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INFINITELY MANY POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this article, we study a class of fractional differential inclusions problem. By nonsmooth variational methods and the theory of the fractional derivative spaces, we establish the existence of infinitely many positive solutions of the problem under suitable oscillatory assumptions on the potential F at zero or at infinity.

1. Introduction

In this article, we consider the existence and multiplicity of solutions for the fractional differential inclusion

$$\begin{gathered} \frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_0D_T^{-\beta}(u'(t))\Big) \in \partial F(t,u(t)), \quad \text{a.a. } t\in[0,T], \\ u(0)=u(T)=0, \end{gathered} \tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_0D_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0\le\beta<1$, respectively, $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is locally Lipschitz function in the t-variable integrand (in general it can be nonsmooth), and $\partial F(t,x)$ is the subdifferential with respect to the t-variable in the sense of Clarke [4].

Fractional differential equations and inclusions have been proved that they are very valued tools in the modeling of many phenomena in various fields of science and engineering, such as, viscoelasticity, electrochemistry, electromagnetism, economics, optimal control, porous media, etc. In consequence, the subject of fractional differential equations and inclusions is gaining much importance and attention. For details and examples, see [2, 3, 13, 14, 21], and the references therein.

Recently, variational methods have turned out to be a very effective analytical tool in the study of nonlinear problems. The classical point theory for C^1 functional was developed in the sixties and seventies, see [1, 5, 16, 18]. The need of specific applications (such as nonsmooth mechanics, nonsmooth gradient systems, etc.) and the impressive progress in nonsmooth analysis and multivalued analysis led to extensions of the critical point theory to nondifferentiable functions, locally Lipschitz functions in particular. The nonsmooth critical point theory for locally Lipschitz functions started with the work of Chang [5]. Chang proposed a generalization of the well-known Palais-Smale condition and obtained various minimax

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principles concerning the existence and characterization of critical points for locally Lipschitz functions. Chang used his theory to study semilinear elliptic boundary value problem with a discontinuous nonlinearity.

There are some papers which are devoted to the boundary value problems for fractional differential inclusion, see [6, 17, 20, 22]. And the main tools they use are fixed point theory for multi-valued contractions. In particular, if $F(x,\cdot) \in C^1(\mathbb{R}^N)$ for a.a. $x \in \mathbb{R}^N$, then problem (1.1) becomes

$$\begin{gathered} \frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_0D_T^{-\beta}(u'(t))\Big) = \nabla F(t,u(t)), \quad \text{a.a. } t \in [0,T], \\ u(0) = u(T) = 0. \end{gathered} \tag{1.2}$$

Thus a solution u of (1.1) is a weak solution to the problem (1.2). So, in some sense, the solutions of (1.1) can be considered as generalized solutions of (1.2), thus, the formulation of (1.1) is completely justified.

In the past decade, there are many papers dealing with the existence of multiple solutions of fractional boundary value problems [7, 8, 9, 10, 11, 12, 15, 19] and the references therein. For example, Jiao and Zhou [11] got one nontrivial solutions for problem (1.2) using the mountain pass theorem. Chen and Tang [7] studied the existence and multiplicity of solutions for the system (1.2) when the nonlinearity $F(t,\cdot)$ are superquadratic, asymptotically quadratic, and subquadratic, respectively. In [8], by using the minmax methods in critical point theory, the authors proved the existence of infinitely many solutions under suitable conditions. Inspired by the above-mentioned papers, we study problem (1.1) from a more extensive viewpoint. So we deal with the existence of infinitely many solutions for problem (1.1) with the potential $F(x,t)$ exhibits an oscillation at the origin or at infinity. Indeed, our main results (see Theorems 3.3 and 3.6 below) give sufficient conditions on the oscillatory terms such that problem (1.1) has infinitely many positive solutions. As a byproduct, these solutions can be constructed in such a way that their norms in E^α tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively).

This article is organized as follows. In section 2, we present some necessary preliminary knowledge on the fractional derivative space $E_0^{\alpha,p}$ and generalized gradient of the locally Lipschitz function. In section 3, we give the main results of this paper.

2. Preliminaries

In this part, we recall some definitions and display the variational setting which has been established for our problem.

Definition 2.1 ([17]). *Let $f(t)$ be a function defined on $[a,b]$ and $\tau > 0$. The left and right Riemann-Liouville fractional integrals of order τ for function $f(t)$ denoted by ${}_aD_t^{-\tau}f(t)$ and ${}_tD_b^{-\tau}f(t)$, respectively, are defined by*

$$\begin{gathered} {}_aD_t^{-\tau}f(t) = \frac{1}{\Gamma(\tau)}\int_a^t (t-s)^{\tau-1}f(s)ds,\ t \in [a,b], \\ {}_tD_b^{-\tau}f(t) = \frac{1}{\Gamma(\tau)}\int_t^b (t-s)^{\tau-1}f(s)ds,\ t \in [a,b], \end{gathered} \tag{2.1}$$

provided the right-hand sides are pointwise defined on $[a,b]$, where Γ is the gamma function.

Definition 2.2 ([17]). Let $f(t)$ be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order τ for function $f(t)$ denoted by ${}_aD_t^\tau f(t)$ and ${}_tD_b^\tau f(t)$, respectively, are defined by

$$\begin{gathered} {}_aD_t^\tau f(t) = \frac{d^n}{dt^n}\,{}_aD_t^{\tau-n} f(t) = \frac{1}{\Gamma(n-\tau)}\frac{d^n}{dt^n}\Big(\int_a^t (t-s)^{n-\tau-1} f(s)ds\Big), \\ {}_tD_b^\tau f(t) = (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\tau-n} f(t) = \frac{1}{\Gamma(n-\tau)}\frac{d^n}{dt^n}\Big(\int_t^b (t-s)^{n-\tau-1} f(s)ds\Big), \end{gathered} \tag{2.2}$$

where $t\in[a,b]$, $n-1\le\tau<n$ and $n\in\mathbb{N}$.

The left and the right Caputo fractional derivatives are defined via the above Riemann-Liouville fractional derivatives. In particular, they are defined for the function belong-ing to the space of absolutely continuous functions, which we denote by $AC([a,b],\mathbb{R}^N)$. $AC^k([a,b],\mathbb{R}^N)(k=1,2,\cdots)$ is the space of functions f such that $f\in C^k([a,b],\mathbb{R}^N)$. In particular, $AC([a,b],\mathbb{R}^N)=AC^1([a,b],\mathbb{R}^N)$.

Definition 2.3 ([17]). Let $\tau\ge 0$ and $n\in\mathbb{N}$. If $\tau\in[n-1,n)$ and $f(t)\in AC^n([a,b],\mathbb{R}^N)$, then the left and right Caputo fractional derivative of order τ for function $f(t)$ denoted by ${}_a^cD_t^\tau f(t)$ and ${}_t^cD_b^\tau f(t)$, respectively, exist almost everywhere on $[a,b]$. ${}_a^cD_t^\tau f(t)$ and ${}_t^cD_b^\tau f(t)$ are represented by

$$\begin{gathered} {}_a^cD_t^\tau f(t) = {}_aD_t^{\tau-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\tau)}\Big(\int_a^t (t-s)^{n-\tau-1} f^{(n)}(s)ds\Big), \\ {}_t^cD_b^\tau f(t) = (-1)\,{}_t^nD_b^{\tau-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\tau)}\Big(\int_t^b (t-s)^{n-\tau-1} f^{(n)}(s)ds\Big), \end{gathered} \tag{2.3}$$

respectively, where $t\in[a,b]$.

Definition 2.4 ([6]). Define $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0^cD_t^\alpha u(t)|^p dt\Big)^{1/p}, \quad \forall u\in E_0^{\alpha,p}, \tag{2.4}$$

where $C_0^\infty([0,T],\mathbb{R}^N)$ denotes the set of all functions $u\in C^\infty([0,T],\mathbb{R}^N)$ with $u(0)=u(T)=0$. It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u\in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u\in L^p([0,T],\mathbb{R}^N)$ and $u(0)=u(T)=0$.

Proposition 2.5 ([6]). *Let $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable space.*

Proposition 2.6 ([6]). Let $0<\alpha\le 1$ and $1<p<\infty$. For all $u\in E_0^{\alpha,p}$, we have

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^cD_t^\alpha u\|_{L^p}. \tag{2.5}$$

Moreover, if $\alpha>\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\|u\|_\infty \le \frac{T^{\frac{\alpha-1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}\|{}_0^cD_t^\alpha u\|_{L^p}. \tag{2.6}$$

According to [6], we can consider $E_0^{\alpha,p}$ with respect to the norm

$$\|u\|_{\alpha,p} = \|{}_0^cD_t^\alpha u\|_{L^p} = \Big(\int_0^T |{}_0^cD_t^\alpha u|^p dt\Big)^{\frac{1}{p}}. \tag{2.7}$$

Proposition 2.7 ([6]). *Define $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence u_k converges weakly to $u \in E_0^{\alpha,p}$, i.e. $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, i.e. $\|u_k - u\|_\infty \to 0$, as $k \to \infty$.*

Using Definition 2.3, for any $u \in AC([0,T],\mathbb{R}^N)$, problem (1.1) is equivalent to the problem

$$\begin{gathered}\frac{d}{dt}\Big(\frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\Big) \in \partial F(t,u(t)), \quad \text{a.e. } t \in [0,T],\\ u(0) = u(T) = 0,\end{gathered} \tag{2.8}$$

where $\alpha = 1-\beta \in (\frac{1}{2},1]$. In the following, we will treat problem (1.2) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$.

Definition 2.8 ([6]). *A function $u \in AC([0,T],\mathbb{R}^N)$ is called a solution of (1.1) if*

(i) *$D^\alpha(u(t))$ is derivative for almost every $t \in [0,T]$, and*
(ii) *u satisfies (1.1),*

where $D^\alpha(u(t)) := \frac{1}{2}{}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - \frac{1}{2}{}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))$.

Proposition 2.9 ([6]). *If $\frac{1}{2} < \alpha \le 1$, then for any $u \in E^\alpha$, we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \le -\int_0^T \Big({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)\Big)dt \le \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.9}$$

Proposition 2.10 ([6]). *Let $1/2 < \alpha \le 1$ be satisfied. If $u \in E^\alpha$, then the functional $J : E^\alpha \to \mathbb{R}$ defined by*

$$J(u) = -\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))dt$$

is convex and continuous on E^α.

Let X be a Banach space and X^* be its topological dual space and we denote $\langle\cdot,\cdot\rangle$ as the duality bracket for pair (X^*, X). A function $\varphi : X \mapsto \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find a neighbourhood U of x and a constant $k > 0$(depending on U), such that $|\varphi(y) - \varphi(z)| \le k\|y - z\|, \forall y, z \in U$.

For a locally Lipschitz function $\varphi : X \mapsto \mathbb{R}$ we define

$$\varphi^0(x;h) = \limsup_{x'\to x;\lambda\downarrow 0} \frac{\varphi(x'+\lambda h) - \varphi(x')}{\lambda}.$$

It is obvious that the function $h \mapsto \varphi^0(x;h)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \{x^* \in X^*; \langle x^*, h\rangle \le \varphi^0(x;h),\ \forall h \in X\}.$$

The multifunction $\partial\varphi : X \mapsto 2^{X^*}$ is called the generalized subdifferential of φ.

If φ is also convex, then $\partial\varphi(x)$ coincides with subdifferential in the sense of convex analysis, defined by

$$\partial_C\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi(x+h) - \varphi(x) \text{ for } h \in X\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

A point $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. It is easily seen that, if $x \in X$ is a local minimum of φ, then $0 \in \partial\varphi(x)$.

Lemma 2.11. *The functional*

$$\varphi(u)=\int_0^T\big[-\frac{1}{2}({}_0^cD_t^\alpha u(t),\,{}_t^cD_T^\alpha u(t))\big]dt-\int_0^T F(t,u(t))dt \tag{2.10}$$

is locally Lipschitz on E^α. Moreover, for $u,v\in E^\alpha$, we have

$$\begin{aligned}\langle\zeta,v\rangle=&-\int_0^T\frac{1}{2}\big[({}_0^cD_t^\alpha u(t),\,{}_t^cD_T^\alpha v(t))+({}_t^cD_T^\alpha u(t),\,{}_0^cD_t^\alpha v(t))\big]dt\\&-\int_0^T(q(t),v(t))dt,\end{aligned} \tag{2.11}$$

where $\zeta\in\partial\varphi(u)$ and $q(t)\in\partial(F(t,u(t)))$.

Proof. Let $I(u)=\int_0^T F(t,u(t))dt$, then $\varphi(u)=J(u)-I(u)$. Obviously, $J(u)$ is locally Lipschitz. For ε is smaller enough, there existent $B_\varepsilon(0)\subset\mathbb{N}$. For any $u_1(t),u_2(t)\in B_\varepsilon(0)$ we have

$$F(t,u_1(t))-F(t,u_2(t))=\langle\partial F(t,\bar u(t)),u_1(t)-u_2(t)\rangle,$$

where $\bar u(t)=\lambda u_1(t)+(1-\lambda)u_2(t)$, for $\lambda\in(0,1)$. Furthermore,

$$\|\bar u\|_{E^\alpha}=\|\lambda u_1+(1-\lambda)u_2\|_{E^\alpha}\le\|\lambda u_1\|_\alpha+\|(1-\lambda)u_2\|_\alpha\le\|u_1\|_\alpha+\|u_2\|_\alpha\le 2\varepsilon.$$

Thus, we obtain

$$\begin{aligned}|I(u_1)-I(u_2)|&\le\int_0^T c(1+|\bar u(t)|^{\alpha(t)-1})|u_1(t)-u_2(t)|dt\\&\le c\int_0^T|u_1(t)-u_2(t)|dt+c\int_0^T||\bar u(t)|^{\alpha_1-1}|u_1(t)-u_2(t)|dt\\&\le c_1\|u_1-u_2\|_{E^\alpha}+c_2\|\bar u\|_{E^\alpha}^{\alpha_1-1}\|u_1-u_2\|_{E^\alpha}\\&\le c_1\|u_1-u_2\|_{E^\alpha}+c_2(2\varepsilon)^{\alpha_1-1}\|u_1-u_2\|_{E^\alpha}\\&\le L\|u_1-u_2\|_{E^\alpha},\end{aligned}$$

where $\alpha_1=\min_{t\in[0,T]}\alpha(t)$, and c_1,c_2 are positive contents. □

Proposition 2.12 ([4])**.** *Let x and y be point in Banach space X, and suppose that f is Lipschitz on an open set containing the line segment $[x,y]$. Then there exists a point u in (x,y) such that*

$$f(y)-f(x)\in\langle\partial f(u),y-x\rangle.$$

3. Main results and their proofs

Now we are in a position to state our first main result which deals with the case when the nonlinearity $F(x,t)$ exhibits an oscillation at the origin. Our hypotheses on nonsmooth potential $F(x,t)$ are listed as follows.

(H1) $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is a function, $F(t,0)=0$ for almost all $t\in[0,T]$ and satisfies the following facts:

(1) For all $x\in\mathbb{R}^N$, $t\mapsto F(t,x)$ is measurable;
(2) For almost all $t\in[0,T]$, $x\mapsto F(t,x)$ is locally Lipschitz;

(3) There exist a positive constant c such that for almost all $x \in \mathbb{R}^N$, all $t \in [0,T]$ and $\omega \in \partial F(t,x)$
$$|\omega| \le c(1+|x|^{\alpha(t)-1})$$
where $1 < \alpha(t) < +\infty$;

(4) $-\infty < \liminf_{|x|\to 0^+} \frac{F(t,x)}{|x|^2} \le \limsup_{|x|\to 0^+} \frac{F(t,x)}{|x|^2} = +\infty$ uniformly for a.e. $t \in [0,T]$;

(5) For every $k \in \mathbb{N}$, there exists $e_k \in \mathbb{R}^N$ with $|e_k| = 1$ and there are two sequences $\{a_k\}$ and $\{b_k\}$ in $(0,+\infty)$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$ such that
$$\sup\{\omega \cdot e_k : \omega \in \partial F(t,x), \quad \text{a.e. } t \in [0,T],\ x \in [a_k, b_k]e_k\} \ge 0.$$

Remark 3.1. Hypotheses (H1)(4) and (H1)(5) imply an oscillatory behaviour of F near the origin.

Remark 3.2. A simple example of a nonsmooth potential function satisfying
$$F(t,x) = \begin{cases} 0, & \text{if } |x| = 0 \text{ or } |x| \in [\frac{1}{2\pi}, +\infty), \\ |x|^{\beta(t)} \sin\frac{1}{|x|}, & \text{if } |x| \in [\frac{1}{(2k+1)\pi}, \frac{1}{2k\pi}), \\ |x|^{\alpha(t)} \sin\frac{1}{|x|}, & \text{if } |x| \in [\frac{1}{(2k+2)\pi}, \frac{1}{(2k+1)\pi}], \end{cases}$$
where $k \in N$ with $k \ge 1$, $1 < \beta(t) < 2 < \alpha(t)$.

Proof. Obviously, (H)(1) and (H1)(2) are satisfied. It is also obvious that $x \mapsto F(t,x)$ is locally Lipschitz. Then
$$\partial F(t,x) = \begin{cases} 0, & \text{if } |x| = 0 \text{ or } |x| > \frac{1}{2\pi}, \\ \alpha(t)|x|^{\beta(t)-2}x \sin\frac{1}{|x|} - |x|^{\beta(t)-3}x\cos\frac{1}{|x|}, & \text{if } |x| \in \big(\frac{1}{(2k+1)\pi}, \frac{1}{2k\pi}\big), \\ \beta(t)|x|^{\alpha(t)-2}x \sin\frac{1}{|x|} - |x|^{\alpha(t)-3}x\cos\frac{1}{|x|}, & \text{if } |x| \in \big(\frac{1}{(2k+2)\pi}, \frac{1}{(2k+1)\pi}\big), \\ [|x|^{\beta(t)-3}x, |x|^{\alpha(t)-3}x], & \text{if } |x| = \frac{1}{(2k+1)\pi}, \\ [-|x|^{\beta(t)-3}x, -|x|^{\alpha(t)-3}x], & \text{if } |x| = \frac{1}{(2k+2)\pi}, \\ [-|x|^{\beta(t)-3}x, 0], & \text{if } |x| = \frac{1}{2\pi}, \end{cases}$$
Hence, there exists a constant $c > 0$ such that
$$|w| \le c(1+|x|^{\alpha(t)-1}) \quad \text{for all } w \in \partial F(t,x).$$
So condition (H1)(3) holds. Then, for any $1 \le k \in N$, we can choose
$$a_k := \frac{1}{(2k+2)\pi}, \quad b_k := \frac{1}{(2k+\frac{3}{2})\pi},$$
which means $a_k < b_k$, $\lim_{k\to+\infty} b_k = 0$ and
$$\sup\{w \cdot e_k : w \in \partial F(t,x),\ \text{a.e. } t \in [0,T] \text{ and } x \in [a_k, b_k]e_k\} \le 0.$$
So condition (H1)(5) is satisfied.

On the other hand, for any $1 \le k \in N$, we can choose $c_k := \frac{1}{(2k+\frac{1}{2})\pi}$, which implies $\lim_{k\to+\infty} c_k = 0$,
$$\limsup_{k\to+\infty} \frac{F(t,c_k e_k)}{|c_k e_k|^2} = \limsup_{k\to+\infty} \frac{|c_k e_k|^{\beta(t)} \sin\frac{1}{|c_k e_k|}}{|c_k e_k|^2} = \limsup_{k\to+\infty} \frac{1}{|c_k e_k|^{2-\beta(t)}} = +\infty,$$
$$-\infty < -1 \le \liminf_{|x|\to 0^+} \frac{F(t,x)}{|x|^2} = \liminf_{|x|\to 0^+} \frac{|x|^{\alpha(t)} \sin\frac{1}{|x|}}{|x|^2} = \liminf_{|x|\to 0^+} |x|^{\alpha(t)-2} \sin\frac{1}{|x|} \le 0$$

uniformly for a.e. $t \in [0,T]$. So condition (H1)(4) holds. □

Theorem 3.3. *Suppose that* (H1) *holds. Then there exists a sequence* $\{u_n\} \subset E^\alpha$ *of distinct positive solution of problem* (1.1) *such that*

$$\lim_{n\to+\infty} \|u_n\|_\alpha = \lim_{n\to+\infty} |u_n|_\infty = 0.$$

Proof. For every fixed $k \in \mathbb{N}$, consider the set

$$S_k = \{u \in E^\alpha : u(t) \neq 0 \text{ and } u(t) \in [0, b_k]e_k \quad \text{a.e. } t \in [0,T]\},$$

where b_k is from (H1)(5). The proof is divided into four steps as follows.

Step 1. We claim that φ is bounded from below on S_k and its infimum m_k on S_k is attained at $u_k \in S_k$.

On account of (H1)(3) and Proposition 2.12, for every $u \in S_k$, we have

$$F(t,x) - F(t,0) \in \langle \partial F(t,\xi), x\rangle,$$

where $\xi = \lambda x$, and $\lambda \in (0,1)$. Furthermore, we have

$$|\omega| \le c(1 + |\xi|^{\alpha(t)-1}) = c(1 + |\lambda|^{\alpha(t)-1}|x|^{\alpha(t)-1}) \le c(1 + |x|^{\alpha(t)-1}). \tag{3.1}$$

Applying the Mean Value Theorem and (2.3), for any $\omega \in \partial F(t,\xi)$, we have

$$|F(t,x) - F(t,0)| = |\langle \omega, x\rangle| \le |\omega| \cdot |x| \le c(|x| + |x|^{\alpha(t)}),$$

That is,

$$|F(t,x)| \le c(|x| + |x|^{\alpha(t)}) \le c(1 + |x|^{\alpha(t)}). \tag{3.2}$$

Thus,

$$\begin{aligned}
\varphi(u) &= \int_0^T \big[-\frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\big]dt - \int_0^T F(t,u(t))dt \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \int_0^T c(1 + |u(t)|^{\alpha(t)})dt \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \int_0^T c(1 + |u(t)|^{\alpha_0})dt \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - cT - c\int_0^T |u(t)|^{\alpha_0}dt \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - cT - cT|b_k|^{\alpha_0} \\
&\ge -cT - cT|b_k|^{\alpha_0},
\end{aligned} \tag{3.3}$$

where $\alpha_0 = \inf_{t\in[0,T]} \alpha(t)$. It is clear that S_k is convex and closed, thus weakly closed in E^α. Let $m_k = \inf_{S_k} \varphi$, and $\{u_k^n\}_{n=1}^\infty$ be a sequence in S_k such that $m_k \le \varphi(u_k^n) \le m_k + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
m_k + \frac{1}{n} &\ge \varphi(u_k^n) \\
&= \int_o^T \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k^n(t), {}_t^cD_T^\alpha u_k^n(t))\big]dt - \int_0^T F(t,u_k^n(t))dt,
\end{aligned} \tag{3.4}$$

which implies

$$\begin{aligned}\frac{|\cos(\pi\alpha)|}{2}\|u_k^n\|_\alpha^2 &\le \int_0^T \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k^n(t), {}_t^cD_T^\alpha u_k^n(t))\big]dt\\ &\le m_k+\frac{1}{n}+\int_0^T F(t,u_k^n(t))dt\\ &\le m_k+\frac{1}{n}+\int_0^T c(1+|u_k^n(t)|^{\alpha_0})dt\\ &\le m_k+\frac{1}{n}+cT+cT|b_k|^{\alpha_0},\end{aligned}\tag{3.5}$$

for all $n\in\mathbb{N}$, thus $\{u_k^n(t)\}_{n=1}^\infty$ is bounded in E^α.

By Proposition 2.5, one can easily see that there exists $\{u_k^n\}_{n=1}^\infty\in E^\alpha$ such that $u_k^n\rightharpoonup u_k$ in E^α. We will show that φ is weak lower semicontinuous. Let $u_k^n\rightharpoonup u_k$ weakly in E^α, and by Proposition 2.7, we obtain the following results:

$$\begin{gathered}E^\alpha\hookrightarrow L^p(\mathbb{R}^N),\\ u_k^n(t)\to u_k(t)\text{ a.e. } t\in[0,T],\\ F(t,u_k^n(t))\to F(t,u_k(t))\text{ a.e. } t\in[0,T].\end{gathered}$$

By Fatou's lemma,

$$\limsup_{n\to\infty}\int_0^T F(t,u_k^n(t))dt\le\int_0^T F(t,u_k(t))dt.$$

On the other hand, by Proposition 2.10, we have $\lim_{n\to\infty}J(u_k^n)=J(u_k)$; that is,

$$\lim_{n\to\infty}\int_0^T[-\frac{1}{2}({}_0^cD_t^\alpha u_k^n(t),\, {}_t^cD_T^\alpha u_k^n(t))]dt=\int_0^T[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\, {}_t^cD_T^\alpha u_k(t))]dt.$$

Thus,

$$\begin{aligned}\liminf_{n\to\infty}\varphi(u_k^n) &= \liminf_{n\to\infty}\int_0^T[-\frac{1}{2}({}_0^cD_t^\alpha u_k^n(t),\, {}_t^cD_T^\alpha u_k^n(t))]dt\\ &\quad -\limsup_{n\to\infty}\lambda\int_0^T F(t,u_k^n(t))dt\\ &\ge\int_0^T[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\, {}_t^cD_T^\alpha u_k(t))]dt-\lambda\int_0^T F(t,u_k^n(t))dt\\ &=\varphi(u_k).\end{aligned}\tag{3.6}$$

Then φ is weak lower semicontinuous, and

$$m_k\le\varphi(u_k)\le\varliminf_{n\to+\infty}\varphi(u_k^n)\le m_k+\frac{1}{n},$$

which implies $\varphi(u_k)=m_k$. Hence, u_k is a minimum point of φ over S_k.

Step 2. We show that $u_k(t)\in[0,a_k]e_k$ a.e. $t\in[0,T]$. Let $A=\{t\in[0,T]:u_k(t)\notin[0,a_k]e_k\}=\{t\in[0,T]:u_k(t)\in[a_k,b_k]e_k\}$. We will prove that $\mathrm{meas}(A)=0$. Define the function $h:[0,+\infty)e_k\to[0,+\infty)e_k$ by

$$h(s)=\begin{cases}a_ke_k, & \text{if } s\in[a_k,+\infty]e_k,\\ s, & \text{if } s\in[0,a_k]e_k.\end{cases}$$

Now, we set $v_k = h \circ u_k$. Since h is a Lipschitz function and $h(0) = 0$, the theorem of Marcus-Mizel [11] shows that $v_k \in E^\alpha$. Moreover, $v_k(t) \in [0, a_k]e_k$ for a.e. $t \in [0, T]$. Consequently, $v_k \in S_k$ and

$$v_k(t) = \begin{cases} u_k(t), & \text{if } t \in [0,T]\backslash A, \\ a_k e_k, & \text{if } t \in A. \end{cases}$$

By straightforward computations, we obtain

$$\begin{aligned}
&\varphi(v_k) - \varphi(u_k) \\
&= \int_{[0,T]} \big[-\frac{1}{2}({}_0^cD_t^\alpha v_k(t),\ {}_t^cD_T^\alpha v_k(t))\big]dt - \int_{[0,T]} F(t, v_k(t))dt \\
&\quad - \int_{[0,T]} \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\ {}_t^cD_T^\alpha u_k(t))\big]dt + \int_{[0,T]} F(t, u_k(t))dt \\
&= \int_{[0,T]\backslash A} \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\ {}_t^cD_T^\alpha u_k(t))\big]dt \\
&\quad + \int_A \big[-\frac{1}{2}({}_0^cD_t^\alpha a_k e_k,\ {}_t^cD_T^\alpha a_k e_k)\big]dt - \int_{[0,T]\backslash A} F(t, u_k(t))dt \\
&\quad - \int_A F(t, a_k e_k)dt - \int_{[0,T]\backslash A} \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\ {}_t^cD_T^\alpha u_k(t))\big]dt \\
&\quad - \int_A \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\ {}_t^cD_T^\alpha u_k(t))\big]dt + \int_{[0,T]\backslash A} F(t, u_k(t)) \\
&\quad + \int_A F(t, u_k(t))dt \\
&= -\int_A \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\ {}_t^cD_T^\alpha u_k(t))\big]dt - \int_A [F(t, a_k e_k) - F(t, u_k(t))]dt.
\end{aligned} \tag{3.7}$$

For every $t \in A$, $u_k(t) \in [a_k, b_k]e_k$, there exists a map $\lambda : A \to [0, 1]$ such that $u_k(t) = a_k e_k + \lambda(t)(b_k - a_k)e_k$.

By the Mean Value Theorem, it holds

$$\begin{aligned}
&\int_A [F(t, a_k e_k) - F(t, u_k(t))]dt \\
&= \int_A \xi_k(t) \cdot (a_k e_k - u_k(t))dt \\
&= \int_A \xi_k(t) \cdot [a_k e_k - a_k e_k - \lambda(t)(b_k - a_k)e_k]dt \\
&= \int_A \xi_k(t) \cdot \lambda(t)(a_k - b_k)e_k dt,
\end{aligned} \tag{3.8}$$

where $\xi_k(t) \in \partial F(t, \tau_k(t))$ for some $\tau_k(t) \in [a_k e_k, u_k(t)] \subseteq [a_k, b_k]e_k$ for a.e. $t \in A$.

By (H1)(5), we have $\xi_k(t) \cdot e_k \le 0$ for a.e. $t \in A$. Consequently,

$$\int_A [F(t, a_k e_k) - F(t, u_k(t))]dt \ge 0. \tag{3.9}$$

In conclusion, every term of the expression $\varphi(v_k) - \varphi(u_k) \le 0$. On the other hand, since $v_k \in S_k$, then $\varphi(v_k) \ge \varphi(u_k) = \inf_{S_k} \varphi$. So, $\varphi(v_k) - \varphi(u_k) = 0$. Namely,

$$-\int_A \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t), {}_t^cD_T^\alpha u_k(t))\big]dt - \int_A [F(t, a_k e_k) - F(t, u_k(t))]dt = 0, \tag{3.10}$$

which implies that meas$(A) = 0$.

Step 3. We show that u_k is a local minimum point in E^α for every $k \in \mathbb{N}$. Let $A' = \{t \in [0,T] : u(t) \notin [0, a_k]e_k\} = \{t \in [0,T] : u(t) \in (a_k, b_k]e_k\}$. Set $v = h \circ u$, then we have

$$
\begin{aligned}
&\varphi(u) - \varphi(v) \\
&= \int_{[0,T]} \big[- \frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big] dt - \int_{[0,T]} F(t,u(t))dt \\
&\quad - \int_{[0,T]} [-\frac{1}{2}({}_0^cD_t^\alpha v(t),\ {}_t^cD_T^\alpha v(t))] dt + \int_{[0,T]} F(t,v(t))dt \\
&= \int_{[0,T]\setminus A'} \big[- \frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big] dt \\
&\quad + \int_{A'} \big[- \frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big] dt \\
&\quad - \int_{[0,T]\setminus A'} F(t,u(t))dt - \int_{A'} F(t,u(t))dt \\
&\quad - \int_{[0,T]\setminus A'} \big[- \frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big] dt \\
&\quad - \int_{A'} \big[- \frac{1}{2}({}_0^cD_t^\alpha a_ke_k,\ {}_t^cD_T^\alpha a_ke_k)\big] dt \\
&\quad + \int_{[0,T]\setminus A'} F(t,u(t)) + \int_{A'} F(t,a_ke_k)dt \\
&= \int_{A'} \big[- \frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big] dt + \int_{A'} [F(t,a_ke_k) - F(t,u(t))]dt.
\end{aligned} \tag{3.11}
$$

From assumption (H1)(5), we have

$$
\int_{A'} [F(t,a_ke_k) - F(t,u(t))]dt = \int_{A'} \xi_k(t) \cdot (a_ke_k - u(t))dt \geq 0, \tag{3.12}
$$

for a.e. $t \in A'$, where $\xi_k(t) \in \partial F(t, \tau(t))$, $\tau(t) \in [a_ke_k, u(t)] \subseteq [a_k, b_k]e_k$, a.e. $t \in A'$. Consequently,

$$
\varphi(u) - \varphi(v) \geq 0. \tag{3.13}
$$

On the other hand, by $v \in S_k$, we have

$$
\varphi(v) \geq \varphi(u_k). \tag{3.14}
$$

In view of (3.11), we derive

$$
\varphi(u) - \varphi(v) \geq \int_{A'} \big[- \frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big] dt. \tag{3.15}
$$

Moreover, we have

$$
\begin{aligned}
\varphi(u) &\geq \varphi(v) + \int_{A'} \big[-\frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big]\,dt \\
&\geq \varphi(u_k) + \int_{A'} \big[-\frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big]\,dt \\
&\geq \varphi(u_k) + \int_{[0,T]} \big[-\frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big]\,dt \\
&\quad - \int_{[0,T]\setminus A'} \big[-\frac{1}{2}({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t))\big]\,dt \\
&\geq \varphi(u_k) + \int_{[0,T]} \big[-\frac{1}{2}({}_0^cD_t^\alpha (u(t)-v(t)),\ {}_t^cD_T^\alpha (u(t)-v(t))\big]\,dt \\
&\geq \varphi(u_k) + \frac{|\cos(\pi\alpha)|}{2}\|u-v\|_\alpha^2 .
\end{aligned}
\tag{3.16}
$$

Since h is continuous, there exists $\delta > 0$ such that, for every $u \in E^\alpha$ with $\|u - v\|_\alpha < \delta$, which implies that u_k is a local minimum of φ.

Step 4. We prove that $m_k = \inf_{S_k} \varphi < 0$ and $\lim_{k\to+\infty} m_k = 0$. Let $B_{r_0}(t_0) \subset [0,T]$ be the ball with radius $r_0 \in (0,1)$ and center $t_0 \in [0,T]$. For $\xi \in \mathbb{R}^N$, define

$$
\eta_\xi(t) = \begin{cases} 0, & \text{if } t \in [0,T]\setminus B_{r_0}(t_0), \\ \xi, & \text{if } t \in B_{\frac{r_0}{2}}(t_0), \\ \frac{2\xi}{r_0}(r_0 - |t-t_0|), & \text{if } t \in B_{r_0}(t_0)\setminus B_{\frac{r_0}{2}}(t_0). \end{cases}
\tag{3.17}
$$

It is clear that $\eta_\xi \in E^\alpha$ and

$$
|\eta_\xi(t)| \leq \frac{2|\xi|}{r_0},
\tag{3.18}
$$

$$
\begin{aligned}
|{}_0^cD_t^\alpha \eta_\xi(t)| &= \big|\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\eta_\xi' ds\big| \\
&\leq \frac{1}{\Gamma(1-\alpha)}\Big(\int_0^t (t-s)^{-\alpha}|\eta_\xi'| ds\Big) \\
&\leq \frac{1}{\Gamma(1-\alpha)}\frac{2|\xi|}{r_0}\frac{t^{1-\alpha}}{1-\alpha}ds,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\|\eta_\xi\|_\alpha^2 &= \int_0^T |{}_0^cD_t^\alpha \eta_\xi(t)|^2 dt \\
&\leq \int_0^T \frac{1}{\Gamma^2(1-\alpha)}\frac{4|\xi|^2}{r_0^2}\frac{t^{2-2\alpha}}{(1-\alpha)^2}dt \\
&\leq \frac{1}{\Gamma^2(1-\alpha)}\frac{4\xi^2}{r_0^2}\frac{1}{(1-\alpha)^2}\int_0^T t^{2-2\alpha}dt \\
&\leq \frac{4|\xi|^2}{\Gamma^2(1-\alpha)r_0^2(1-\alpha)^2(3-2\alpha)}T^{3-2\alpha}.
\end{aligned}
\tag{3.20}
$$

From the left part of (H1)(4) we deduce that the existence of some $l_0 > 0$ and $\lambda_0 \in [0, a_k]e_k$, such that

$$
\operatorname{ess\,inf}_{t\in[0,T]} F(t,x) \geq -l_0|x|^2 \quad \text{for all } x \in [0,\lambda_0]e_k.
\tag{3.21}
$$

There exist $L_0 > 0$ large enough to enable

$$C(r_0, \alpha, T) + l_0 T < \frac{1}{3} L_0 r_0, \quad C(r_0, \alpha, T) = \frac{1}{2|\cos(\pi\alpha)|} \frac{4T^{3-2\alpha}}{\Gamma^2(1-\alpha) r_0^2 (3-2\alpha)}. \tag{3.22}$$

Taking into account the right part of (H1)(4), there is a sequence $\{\xi_k\} \in [0, \lambda_0]$ such that $\{\xi_k\} \in [0, a_k] e_k$ and

$$\operatorname{ess\,sup}_{t\in[0,T]} F(t, \xi_k) > L_0 |\xi_k|^2 \quad \text{for all } k \in N. \tag{3.23}$$

Note that $\frac{2\xi_k}{r_0}(r_0 - |t - t_0|) \in [0, \xi_k] \subset [0, \lambda_0] e_k$, for every $t \in B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0)$, because of $|t - t_0| \in (\frac{r_0}{2}, r_0)$ and $r_0 - |t - t_0| \in (0, \frac{r_0}{2})$, $\forall t \in B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0)$.

In view of proposition 2.9 and (3.20), we deduce

$$\begin{aligned} &\int_0^T \big[-\frac{1}{2} ({}_0^c D_t^\alpha \eta_{\xi_k}(t),\ {}_t^c D_T^\alpha \eta_{\xi_k}(t)) \big] dt \\ &\le \frac{1}{2|\cos(\pi\alpha)|} \|\eta_{\xi_k}(t)\|_\alpha^2 \\ &\le \frac{1}{2|\cos(\pi\alpha)|} \frac{4T^{3-2\alpha}}{\Gamma^2(1-\alpha) r_0^2 (3-2\alpha)} |\xi_k|^2 \\ &= C(r_0, \alpha, T) |\xi_k|^2, \end{aligned} \tag{3.24}$$

And combining (3.21) with (3.23), we obtain

$$\begin{aligned} &\int_0^T F(t, \eta_\xi(t)) dt \\ &= \int_{B_{\frac{r_0}{2}}(t_0)} F(t, \eta_{\xi_k}(t)) dt + \int_{B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0)} F(t, \eta_{\xi_k}(t)) dt \\ &\ge \int_{B_{\frac{r_0}{2}}(t_0)} F(t, \xi_k(t)) dt + \int_{B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0)} F(t, \frac{2\xi_k}{r_0}(r_0 - |t - t_0|)) dt \\ &\ge \int_{B_{\frac{r_0}{2}}(t_0)} -l_0 |\xi_k|^2 dt + \int_{B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0)} L_0 |\frac{2\xi_k}{r_0}(r_0 - |t - t_0|)|^2 dt \\ &= L_0 \frac{4|\xi_k|^2}{r_0^2} [\int_{t_0 - r_0}^{t_0 - \frac{r_0}{2}} (r_0 - |t - t_0|)^2 dt + \int_{t_0 + \frac{r_0}{2}}^{t_0 + r_0} (r_0 - |t - t_0|)^2] - l_0 r_0 |\xi_k|^2 \\ &= L_0 \frac{4|\xi_k|^2}{r_0^2} [\int_{t_0 - r_0}^{t_0 - \frac{r_0}{2}} (r_0 + t - t_0)^2 dt + \int_{t_0 + \frac{r_0}{2}}^{t_0 + r_0} (r_0 - t + t_0)^2] - l_0 r_0 |\xi_k|^2 \\ &\ge \frac{1}{3} L_0 r_0 |\xi_k|^2 - l_0 T |\xi_k|^2. \end{aligned} \tag{3.25}$$

Let $k \in \mathbb{N}$ be a fixed number and let $\eta_{\xi_k} \in E^\alpha$ be the function from (3.17) corresponding to the value $|\xi_k| > 0$. Then $\eta_{\xi_k} \in S_k$, and on account of (3.22), (3.24) and (3.25), one has

$$
\begin{aligned}
\varphi(\eta_{\xi_k}) &= \int_0^T \big[-\frac{1}{2}({}_0^cD_t^\alpha\eta_{\xi_k}(t),\ {}_t^cD_T^\alpha\eta_{\xi_k}(t))\big]dt - \int_0^T F(t,\eta_\xi(t))dt \\
&\le C(r_0,\alpha,T)|\xi_k|^2 - \frac{1}{3}L_0r_0|\xi_k|^2 + l_0T|\xi_k|^2 \\
&\le (C(r_0,\alpha,T) + l_0T - \frac{1}{3}L_0r_0)|\xi_k|^2 < 0.
\end{aligned} \tag{3.26}
$$

From Step 3 and (3.26), we deduce

$$
m_k = \varphi(u_k) = \inf_{S_k}\varphi \le \varphi(\eta_{\xi_k}) < 0. \tag{3.27}
$$

Now we prove that $\lim_{k\to+\infty} m_k = 0$. Observe that by assumption (H1)(3), one can find a positive constant c and $\omega \in \partial F(t,x)$ such that

$$
|\omega| \le c(1+|x|^{\alpha_0}), \quad \forall t\in[0,T], x\in\mathbb{R}^N. \tag{3.28}
$$

where $\alpha_1 = \max_{t\in[0,T]}\alpha(t)$.

Applying the Mean Value Theorem and Step 1, for every $x \in [0,a_k]e_k$ and all $t\in[0,T]$, there exists a constant $c>0$ such that

$$
|F(t,x)| = |F(t,x) - F(t,0)| \le c(1+|x|^{\alpha_1}). \tag{3.29}
$$

Therefore

$$
\begin{aligned}
m_k &= \varphi(u_k) \\
&= \int_0^T \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),{}_t^c D_T^\alpha u_k(t))\big]dt - \int_0^T F(t,u_k(t))dt \\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u_k\|_\alpha^2 - \int_0^T F(t,u_k(t))dt \\
&\ge -\int_0^T F(t,u_k(t))dt \\
&\ge -\int_0^T \big[c|u_k(t)| + c|u_k(t)|^{\alpha_1}\big]dt \\
&\ge -cT(|b_k| + |b_k|^{\alpha_1}).
\end{aligned} \tag{3.30}
$$

Since $\lim_{k\to+\infty} b_k = 0$, we have $\lim_{k\to+\infty} m_k \ge 0$. Note that $m_k<0$, hence $\lim_{k\to+\infty} m_k = 0$.

Finally, since u_k are local minima of φ, they are critical points of φ, thus weak solutions of (1.1). Due to Step 2, there are infinitely many distinct u_k with $\lim_{k\to+\infty}|u_k|_\infty = 0$. Moreover, we have

$$
\begin{aligned}
\frac{|\cos(\pi\alpha)|}{2}\|u_k\|_\alpha^2 &\le \int_0^T \big[-\frac{1}{2}({}_0^cD_t^\alpha u_k(t),\ {}_t^cD_T^\alpha u_k(t))\big]dt \\
&= m_k + \int_0^T F(t,u_k(t))dt \\
&\le m_k + cT(|b_k| + |b_k|^{\alpha_1}),
\end{aligned} \tag{3.31}
$$

which means that $\lim_{k\to+\infty}\|u_k\|_\alpha = 0$. □

Next, we will state the counterpart of Theorem 3.3 when the nonlinearity oscillates at infinity. The hypotheses on the nonsmooth potential $F(x,t)$ are the following:

Our hypotheses on nonsmooth potential $F(x,t)$ are as follows.

(H2) $F:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ is a function, $F(t,0)=0$ for almost all $t\in[0,T]$ and satisfies the following facts:

(1) For all $x\in\mathbb{R}^N$, $t\mapsto F(t,x)$ is measurable;
(2) For almost all $t\in[0,T]$, $x\mapsto F(t,x)$ is locally Lipschitz;
(3) There exist a positive constant c such that for almost all $x\in\mathbb{R}^N$, all $t\in[0,T]$ and $\omega\in\partial F(t,x)$
$$|\omega|\le c(1+|x|^{\alpha(t)-1})$$
where $1<\alpha(t)<+\infty$;
(4)
$$-\infty<\liminf_{|x|\to+\infty}\frac{F(t,x)}{|x|^2}\le\limsup_{|x|\to+\infty}\frac{F(t,x)}{|x|^2}=+\infty$$
uniformly for a.e. $x\in\mathbb{R}^N$;
(5) For every $k\in\mathbb{N}$, there exists $e_k\in\mathbb{R}^N$ with $|e_k|=1$ and there are two sequences $\{a_k\}$ and $\{b_k\}$ in $(0,+\infty)$ with $a_k<b_k$, $\lim_{k\to+\infty}b_k=0$ such that
$$\sup\{\omega\cdot e_k:\omega\in\partial F(t,x),\ \text{a.e. } t\in[0,T],\ x\in[a_k,b_k]e_k\}\le 0.$$

Remark 3.4. Hypotheses (H2)(4) and (H2)(5) imply an oscillatory behaviour of F near the infinity.

Remark 3.5. A simple example of a nonsmooth potential function satisfying (H2) is
$$F(t,x)=\begin{cases}|x|^{\alpha(t)}\sin|x|, & \text{if } |x|\in\big[2k\pi,(2k+1)\pi\big),\\ |x|^{\beta(t)}\sin|x|, & \text{if } |x|\in\big[(2k+1)\pi,(2k+2)\pi\big],\end{cases}$$
where $k\in N$ with $k\ge1$, $1<\beta(t)<2<\alpha(t)<\infty$.

Proof. Obviously, Hypothesis (H2)(1) and (H2)(2) are satisfied. Clearly, $x\mapsto F(t,x)$ is locally Lipschitz. Then for any $1\le k\in N$,
$$\partial F(t,x)$$
$$=\begin{cases}\alpha(t)|x|^{\alpha(t)-2}x\sin|x|+|x|^{\alpha(t)-1}x\cos|x|, & \text{if } |x|\in\big(2k\pi,(2k+1)\pi\big),\\ \beta(t)|x|^{\beta(t)-2}x\sin|x|+|x|^{\beta(t)-1}x\cos|x|, & \text{if } |x|\in\big((2k+1)\pi,(2k+2)\pi\big),\\ [-x|x|^{\alpha(t)-1},-x|x|^{\beta(t)-1}\}, & \text{if } |x|=(2k+1)\pi,\\ [x|x|^{\alpha(t)-1},x|x|^{\beta(t)-1}\}, & \text{if } |x|=2k\pi,\end{cases}$$
where $\{\gamma,\delta\}=\{\xi:\xi=\lambda\gamma+(1-\lambda)\delta,\lambda\in[0,1]\}$. Then, there exists a constant $c>0$ and $\theta(t)=\alpha(t)+1$ such that
$$|w|\le c(1+|x|^{\theta(t)-1})\quad\text{for all } w\in\partial F(t,x).$$
So condition (H2)(3) holds. Then, for any $1\le k\in N$, we can choose
$$a_k:=(2k+1)\pi,\quad b_k:=(2k+\frac{3}{2})\pi,$$
which implies $a_k<b_k$, $\lim_{k\to+\infty}a_k=+\infty$ and
$$\sup\{w\cdot e_k:w\in\partial F(x,t),\ \text{a.e. } t\in[0,T] \text{ and } x\in[a_k,b_k]e_k\}\le 0.$$
So condition (H2)(5) is satisfied.

On the other hand, for any $1 \le k \in N$, we can choose $c_k := (2k + \frac{1}{2})\pi$, which means $\lim_{k\to+\infty} c_k = +\infty$,

$$\limsup_{k\to+\infty} \frac{F(t, c_k e_k)}{|c_k|^2} = \limsup_{k\to+\infty} |c_k|^{\alpha(t)-2} \sin|c_k| = \limsup_{k\to+\infty} |c_k|^{\alpha(t)-2} = +\infty,$$

$$-\infty < 1 \le \liminf_{|x|\to+\infty} \frac{F(t,x)}{|x|^2} = \liminf_{|x|\to+\infty} \frac{|x|^{\beta(t)} \sin|x|}{|x|^2} = \liminf_{|x|\to+\infty} |x|^{\beta(t)-2} \sin|x| \le 0$$

uniformly for a.e. $t \in [0, T]$. So condition (H2)(4) holds. $\square$

Theorem 3.6. *Suppose that (H2) holds. Then there exists a sequence $\{u_n\} \subset E^\alpha$ of distinct positive solution of problem (1.1) such that*

$$\lim_{n\to+\infty} \|u_n\|_\alpha = \lim_{n\to+\infty} |u_n|_\infty = +\infty.$$

Proof. For every fixed $k \in \mathbb{N}$, consider the set

$$T_k = \{u \in E^\alpha : u(x) \neq 0 \text{ and } u(x) \in [0, b_k]e_k \text{ a.e. } x \in \mathbb{R}^N\},$$

where b_k is from (H2)(5). The first part of the proof is similar to that of Theorem 3.3. Indeed, we can prove that the functional φ is bounded from below on T_k and its infimum on T_k is attained (see Step 1 of Theorem 3.3). Moreover, if $u_k \in T_k$ is chosen such that $\varphi(u_k) = \inf_{T_k}$, then $u_k(t) \in [0, a_k]e_k$ a.e. $t \in [0, T]$ (see Step 2 of Theorem 3.3), and u_k is a local minimum point of φ in E^α (see Step 3 of Theorem 3.3). Instead of Step 4, we prove

Step 4. Let $\vartheta_k = \inf_{T_k} \varphi = \varphi(u_k)$, then $\lim_{k\to+\infty} \vartheta_k = -\infty$. From (H2)(4), we deduce that there exist $l_\infty > 0$ and $\lambda_\infty > 0$ such that

$$\operatorname{ess\,inf}_{t\in[0,T]} F(t,x) \ge -l_\infty |x|^2 \quad \text{for all } |x| > \lambda_\infty. \tag{3.32}$$

There exist $L_\infty > 0$ be large enough to enable

$$C(r_0, \alpha, T) + l_\infty T < L_\infty r_0. \tag{3.33}$$

From the right hand side of (H2)(4), we deduce that there is a sequence $\{\xi_k\} \subset \mathbb{R}^N$ such that $\lim_{k\to+\infty} |\xi_k| = +\infty$, and

$$\operatorname{ess\,inf}_{t\in[0,T]} F(t, \xi_k) > L_\infty |\xi_k|^2 \quad \text{for all } k \in \mathbb{N}. \tag{3.34}$$

It is easy to see that

$$|\eta_{\xi_k}(t)| \le |\xi_k|, \quad \forall t \in B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0), \tag{3.35}$$

since

$$\eta_{\xi_k}(t) = \frac{2\xi_k}{r_0}(r_0 - |t - t_0|), \ \forall\, t \in B_{r_0}(t_0) \backslash B_{\frac{r_0}{2}}(t_0).$$

Let $k \in \mathbb{N}$ be fixed and let $\eta_{\xi_k} \in E^\alpha$ be the function from (3.17) corresponding to the value $\xi_k \in \mathbb{R}^N$. Then $\eta_{\xi_k} \in T_{b_k}$, and on account of (3.32) and (3.34), we have

$$
\begin{aligned}
\varphi(\eta_{\xi_k}) &= \int_0^T \big[-\frac{1}{2}({}_0^cD_t^\alpha \eta_{\xi_k}(t),\ {}_t^cD_T^\alpha \eta_{\xi_k}(t))\big]dt - \int_0^T F(t,\eta_\xi(t))dt \\
&\le \frac{1}{2|\cos(\pi\alpha)|}\|\eta_{\xi_k}\|_\alpha^2 - \int_{B_{\frac{r_0}{2}}(t_0)} F(t,\eta_{\xi_k}(t))dt \\
&\quad - \int_{(B_{r_0}(t_0)\setminus B_{\frac{r_0}{2}}(t_0))\bigcap\{t:|\eta_{\xi_k}(t)|>\lambda_\infty\}} F(t,\eta_{\xi_k}(t))dt \\
&\quad - \int_{(B_{r_0}(t_0)\setminus B_{\frac{r_0}{2}}(t_0))\bigcap\{t:|\eta_{\xi_k}(t)|\le\lambda_\infty\}} F(t,\eta_{\xi_k}(t))dt \\
&\le \frac{1}{2|\cos(\pi\alpha)|}\frac{4T^(3-2\alpha)}{\Gamma^2(1-\alpha)r_0^2(3-2\alpha)}|\xi_k|^2 - \frac{1}{3}L_\infty r_0|\xi_k|^2 \\
&\quad + l_\infty T|\xi_k|^2 - cT(1+\lambda_\infty^{\alpha_0}) \\
&= (C(r_0,\alpha,T) - L_\infty r_0 + l_\infty T)|\xi_k|^2 + cT\lambda_\infty^{\alpha_1}.
\end{aligned}
\tag{3.36}
$$

From (3.33), (3.36) and $\lim_{k\to+\infty}|\xi_k| = +\infty$, we conclude that

$$
\lim_{k\to+\infty}\varphi(\eta_{\xi_k}) = -\infty. \tag{3.37}
$$

On the other hand, from $\varphi(u_{m_k}) = \min_{T_{b_{m_k}}}\varphi$, we have

$$
\varphi(u_{m_k}) \le \varphi(\eta_{\xi_k}(t)).
$$

On account of (3.37), we have

$$
\lim_{k\to+\infty}\varphi(u_{m_k}) = -\infty. \tag{3.38}
$$

Since the sequence $\{\varphi(u_k)\}$ is non-increasing, so, we have

$$
\lim_{k\to+\infty}\vartheta_k = \lim_{k\to+\infty}\varphi(u_k) = -\infty.
$$

Step 5. We prove that

$$
\lim_{k\to+\infty}|u_k|_\infty = \lim_{k\to+\infty}\|u_k\|_\alpha = +\infty\,.
$$

Arguing by contradiction, assume that there exists a subsequence $\{u_{n_k}\}$ of $\{u_k\}$ such that $|u_{n_k}|_\infty \le M$ for some $M > 0$. In particular, $\{u_{n_k}\}\subset T_{b_l}$ for some $l\in\mathbb{N}$. Thus, for every $n_k > l$, we have

$$
\vartheta_l \ge \vartheta_{n_k} = \inf_{T_{n_k}}\varphi = \varphi(u_{n_k}) \ge \inf_{T_l}\varphi = \vartheta_l. \tag{3.39}
$$

So, $\vartheta_{n_k} = \vartheta_l$ for every $n_k > l$. This fact contradicts with (3.38) which completes the first part of the proof.

Next, we prove that $\lim_{k\to+\infty}\|u_k\|_\alpha = +\infty$. Note that $1 < \alpha_1 < +\infty$, then by Proposition 2.7, we have $E^\alpha \hookrightarrow C([0,T],\mathbb{R}^N)$ (compact embedding). Furthermore, there exists $c_1 > 0$ such that $|u_k|_\infty \le c_1\|u_k\|_\alpha$. Hence, there exists a constant

$c_2 > 0$, such that

$$\begin{aligned} \int_0^T F(t, u_k(t))dt &\leq \int_0^T c(1 + |u_k(t)|^{\alpha_1})dt \\ &\leq cT + c|u_k(t)|_\infty^{\alpha_1} T \\ &\leq cT + cc_1^{\alpha_1} \|u_k\|_\alpha^{\alpha_1} T \\ &\leq cT + c_2 \|u_k\|_\alpha^{\alpha_1} T. \end{aligned} \tag{3.40}$$

Let us assume that there exists a subsequence $\{u_{n_k}\}$ of $\{u_k\}$ such that for some $M > 0$, we have $\|u_{n_k}\|_\alpha \leq M$. In particular, by the above inequality,

$$\begin{aligned} |\varphi(u_{n_k})| &= \Big| \int_0^T \big[-\frac{1}{2} ({}_0^cD_t^\alpha u_{n_k}(t),\ {}_t^cD_T^\alpha u_{n_k}(t)) \big] dt - \int_0^T F(t, u_{n_k}(t))dt \Big| \\ &\leq \int_0^T \big[-\frac{1}{2} ({}_0^cD_t^\alpha u_{n_k}(t),\ {}_t^cD_T^\alpha u_{n_k}(t)) \big] dt + \Big| \int_0^T F(t, u_{n_k}(t))dt \Big| \\ &\leq \frac{1}{2|\cos(\pi\alpha)|} \|u_{n_k}\|_\alpha^2 + cT + c_2 \|u_k\|_\alpha T \end{aligned} \tag{3.41}$$

is bounded. Hence $\vartheta_{n_k} = \varphi(u_{n_k})$ is also bounded. This fact contradicts with $\lim_{k\to+\infty} \vartheta_k = -\infty$. $\square$

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