\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))' H_0^1(\Omega^*_{\varepsilon,\delta})} \varphi(t)\,dt \\
&= \int_{\Omega^*_{\varepsilon,\delta}} (u'_{\varepsilon,\delta}(x,t)\varphi(t))\big|_0^T v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi'(t)\,dx\,dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,0) v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi(t)'\,dx\,dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u^1_{\varepsilon,\delta}(x) v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi(t)'\,dx\,dt.
\end{aligned}
$$

In view of (7.11)-(7.13) and (6.4), passing to the limit in this equation yields

$$
\begin{aligned}
& v(B)\int_0^T \int_{\Omega\times Y} f(x,t)\psi(x)\varphi(t)\,dx\,dy\,dt \\
& + k^* \int_0^T \int_{\Omega\times(\mathbb{R}^N\setminus B)} A^0(x,z)\nabla_z U(x,z,t)\nabla_z v(z)\psi(x)\varphi(t)\,dx\,dz\,dt \\
& - v(B)\int_0^T \int_{\Omega\times Y} A(x,y)(\nabla_x u(x,t) + \nabla_y \widehat{u}(x,y,t))\nabla\psi(x)\varphi(t)\,dx\,dy\,dt \\
& = -v(B)\int_\Omega u^1(x)\psi(x)\,dx - v(B)\int_0^T \int_\Omega u'(x,t)\psi(x)\varphi'(t)\,dx\,dt \\
& = -v(B)\int_\Omega u^1(x)\psi(x)\,dx + v(B)\int_\Omega u'(x,0)\psi(x)\,dx \\
& \quad + v(B)\int_0^T \langle u''(x,t),\psi(x)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\varphi(t)\,dt.
\end{aligned}
$$

Combining this equation with (7.14) yields

$$
-\int_\Omega u^1(x)\psi(x)\,dx + \int_\Omega u'(x,0)\psi(x)\,dx = 0, \quad \forall \psi \in \mathcal{D}(\Omega), \tag{7.15}
$$

which implies $u'(x,0) = u^1(x)$.

For the first initial condition, let us now choose $\varphi \in C^\infty([0,T])$ with $\varphi(0) = \varphi(T) = \varphi'(T) = 0$ and $\varphi'(0) = 1$. Let us take again $v_\varepsilon\varphi$ as a test function in (6.5). Using the initial conditions in (6.5) and by integration by parts, we have

$$
\begin{aligned}
&\int_0^T \int_{\Omega^*_{\varepsilon,\delta}} f_{\varepsilon,\delta}(x,t) v_\varepsilon(x,z)\varphi(t)\,dx\,dt - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} A^\varepsilon(x)\nabla u_{\varepsilon,\delta}(x,t)\nabla v_\varepsilon(x,z)\varphi(t)\,dx\,dt \\
&= \int_0^T \langle u''_{\varepsilon,\delta}(x,t), v_\varepsilon(x,z)\rangle_{(H_0^1(\Omega^*_{\varepsilon,\delta}))' H_0^1(\Omega^*_{\varepsilon,\delta})} \varphi(t)\,dt \\
&= \int_{\Omega^*_{\varepsilon,\delta}} (u'_{\varepsilon,\delta}(x,t)\varphi(t))\big|_0^T v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u'_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi'(t)\,dx \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} (u_{\varepsilon,\delta}(x,t)\varphi'(t))\big|_0^T v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi''(x,t)\,dx\,dt \\
&= -\int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,0) v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) v_\varepsilon(x)\varphi''(x,t)\,dx\,dt
\end{aligned}
$$

$$= -\int_{\Omega^*_{\varepsilon,\delta}} u^0_{\varepsilon,\delta}(x) v_\varepsilon(x)\,dx - \int_0^T \int_{\Omega^*_{\varepsilon,\delta}} u_{\varepsilon,\delta}(x,t) v_\varepsilon(x) \varphi''(x,t)\,dx\,dt$$

By similar argument as those used to obtain (7.15), in view of (7.11)-(7.13), the initial conditions in (6.5) together with (6.4), passing to the limit and combining with (7.14) gives

$$-\int_\Omega u^0(x)\psi(x)\,dx + \int_\Omega u(x,0)\psi(x)\,dx = 0, \quad \forall \psi \in \mathcal{D}(\Omega),$$

which implies $u(x,0) = u^0(x)$. This concludes the proof.

Proof of Corollary 6.7. Let us show first that $\hat{u}$ can be expressed as function of u. This is a standard procedure in homogenization, see for instance [2] or [8]. To do so, let us have a look at equation (6.16). Recalling the cell problems (6.19) defining the functions $\hat{\chi}_j$, $j = 1, \dots, N$, this equation allows as to write $\hat{u}$ in the form

$$\widehat{u}(x,y) = -\sum_{j=1}^{n} \widehat{\chi}_j(y) \frac{\partial u_0}{\partial x_j} + \tilde{u}(x),$$

with $\tilde{u}$ unknown.

Plugging this formula in the second integral from (6.18) yields

$$\begin{gathered} \langle u'', \psi \rangle_{(H^1_0(\Omega))', H^1_0(\Omega)} - k^* \int_{\Omega\times\partial B} A^0 \nabla_z U \nu_B \psi\,d\sigma_z \\ + \int_\Omega \mathcal{A}^{\rm hom} \nabla u \nabla \psi\,dx = \int_\Omega f\psi\,dx, \end{gathered} \tag{7.16}$$

for a.e. $t \in]0,T[$ and where $\mathcal{A}^{\rm hom}$ is given by (6.22).

Taking into account the initial conditions of u, we derive that (7.16) is the variational formulation of the problem

$$\begin{gathered} u'' - k^* \int_{\partial B} A^0 \nabla_z U \nu_B\,d\sigma_z + \operatorname{div}(\mathcal{A}^{\rm hom} \nabla u) = f \quad \text{in } \Omega\times]0,T[, \\ u = 0 \quad \text{in } \partial\Omega\times]0,T[, \\ u(x,0) = u^0, \quad u'(x,0) = u^1 \quad \text{in } \Omega, \end{gathered} \tag{7.17}$$

where $u'' \in L^2(0,T;H^{-1}(\Omega))$. Classical results give

$$u \in C^0([0,T];L^2(\Omega)) \quad \text{and} \quad u' \in C^0([0,T];H^{-1}(\Omega)).$$

Finally, the same computation as in [6] shows that the second term in the first equation of (7.17) satisfies

$$\int_{\partial B} A^0 \nabla_z U \nu_B\,d\sigma_z = -k^* u \Big(\int_{\partial B} {}^t A^0 \nabla_z \theta \nu_B\,d\sigma_z \Big), \tag{7.18}$$

for a.e. $t \in]0,T[$, where θ is the solution of (6.20). Thus, problem (7.17) can be rewritten as (6.21) where Θ is given by (6.23). $\square$

Acknowledgments. The first author would like to extend her gratitude to Professors A. Damlamian and G. Griso for some meaningful discussions on the topic and to the Office of the Vice President for Academic Affairs of the University of the Philippines Diliman (under ECWRG) for the financial support for this project.

The authors are also deeply indebted to D. Cioranescu for several discussions and comments, as well as with David Gómez-Castro for a careful lecture of a preliminary version of the paper and several suggestions, that improved the paper.

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