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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR SOME NONLINEAR INTEGRAL EQUATIONS ON AN UNBOUNDED INTERVAL

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Abstract. The goal in this paper is to prove an existence theorem for the solutions of a class of functional integral equations which contain a number of classical nonlinear integral equations as special cases. Our investigations will be carried out in the space of continuous and bounded functions on an unbounded interval. The main tools are some techniques in analysis and the Schauder fixed point theorem via measures of noncompactness. Our results extend and improve some known results in the recent literature. Three nontrivial examples explain the generalizations and applications of our results.

1. Introduction

It is well known that the theory of nonlinear integral equations of various types appears in many applications that arise in the fields of mathematical analysis, nonlinear functional analysis, mathematical physics, and engineering (see [6, 7, 12]). There has been a significant development in ordinary and partial fractional differential and integral equations in recent years.

Agarwal and O'Regan [1] gave the existence of solutions for the nonlinear integral equation

$$x(t) = \int_0^\infty k(t,s) f(s, x(s)) ds, \quad t \in \mathbb{R}^+, \tag{1.1}$$

in the space of bounded and continuous functions $\mathrm{C}_l[0,\infty)$ which have limit at infinity.

Meehan and O'Regan [13, 14] discussed the existence of solutions for the nonlinear integral equation

$$x(t) = h(t) + \mu \int_0^\infty k(t,s) f(s, x(s)) ds, \quad t \in \mathbb{R}^+, \tag{1.2}$$

in the space $\mathrm{C}_l[0,\infty)$ and the existence of solutions for the nonlinear integral equation

$$x(t) = h(t) + \int_0^\infty k(t,s)[f(x(s)) + g(x(s))] ds, \quad t \in \mathbb{R}^+, \tag{1.3}$$

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in the space the space $BC(\mathbb{R}^+,\mathbb{R})$ of bounded and continuous functions on $\mathbb{R}^+$. Later in [15] they established the existence of at least one positive solution for the nonlinear integral equation

$$x(t) = h(t) + \int_0^\infty k(t,s)f(s,x(s))ds, \quad t \in \mathbb{R}^+, \tag{1.4}$$

in the space $L^p(\mathbb{R}^+)$, in 2001.

In 2004, Banaś and Poludniak [3] investigated the monotonic solutions for the nonlinear integral equation

$$x(t) = f(t) + \int_0^\infty u(t,s,x(s))ds, \quad t \in \mathbb{R}^+, \tag{1.5}$$

in the space of Lebesque integrable functions on the halfaxis $\mathbb{R}^+$ by using the Darbo fixed point theorem and the measure of noncompactness (both in strong and weak sense).

Banaś and Martin [4] studied the existence and asymptotic stability of solutions for the nonlinear integral equation

$$x(t) = g(t) + f(t,x(t)) \int_0^\infty K(t,s)h(s,x(s))ds, \quad t \in \mathbb{R}^+, \tag{1.6}$$

in the Banach space $BC(\mathbb{R}^+,\mathbb{R})$, in 2006.

In 2004, Cabellaro and others [8], in 2008, Banaś and Olszowy [5] and more recently in 2013, Darwish and others [9] studied the existence of solutions for the Urysohn integral equation defined on an unbounded interval

$$x(t) = a(t) + f(t,x(t)) \int_0^\infty u(t,s,x(s))ds, \quad t \in \mathbb{R}^+, \tag{1.7}$$

in the space $BC(\mathbb{R}^+,\mathbb{R})$.

In those papers, different conditions and various measures of noncompactness were applied in proving existence theorems.

Olszowy [16, 17, 18] studied (1.7) in the Frèchet space of real functions being defined and continuous on $\mathbb{R}^+$ and has given results about monotonocity of the solutions of the integral equation (1.7).

In 2010, Karoui and others [10] studied (1.7) in the space $L^p(\mathbb{R}^+)$ by means of Schauder's fixed point theorem. Recently, Khosravi and others [11] studied the existence of solutions for nonlinear functional integral equations of convolution type

$$x(t) = f(t,x(t)) + \int_0^\infty k(t-s)(Qx)(s)ds, \quad t \in \mathbb{R}^+, \tag{1.8}$$

in the space $L^p(\mathbb{R}^+)$, in 2015.

Motivated by recent researches in this field, we study the more general nonlinear integral equation

$$x(t) = (T_1x)(t) + (T_2x)(t) \int_0^\infty u(t,s,x(s))ds, \quad t \in \mathbb{R}^+, \tag{1.9}$$

where the functions $u(t,s,x)$ and the operators T_ix, $(i=1,2)$ appearing in (1.9) are given, while $x = x(t)$ is an unknown function. Using the technique of a suitable measure of noncompactness, we prove an existence theorem for (1.9). We give three nontrivial examples that explain the generalizations and applications of our results. So our work improves directly results obtained in [3, 4, 8, 9, 10] and completes some results mentioned before. To the best of our knowledge, (1.9) is more general than those investigated up to now and includes (1.2)-(1.8) as special cases.

2. Auxiliary facts and notation

In this section, we give a collection of auxiliary facts which will be needed further on. Assume that $(E, \|\cdot\|)$ is a real Banach space with zero element θ. Let $B(x,r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, then $\overline{X}$ and $\operatorname{conv} X$ denote the closure and convex closure of X, respectively. With the symbols λX and $X+Y$, we denote the standard algebraic operations on sets. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact subsets. The definition of the concept of a measure of noncompactness presented below comes from [2].

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}^+ = [0,\infty)$ is said to be a measure of noncompactness in E if it satisfies following conditions:

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(\operatorname{conv} X) = \mu(X)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0,1]$.
(5) If $\{X_n\}$ is a sequence of nonempty, bounded, closed subsets of the set E such that $X_{n+1} \subset X_n, (n = 1, 2, \dots)$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

Notice that the intersection set X_∞ from 5 belongs to $\ker \mu$. In fact, from the inequality $\mu(X_\infty) \le \mu(X_n)$ for any $n = 1, 2, \dots$ we have that $\mu(X_\infty) = 0$. This property of the set X_∞ will be crucial later.

In the sequel, we will work in the Banach space $BC(\mathbb{R}^+, \mathbb{R})$ which is equipped with the standard norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}^+\}$.

We will use a measure of noncompactness in the space $BC(\mathbb{R}^+, \mathbb{R})$. To define this measure let us fix a nonempty and bounded subset X of $BC(\mathbb{R}^+, \mathbb{R})$. For $x \in X$, $\varepsilon \ge 0$ and $L > 0$, we denoted by the modulus of continuity for function x, as

$$w^L(x,\varepsilon) = \sup\{|x(s) - x(t)| : t, s \in [0, L] \text{ and } |t-s| \le \varepsilon\}.$$

Further let us put

$$\begin{aligned} w^L(X,\varepsilon) &= \sup\{w^L(x,\varepsilon) : x \in X\}, \\ w_0^L(X) &= \lim_{\varepsilon\to 0} w^L(X,\varepsilon), \\ w_0(X) &= \lim_{L\to\infty} w_0^L(X). \end{aligned} \tag{2.1}$$

Next we define

$$\begin{aligned} \beta^L(x) &= \sup\{|x(t)| : t \ge L\}, \\ \beta^L(X) &= \sup\{\beta^L(x) : x \in X\}, \\ \beta(X) &= \lim_{L\to\infty} \beta^L(X). \end{aligned} \tag{2.2}$$

Finally, let us define the function μ as

$$\mu(X) = w_0(X) + \beta(X).$$

It is shown in [2] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}^+, \mathbb{R})$. Moreover, the kernel $\ker\mu$ of this measure contains nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}^+$ and tend to zero at infinity uniformly with respect to the set X; i.e., for each $\varepsilon \geq 0$ there exists $L > 0$ with the property that

$$|x(t)| \leq \varepsilon; \quad \text{for all } x \in X \text{ and } t \text{ with } t \geq L.$$

This property of $\ker\mu$ will be important in our further study.

3. Main Result

We shall study the existence of solutions to (1.9) assuming that following conditions are satisfied:

- (i) The operators $T_i : BC(\mathbb{R}^+, \mathbb{R}) \to BC(\mathbb{R}^+, \mathbb{R})$ are continuous and there exist continuous nondecreasing functions $d_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$|(T_i x)(t)| \leq d_i(\|x\|)$$
for all $x \in BC(\mathbb{R}^+, \mathbb{R})$ and $t \in \mathbb{R}^+$, $(i = 1, 2)$.
- (ii) $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist a continuous function $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and a continuous nondecreasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$|u(t, s, x)| \leq g(t, s)h(|x|)$$
for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$.
- (iii) For every $t \geq 0$ the function $s \to g(t, s)$ is integrable on $\mathbb{R}^+$ and the function $t \to \int_0^\infty g(t, s)ds$ is bounded on $\mathbb{R}^+$.
- (iv) The inequality $d_1(r) + d_2(r)h(r)G \leq r$ has a positive solution r_0, where $G = \sup\{\int_0^\infty g(t, s)ds : t \geq 0\} < \infty$.
- (v) There exist the nonnegative constants k_i and m_i for r_0 such that the inequalities
$$\omega_0(T_i X) \leq k_i \omega_0(X),$$
$$\beta(T_i X) \leq m_i \beta(X)$$
hold for all nonempty and bounded subset X of the ball B_{r_0}, $(i = 1, 2)$, where w_0 and β are defined by (2.1) and (2.2).
- (vi) $\max\big\{k_1 + k_2 G h(r_0), m_1 + m_2 G h(r_0)\big\} < 1$.
- (vii) $\lim_{L\to\infty} \big\{\sup\{\int_L^\infty g(t, s)ds : t \in [0, L]\}\big\} = 0$.

Now we can formulate an existence result concerning the functional integral equation (1.9).

Theorem 3.1. *Under assumptions* (i)–(vii)*, there exists at least one solution $x = x(t)$ of* (1.9) *in the space $BC(\mathbb{R}^+, \mathbb{R})$ such that $x(t) \to 0$ as $t \to \infty$.*

Proof. We define an operator F on $BC(\mathbb{R}^+, \mathbb{R})$ as follows

$$(Fx)(t) = (T_1 x)(t) + (T_2 x)(t) \int_0^\infty u(t, s, x(s))ds.$$

Notice that in view of assumptions (i) and (ii) the function $t \to (Fx)(t)$ is well defined on the interval $\mathbb{R}^+$. At first we show that the function Fx is continuous on

$\mathbb{R}^+$. To do this fix arbitrarily $L > 0$ and $\varepsilon \geq 0$. Take arbitrary numbers $t, t_0 \in [0, L]$ with $|t - t_0| \leq \varepsilon$. Then in view of assumptions we obtain

$$
\begin{aligned}
&|(Fx)(t) - (Fx)(t_0)| \\
&= \Big|(T_1x)(t) + (T_2x)(t) \int_0^\infty u(t,s,x(s))ds - (T_1x)(t_0) \\
&\quad - (T_2x)(t_0) \int_0^\infty u(t_0,s,x(s))ds\Big| \\
&\leq |(T_1x)(t) - (T_1x)(t_0)| + |(T_2x)(t) - (T_2x)(t_0)| \Big|\int_0^\infty u(t,s,x(s))ds\Big| \\
&\quad + |(T_2x)(t_0)| \Big|\int_0^\infty [u(t,s,x(s)) - u(t_0,s,x(s))]ds\Big| \\
&\leq \omega^L(T_1x,\varepsilon) + \omega^L(T_2x,\varepsilon) \int_0^\infty |u(t,s,x(s))|ds \\
&\quad + d_2(\|x\|) \int_0^\infty |u(t,s,x(s)) - u(t_0,s,x(s))|ds \\
&\leq \omega^L(T_1x,\varepsilon) + \omega^L(T_2x,\varepsilon) \int_0^\infty g(t,s)h(|x(s)|)ds \\
&\quad + d_2(\|x\|)\Big\{ \int_0^L |u(t,s,x(s)) - u(t_0,s,x(s))|ds \\
&\quad + \int_L^\infty |u(t,s,x(s)) - u(t_0,s,x(s))|ds\Big\} \\
&\leq \omega^L(T_1x,\varepsilon) + \omega^L(T_2x,\varepsilon)Gh(\|x\|) + d_2(\|x\|)L\omega^L_{\|x\|}(u,\varepsilon) \\
&\quad + 2d_2(\|x\|)h(\|x\|) \sup\Big\{ \int_L^\infty g(t,s)ds : t \in [0,L]\Big\},
\end{aligned}
\tag{3.1}
$$

where

$$
\omega^L(T_ix,\varepsilon) = \sup\{|(T_ix)(s) - (T_ix)(t)| : t,s \in [0,L] \text{ and } |t-s| \leq \varepsilon\}
$$

for $i = 1, 2$ and

$$
\begin{aligned}
\omega^L_{\|x\|}(u,\varepsilon) = \sup\big\{&|u(t,s,y) - u(t_0,s,y)| : t, t_0, s \in [0,L], \\
&y \in [-\|x\|, \|x\|] \text{ and } |t - t_0| \leq \varepsilon\big\}.
\end{aligned}
$$

By the uniform continuity of the functions T_ix on the set $[0, L]$ and u on the set $[0, L] \times [0, L] \times [-\|x\|, \|x\|]$, we deduce that $\omega^L(T_ix, \varepsilon) \to 0$ and $\omega^L_{\|x\|}(u, \varepsilon) \to 0$ as $\varepsilon \to 0$. Further observe that by assumption (vii) we can choose a number L so big that the last term of the estimate (3.1) is sufficiently small. Thus we infer that Fx is continuous on the interval $[0, L]$ for any $L > 0$ big enough. This implies that Fx is continuous on the whole interval $\mathbb{R}^+$. Next we show that Fx is bounded on $\mathbb{R}^+$. By our assumptions, for arbitrarily fixed $t \in \mathbb{R}^+$ we have:

$$
\begin{aligned}
|(Fx)(t)| &= \Big|(T_1x)(t) + (T_2x)(t) \int_0^\infty u(t,s,x(s))ds\Big| \\
&\leq |(T_1x)(t)| + |(T_2x)(t)| \int_0^\infty |u(t,s,x(s))|\, ds.
\end{aligned}
\tag{3.2}
$$

By the assumptions and from estimate (3.2) we obtain

$$\begin{aligned}|(Fx)(t)| &\le d_1(\|x\|) + d_2(\|x\|)\int_0^\infty g(t,s)h(|x(s)|)ds\\ &\le d_1(\|x\|) + d_2(\|x\|)h(\|x\|)G.\end{aligned}$$

Hence, we obtain

$$\|Fx\| \le d_1(\|x\|) + d_2(\|x\|)h(\|x\|)G \tag{3.3}$$

which implies that the function Fx is bounded on $\mathbb{R}^+$.

Linking this fact with the continuity of the function Fx on $\mathbb{R}^+$ we conclude that the operator F maps the space $BC(\mathbb{R}^+,\mathbb{R})$ into itself. Furthermore, by estimate (3.3) and assumption (iv) we infer that F is a self mapping of the ball B_{r_0}, where r_0 is mentioned in the assumption (iv).

Next we take a nonempty subset X of the ball B_{r_0}. Fix $\varepsilon \ge 0$ and $L > 0$ and take an arbitrary function $x \in X$. Then, using estimate (3.1), it follows that

$$\begin{aligned}\omega^L(Fx,\varepsilon) &\le \omega^L(T_1x,\varepsilon) + \omega^L(T_2x,\varepsilon)Gh(r_0) + d_2(r_0)L\omega^L_{r_0}(u,\varepsilon)\\ &\quad + 2d_2(r_0)h(r_0)\sup\Big\{\int_L^\infty g(t,s)ds : t\in[0,L]\Big\}.\end{aligned} \tag{3.4}$$

Hence by (3.4) we obtain

$$\begin{aligned}\omega^L(FX,\varepsilon) &\le \omega^L(T_1X,\varepsilon) + \omega^L(T_2X,\varepsilon)Gh(r_0) + d_2(r_0)L\omega^L_{r_0}(u,\varepsilon)\\ &\quad + 2d_2(r_0)h(r_0)\sup\Big\{\int_L^\infty g(t,s)ds : t\in[0,L]\Big\}.\end{aligned} \tag{3.5}$$

Now taking into account the properties of the component involved in the estimate (3.5), we have

$$\begin{aligned}\omega^L_0(FX) &\le \omega^L_0(T_1X) + \omega^L_0(T_2X)Gh(r_0)\\ &\quad + 2d_2(r_0)h(r_0)\sup\Big\{\int_L^\infty g(t,s)ds : t\in[0,L]\Big\}.\end{aligned} \tag{3.6}$$

Combining (3.6) with assumption (vii), we derive the estimate

$$\omega_0(FX) \le [k_1 + k_2Gh(r_0)]\omega_0(X). \tag{3.7}$$

Further taking $x \in X$ and choosing arbitrarily $L > 0$, in view of the estimate (3.2) we obtain that

$$\begin{aligned}&\sup\{|(Fx)(t)| : t \ge L\}\\ &\le \sup\{|(T_1x)(t)| : t\ge L\}\\ &\quad + \sup\{|(T_2x)(t)| : t \ge L\}h(r_0)\sup\Big\{\int_0^\infty g(t,s)ds : t \ge L\Big\}.\end{aligned} \tag{3.8}$$

Hence by (3.8) we obtain

$$\begin{aligned}\beta(FX) &\le \beta(T_1X) + \beta(T_2X)h(r_0)\sup\Big\{\int_0^\infty g(t,s)ds : t \ge L\Big\}\\ &\le m_1\beta(X) + m_2\beta(X)h(r_0)G\\ &\le [m_1 + m_2h(r_0)G]\beta(X).\end{aligned} \tag{3.9}$$

Now, linking (3.7) and (3.9) we derive that

$$\mu(FX) \le k\mu(X), \tag{3.10}$$

where $k = \max\{k_1 + k_2Gh(r_0),\, m_1 + m_2Gh(r_0)\}$.

Next, we consider the sequence of sets $(B^n_{r_0})$, where $B^1_{r_0} = \operatorname{conv} F(B_{r_0})$, $B^2_{r_0} = \operatorname{conv} F(B^1_{r_0})$ and so on. Observe that all sets of this sequence are nonempty, bounded, closed and convex. Moreover, $B^{n+1}_{r_0} \subset B^n_{r_0}$ for $n = 1, 2, \dots$ Further, keeping in mind (3.10) we obtain

$$\mu(B^n_{r_0}) \le k^n \mu(B_{r_0}).$$

Obviously in view of assumption (vi) we have that $k < 1$. Hence, from condition 5 of Definition 2.1 we infer that the set $Y = \cap_{n=1}^{\infty} B^n_{r_0}$ is nonempty, bounded, closed and convex. In fact, since $\mu(Y) \le \mu(B^n_{r_0})$ for any n, we deduce that $\mu(Y) = 0$ and thus $Y \in \ker\mu$. It should be also noted that the operator F maps the set Y into itself. Now we show that F is continuous on the set Y. To do this fix $\varepsilon \ge 0$ and take functions $x, y \in Y$ such that $\|x - y\| \le \varepsilon$. Taking into account the fact that $Y \in \ker\mu$ and the description of sets belonging to ker μ we can find a number $L > 0$ such that for each $z \in Y$ and $t \ge L$ the inequality $|z(t)| \le \varepsilon$ is satisfied. Since $F : Y \to Y$, we have that $Fx, Fy \in Y$. Thus, for $t \ge L$ we obtain that

$$|(Fx)(t) - (Fy)(t)| \le |(Fx)(t)| + |(Fy)(t)| \le 2\varepsilon.$$

On the other hand, for $t \in [0, L]$ we obtain

$$\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&= \Big|(T_1x)(t) + (T_2x)(t)\int_0^\infty u(t,s,x(s))ds - (T_1y)(t) \\
&\quad - (T_2y)(t)\int_0^\infty u(t,s,y(s))ds\Big| \\
&\le |(T_1x)(t) - (T_1y)(t)| + |(T_2x)(t) - (T_2y)(t)|\int_0^\infty |u(t,s,x(s))|ds \\
&\quad + |(T_2y)(t)|\int_0^\infty |u(t,s,x(s)) - u(t,s,y(s))|\,ds \\
&\le \varepsilon + \varepsilon Gh(r_0) + d_2(r_0)\Big\{\int_0^L |u(t,s,x(s)) - u(t,s,y(s))|\,ds \\
&\quad + \int_L^\infty |u(t,s,x(s)) - u(t,s,y(s))|\,ds\Big\} \\
&\le \varepsilon + \varepsilon Gh(r_0) + d_2(r_0)L\bar{\omega}^L_{r_0}(u,\varepsilon) \\
&\quad + 2d_2(r_0)h(r_0)\sup\Big\{\int_L^\infty g(t,s)ds : t \in [0,L]\Big\},
\end{aligned} \tag{3.11}$$

where

$$\bar{\omega}^L_{r_0}(u,\varepsilon) = \sup\{|u(t,s,x) - u(t,s,y)| : t, s \in [0,L];\ x, y \in [-r_0, r_0] \text{ and } |x - y| \le \varepsilon\}.$$

Observe that $\bar{\omega}^L_{r_0}(u,\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover we can choose L in such a way that the last term in estimate (3.11) is small enough. Taking into account the above facts we conclude that the operator F is continuous on the set Y. Finally, linking all above established properties of the set Y and the operator $F : Y \to Y$ and using the Schauder fixed point principle we infer that the operator F has at least one fixed point x in the set Y. Moreover, keeping in mind that $Y \in \ker\mu$ we obtain that $x(t) \to 0$ as $t \to \infty$. $\square$

4. Examples

Example 4.1. Let us consider the integral equation

$$x(t) = \frac{t \sin x(t)}{3t+9} + \frac{t^2 x^2(t)}{3t^2+2} \int_0^\infty \frac{t^2 \exp(-t-s)\sqrt{|x(s)|}}{t^2+1} ds, \tag{4.1}$$

where $t \in \mathbb{R}^+$.

If we put $(T_1x)(t) = t \sin x(t)/(3t+9)$, $(T_2x)(t) = t^2x^2(t)/(3t^2+2)$ and $u(t,s,x) = t^2 \exp(-t-s)\sqrt{|x|}/(t^2+1)$, then (4.1) is a special case of (1.9). It is easily verified that the assumptions of Theorem 3.1 are satisfied. Indeed, T_1 and T_2 are continuous operators on the space $BC(\mathbb{R}^+, \mathbb{R})$. Further for all $t \in \mathbb{R}^+$ and $x \in BC(\mathbb{R}^+, \mathbb{R})$, we have

$$|(T_1x)(t)| \le \frac{1}{3},$$
$$|(T_2x)(t)| \le \frac{x^2(t)}{3}.$$

Hence assumption (i) is satisfied with $d_1(x) = 1/3$ and $d_2(x) = x^2/3$. Now note that the function u is continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Moreover we obtain

$$|u(t,s,x)| = \Big|\frac{t^2 \exp(-t-s)\sqrt{|x|}}{t^2+1}\Big| \le \exp(-t-s)\sqrt{|x|}$$

and if we choose $g(t,s) = \exp(-t-s)$ and $h(x) = \sqrt{x}$, we see that assumption (ii) is satisfied. To check that the assumption (iii) is satisfied let us observe that $s \to \exp(-t-s)$ is integrable on $\mathbb{R}^+$ and $t \to \int_0^\infty \exp(-t-s)ds$ is bounded on $\mathbb{R}^+$. Thus it is easily seen that

$$G = \sup\Big\{\int_0^\infty \exp(-t-s)ds : t \in \mathbb{R}^+\Big\} = \sup\{\exp(-t) : t \in \mathbb{R}^+\} = 1.$$

Now notice that the inequality in assumption (iv) has the form

$$\frac{1}{3} + \frac{r^2\sqrt{r}}{3} \le r. \tag{4.2}$$

It can be easily verified that if $0.3590 \le r_0 < 1$. Then r_0 is the solution of (4.2). Also for $\varepsilon \ge 0$, $L > 0$, $\|x\| \le r_0$ and $t, s \in [0, L]$ such that $|t-s| \le \varepsilon$, we have

$$\begin{aligned}
|(T_1x)(t) - (T_1x)(s)| &= \Big|\frac{t \sin x(t)}{3t+9} - \frac{s \sin x(s)}{3s+9}\Big| \\
&\le \frac{t(s+3)|\sin x(t) - \sin x(s)| + 3|\sin x(s)||t-s|}{3(t+3)(s+3)} \\
&\le \frac{t}{3(t+3)}|x(t) - x(s)| + \frac{\varepsilon|\sin x(s)|}{(t+3)(s+3)} \\
&\le \frac{1}{3}|x(t) - x(s)| + \frac{\varepsilon}{9}.
\end{aligned} \tag{4.3}$$

Further, it can be seen that

$$
\begin{aligned}
|(T_2x)(t)-(T_2x)(s)| &= \Big|\frac{t^2x^2(t)}{3t^2+2}-\frac{s^2x^2(s)}{3s^2+2}\Big| \\
&\le \frac{2r_0t^2(3s^2+2)|x(t)-x(s)|+2r_0^2(t+s)|t-s|}{(3t^2+2)(3s^2+2)} \\
&\le \frac{2r_0t^2}{3t^2+2}|x(t)-x(s)|+\frac{2r_0^2\varepsilon(t+s)}{(3t^2+2)(3s^2+2)} \\
&\le \frac{2r_0}{3}|x(t)-x(s)|+r_0^2\varepsilon L.
\end{aligned} \tag{4.4}
$$

From estimates (4.3) and (4.4) in view of the (2.1) we have

$$
\begin{gathered}
\omega_0(T_1X)\le \frac{1}{3}\omega_0(X),\\
\omega_0(T_2X)\le \frac{2r_0}{3}\omega_0(X),
\end{gathered}
$$

respectively. This implies that assumption (v) is satisfied with $k_1=1/3$ and $k_2=2r_0/3$. Finally we obtain

$$
\begin{aligned}
\sup\{|(T_1x)(t)|:t\ge L\} &= \sup\Big\{\Big|\frac{t\sin x(t)}{3t+9}\Big|:t\ge L\Big\} \\
&= \frac{1}{3}\sup\{|\sin x(t)|:t\ge L\} \\
&\le \frac{1}{3}\sup\{|x(t)|:t\ge L\}.
\end{aligned} \tag{4.5}
$$

Thus from estimate (4.5) we have

$$
\beta(T_1X)\le \frac{1}{3}\beta(X).
$$

Moreover, we derive that

$$
\begin{aligned}
\sup\{|(T_2x)(t)|:t\ge L\} &= \sup\Big\{\Big|\frac{t^2x^2(t)}{3t^2+2}\Big|:t\ge L\Big\} \\
&= \frac{1}{3}\sup\{|x(t)|^2:t\ge L\} \\
&\le \frac{r_0}{3}\sup\{|x(t)|:t\ge L\}.
\end{aligned} \tag{4.6}
$$

Thus from estimate (4.6) we have

$$
\beta(T_2X)\le \frac{r_0}{3}\beta(X).
$$

Therefore $m_1=1/3$ and $m_2=r_0/3$.

Taking into account the above estimates we have

$$
\max\{k_1+k_2Gh(r_0),\, m_1+m_2Gh(r_0)\}=\max\Big\{\frac{1}{3}+\frac{2r_0}{3}\sqrt{r_0},\, \frac{1}{3}+\frac{r_0}{3}\sqrt{r_0}\Big\}<1.
$$

Hence assumption (vi) is satisfied. Finally, we have the following equality for the function $g(t,s)$ appearing in assumption (vii):

$$
\sup\Big\{\int_L^\infty \exp(-t-s)ds:t\in[0,L]\Big\}=\sup\Big\{\frac{1}{\exp(t+L)}:t\in[0,L]\Big\}=\frac{1}{\exp(L)}.
$$

Since $1/\exp(L)\to 0$ as $L\to\infty$, assumption (vii) is satisfied.

Thus we showed that all assumptions of Theorem 3.1 are fulfilled. This yields that (4.1) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}^+, \mathbb{R})$ vanishing at infinity.

Remark 4.2. Note that none of the existence theorems in [13, 14, 15, 3, 1, 4, 8, 5, 9, 16, 17, 18, 10] are applicable to (4.1), since the integral equation (4.1) can not be derived from any of the integral equations handled in mentioned papers.

Example 4.3. Let us consider the following integral equation:

$$x(t) = \frac{t}{10t+1} + \frac{x^2(t)}{t+1} \int_0^\infty \frac{\exp(-t)(e-1)x(s)}{(t+1)(s+e)(s+1)} ds, \tag{4.7}$$

where $t \in \mathbb{R}^+$. Observe that this equation has the form of (1.9) if we put $(T_1x)(t) = t/(10t+1)$, $(T_2x)(t) = x^2(t)/(t+1)$ and $u(t,s,x) = \exp(-t)(e-1)x/((t+1)(s+e)(s+1))$. It is clear that T_1 and T_2 are continuous operators on the space $BC(\mathbb{R}^+, \mathbb{R})$. Additionally for all $t \in \mathbb{R}^+$ and $x \in BC(\mathbb{R}^+, \mathbb{R})$, we have

$$\begin{gathered} |(T_1x)(t)| \le \Big|\frac{t}{10t+1}\Big|, \\ |(T_2x)(t)| \le \Big|\frac{x^2(t)}{t+1}\Big|. \end{gathered}$$

Hence assumption (i) is satisfied with $d_1(x) = 1/10$ and $d_2(x) = x^2$, respectively. The function $u(t,s,x)$ is continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Further we obtain

$$|u(t,s,x)| = \Big|\frac{\exp(-t)(e-1)x}{(t+1)(s+e)(s+1)}\Big| = \frac{\exp(-t)(e-1)|x|}{(t+1)(s+e)(s+1)}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Thus the functions appearing in assumption (ii) have the form $g(t,s) = \exp(-t)(e-1)/((t+1)(s+e)(s+1))$ and $h(x) = x$. Obviously $s \to \exp(-t)(e-1)/((t+1)(s+e)(s+1))$ is integrable on $\mathbb{R}^+$ and $t \to \int_0^\infty \exp(-t)(e-1)ds/((t+1)(s+e)(s+1))$ is bounded on $\mathbb{R}^+$. Moreover, we have

$$G = \sup\Big\{\int_0^\infty \frac{\exp(-t)(e-1)}{(t+1)(s+e)(s+1)} ds : t \in \mathbb{R}^+\Big\} = \sup\Big\{\frac{\exp(-t)}{t+1} : t \in \mathbb{R}^+\Big\} = 1.$$

Now note that the inequality in assumption (iv) has the form:

$$\frac{1}{10} + r^3 \le r. \tag{4.8}$$

It can be easily verified that if $0.1010 \le r_0 < 1/\sqrt{2}$, then r_0 is the solution of (4.8). Also for $\varepsilon \ge 0$, $L > 0$, $\|x\| \le r_0$ and $t, s \in [0, L]$ such that $|t - s| \le \varepsilon$, we have that

$$\begin{aligned} |(T_1x)(t) - (T_1x)(s)| &= \Big|\frac{t}{10t+1} - \frac{s}{10s+1}\Big| \\ &= \frac{|t-s|}{(10t+1)(10s+1)} \\ &\le \frac{\varepsilon}{(10t+1)(10s+1)} \le \varepsilon. \end{aligned} \tag{4.9}$$

Further, it can be seen that

$$\begin{aligned}
|(T_2x)(t)-(T_2x)(s)| &= \Big|\frac{x^2(t)}{t+1}-\frac{x^2(s)}{s+1}\Big| \\
&\le \frac{(t+1)|x^2(t)-x^2(s)|+|x^2(t)||s-t|}{(t+1)(s+1)} \\
&\le \frac{2r_0|x(t)-x(s)|}{s+1}+\frac{r_0^2\varepsilon}{(t+1)(s+1)} \\
&\le 2r_0|x(t)-x(s)|+r_0^2\varepsilon.
\end{aligned} \tag{4.10}$$

From estimates (4.9) and (4.10) in view of (2.1), we have

$$\begin{gathered}
\omega_0(T_1X)=0,\\
\omega_0(T_2X)\le 2r_0\omega_0(X).
\end{gathered}$$

This implies that assumption (v) is satisfied with $k_1=0$ and $k_2=2r_0$. Further, we have

$$\sup\{|(T_1x)(t)|:t\ge L\}=\sup\Big\{\Big|\frac{t}{10t+1}\Big|:t\ge L\Big\}=\frac{1}{10}. \tag{4.11}$$

Moreover, we obtain

$$\begin{aligned}
\sup\{|(T_2x)(t)|:t\ge L\} &= \sup\Big\{\Big|\frac{x^2(t)}{t+1}\Big|:t\ge L\Big\} \\
&= \frac{1}{L+1}\sup\{|x(t)|^2:t\ge L\} \\
&\le \frac{r_0}{L+1}\sup\{|x(t)|:t\ge L\}.
\end{aligned} \tag{4.12}$$

Thus from estimates (4.11) and (4.12) in view of (2.2) we have that $\beta(T_1X)=1/10$ and $\beta(T_2X)=0$. Therefore we obtain $m_1=1/10$ and $m_2=0$. Keeping in mind the above obtained constants k_1,k_2,m_1,m_2 we obtain

$$\max\Big\{2r_0^2,\ \frac{1}{10}\Big\}<1$$

and thus assumption (vi) is satisfied. Finally, we obtain

$$\begin{aligned}
&\sup\Big\{\int_L^\infty \frac{\exp(-t)(e-1)}{(t+1)(s+e)(s+1)}ds:t\in[0,L]\Big\} \\
&= \sup\Big\{\frac{\exp(-t)}{(t+1)}\ln\Big(\frac{L+1}{L+e}\Big):t\in[0,L]\Big\} \\
&= \frac{1}{\exp(L)(L+1)}\ln\Big(\frac{L+1}{L+e}\Big).
\end{aligned} \tag{4.13}$$

Hence, we infer that assumption (vii) is satisfied.

Finally we conclude that the assumptions of Theorem 3.1 are satisfied. This implies that the considered integral equation has a solution $x=x(t)$ belonging to the set Y. Moreover, $x(t)\to 0$ as $t\to 0$.

Remark 4.4. Now we compare the result in Theorem 3.1 with the results in [5, 18, 9].

Notice that in [5] it was assumed that $\lim_{t\to\infty}a(t)=0$. Thus, the result given in [5] is inapplicable to(4.7).

Now if we consider the assumption (iii) of [9, Theorem 8], we have $f(t,x)=x^2/(t+1)$ and it does not satisfy the Lipschitz condition with respect to second variable for all $x \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$. Hence, the existence theorem given in [9] is inapplicable to (4.7).

Further, in assumption (ii) of [18, Theorem 3.1] states that $f(t,x)$ is nondecreasing with respect to both nonnegative variables t and x. Since we have $t \to f(t,x) = x^2/(t+1)$ is a decreasing function in t, [18, Theorem 3.1] is not inapplicable to (4.7).

Example 4.5. Consider the integral equation

$$x(t) = \frac{t}{\exp(t)} + \frac{\sqrt{x^2(t)+1}}{t+1} \int_0^\infty \frac{\sqrt{1+|x(s)|}}{\exp(t+s+1)} ds, \tag{4.14}$$

where $t \in \mathbb{R}^+$. This equation is a special case of (1.9) if we put $(T_1x)(t) = t/\exp(t)$, $(T_2x)(t) = \sqrt{x^2(t)+1}/(t+1)$ and $u(t,s,x) = \sqrt{1+|x|}/\exp(t+s+1)$. Obviously, we have that T_1 and T_2 are continuous operators on the space $BC(\mathbb{R}^+, \mathbb{R})$. Moreover, for all $t \in \mathbb{R}^+$ and $x \in BC(\mathbb{R}^+, \mathbb{R})$, we obtain

$$|(T_1x)(t)| \le \Big|\frac{t}{\exp(t)}\Big| \le \frac{1}{e},$$
$$|(T_2x)(t)| \le \Big|\frac{\sqrt{x^2(t)+1}}{t+1}\Big| \le \sqrt{\|x\|^2+1}.$$

Thus, we conclude that assumption (i) is satisfied with $d_1(x) = 1/e$ and $d_2(x) = \sqrt{x^2+1}$. Observe that the function $u(t,s,x)$ is continuous on the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Further, we obtain

$$|u(t,s,x)| = \Big|\frac{\sqrt{1+|x|}}{\exp(t+s+1)}\Big| = \frac{\sqrt{1+|x|}}{\exp(t+s+1)}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Thus the function $u(t,s,x)$ satisfies assumption (ii) with $g(t,s) = 1/\exp(t+s+1)$ and $h(x) = \sqrt{1+|x|}$. Obviously, $s \to 1/\exp(t+s+1)$ is integrable on $\mathbb{R}^+$ and $t \to \int_0^\infty ds/\exp(t+s+1)$ is bounded on $\mathbb{R}^+$. Further, we have that

$$G = \sup\Big\{\int_0^\infty \frac{1}{\exp(t+s+1)} ds : t \in \mathbb{R}^+\Big\} = \sup\Big\{\frac{1}{\exp(t+1)} : t \in \mathbb{R}^+\Big\} = \frac{1}{e}.$$

Next, observe that the inequality from assumption (iv) has the form

$$\frac{1}{e} + \sqrt{r_0^2+1}\sqrt{1+r_0}\frac{1}{e} \le r_0. \tag{4.15}$$

It can be easily verified that if $1.2532 \le r_0 \le 5.1357$, then r_0 is the solution of (4.15). Also for $\varepsilon \ge 0$, $L > 0$, $\|x\| \le r_0$ and $t, s \in [0, L]$ such that $|t-s| \le \varepsilon$, we have that

$$|(T_1x)(t) - (T_1x)(s)| = \Big|\frac{t}{\exp(t)} - \frac{s}{\exp(s)}\Big|. \tag{4.16}$$

Further, without loss of generality we can assume that $x(t) < x(s)$. Hence, we obtain

$$
\begin{aligned}
|(T_2x)(t) - (T_2x)(s)| &= \Big|\frac{\sqrt{x^2(t)+1}}{t+1} - \frac{\sqrt{x^2(s)+1}}{s+1}\Big| \\
&\le \frac{1}{t+1}\Big|\sqrt{x^2(t)+1} - \sqrt{x^2(s)+1}\Big| \\
&\quad + \sqrt{x^2(s)+1}\Big|\frac{1}{t+1} - \frac{1}{s+1}\Big| \\
&\le \frac{|x(t)-x(s)||2\xi|}{2\sqrt{\xi^2+1}} + \frac{\sqrt{r_0^2+1}|s-t|}{(t+1)(s+1)} \\
&\le |x(t)-x(s)| + \sqrt{r_0^2+1}\varepsilon,
\end{aligned}
\tag{4.17}
$$

where $\xi \in (x(t), x(s))$. In view of (2.1) and the uniform continuity of the function $t \to t\exp(t)$ on the set $[0, L]$, we have by (4.16) and (4.17) that

$$
\begin{gathered}
\omega_0(T_1X) = 0, \\
\omega_0(T_2X) \le \omega_0(X).
\end{gathered}
$$

This implies that assumption (v) is satisfied with $k_1 = 0$ and $k_2 = 1$. Further, we have

$$
\begin{aligned}
\sup\{|(T_1x)(t)| : t \ge L\} &= \sup\Big\{\Big|\frac{t}{\exp(t)}\Big| : t \ge L\Big\} \\
&= \begin{cases} 1/e, & 0 < L \le 1 \\ L/\exp(L), & L > 1. \end{cases}
\end{aligned}
\tag{4.18}
$$

Moreover,

$$
\sup\{|(T_2x)(t)| : t \ge L\} = \sup\Big\{\Big|\frac{\sqrt{x^2(t)+1}}{t+1}\Big| : t \ge L\Big\} \le \frac{\sqrt{r_0^2+1}}{L+1}. \tag{4.19}
$$

Thus from estimates (4.18) and (4.19) in view of (2.2) we have that $\beta(T_1X) = \beta(T_2X) = 0$. Therefore we obtain $m_1 = m_2 = 0$. Keeping in mind the constants k_1, k_2, m_1, m_2 above obtained we have

$$
\max\Big\{\frac{\sqrt{r_0+1}}{e}, 0\Big\} < 1
$$

and assumption (vi) is satisfied. Then, we obtain

$$
\begin{aligned}
&\sup\Big\{\int_L^\infty \frac{1}{\exp(t+s+1)} ds : t \in [0, L]\Big\} \\
&= \sup\Big\{\frac{1}{\exp(t+L+1)} : t \in [0, L]\Big\} \\
&= \frac{1}{\exp(L+1)}.
\end{aligned}
$$

Hence, we infer that assumption (vii) is satisfied.

Finally, we conclude that all of the assumptions of Theorem 3.1 are satisfied. This implies that the integral equation (4.14) has a solution $x = x(t)$ belonging to the set Y. Moreover, $x(t) \to 0$ as $t \to 0$.

Remark 4.6. Observe that if we put $a(t) = t\exp(-t)$, $f(t,x) = \sqrt{x^2+1}/(t+1)$ and $u(t,s,x) = \sqrt{1+|x|}/\exp(t+s+1)$, then (4.14) is a special case of (1.7) which is handled in [9]. It is easily seen that $a \in BC(\mathbb{R}^+, \mathbb{R})$ and $\|a\| = 1/e$. $f(t,0) \in BC(\mathbb{R}^+, \mathbb{R})$ and f satisfies the Lipschitz condition with respect to the second variable for $k = 1$.

On the other hand we have $g(t,s) = 1/\exp(t+s+1)$ and $h(r) = r$ which are imposed in assumption (iv) of [9, Theorem 8] for the inequality

$$|u(t,s,x) - u(t,s,y)| \le g(t,s)h(|x-y|)$$

to be satisfied for all $t, s \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$.

Moreover we obtain $\bar{f} = 1$, $\bar{g} = 1/e$ and $\bar{u} = 1/e$, where

$$\bar{f} = \sup\{|f(t,0)| : t \in \mathbb{R}^+\}, \quad \bar{g} = \sup\Big\{\int_0^\infty g(t,s)ds : t \in \mathbb{R}^+\Big\},$$
$$\bar{u} = \sup\Big\{\int_0^\infty |u(t,s,0)|ds : t \in \mathbb{R}^+\Big\}.$$

Thus assumptions (i)–(vi) of [9, Theorem 8] are fulfilled.

Finally, let us note that the inequality of assumption (vii), given in [9]:

$$\|a\| + k\bar{g}rh(r) + k\bar{u}r + \bar{f}\bar{g}h(r) + \bar{f}\bar{u} \le r$$

takes the form

$$\frac{r^2+2r+2}{e} \le r. \tag{4.20}$$

It can be checked that (4.20) does not have a positive solution. Therefore, [9, Theorem 8] is inapplicable to (4.14),

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