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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR INITIAL-VALUE FRACTIONAL DIFFERENTIAL PROBLEMS

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Abstract. We study the boundedness and asymptotic behavior of solutions for a class of nonlinear fractional differential equations. These equations involve two Riemann-Liouville fractional derivatives of different orders. We determine fairly large classes of nonlinearities and appropriate underlying spaces where solutions are bounded, exist globally and decay to zero as a power type function. Our results are obtained by using generalized versions of Gronwall-Bellman inequality, appropriate regularization techniques and several properties of fractional derivatives. Three examples are given to illustrate our results.

1. Introduction

The field of fractional calculus is concerned with the generalization of the integer order differentiation and integration to an arbitrary real (or complex) order [31, 32, 34]. Many events in diverse fields of engineering can be portrayed better and more accurately by differential equations of non-integer order [15, 16].
In this article, we consider the fractional differential problem

$$\begin{gathered} D_0^\alpha y(t) = f(t, y(t), D_0^\beta y(t)), \quad 0 \le \beta < \alpha < 1,\ t > 0, \\ I_0^{1-\alpha} y(t)\big|_{t=0} = b, \quad b \in \mathbb{R}, \end{gathered} \tag{1.1}$$

in an appropriate space of continuous functions, where f is a continuous nonlinear function with respect to all of its arguments, I_0^σ and D_0^σ are the Riemann-Liouville fractional integral and fractional derivative defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > a, \tag{1.2}$$

$$D_a^\alpha f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t), \quad t > a,\ 0 < \alpha < 1, \tag{1.3}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. When $\alpha = 0$, we define $I_a^0 f = f$. In particular, when $\alpha = 1$ we have $D_a^\alpha f = Df$, $D = \frac{d}{dt}$ and when $\alpha = 0$, $D_a^0 f = f$. When $\beta = 0$, Problem (1.1) reduces to

$$\begin{gathered} D_0^\alpha y(t) = f(t, y(t)), \quad 0 < \alpha < 1, \quad t > 0, \\ I_0^{1-\alpha} y(t)\big|_{t=0} = b. \end{gathered}$$

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The existence and uniqueness of solutions in a weighted space of continuous functions for problem (1.1) have been established in [23, p 168, Theorem 3.14] when f is a real-valued continuous function and satisfies the Lipschitz condition.

The study of the long time behavior of solutions of differential problems is in general extremely useful in applications. It has attracted many researchers, see [14, 28, 29, 36] and many other references in [24].

The question of asymptotic behavior of solutions of general differential problems consists often of determining sufficient conditions ensuring a certain specific (or just exploring the) behavior for large values of time. This task may be simple for simple problems. Things become even more complicated when dealing with nonlinear fractional differential equations. Therefore, observing the behavior through the explicit solution is not always possible.

The behavior of solutions of various classes of FDEs (fractional differential equations) has been considered in many papers in the literature, see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20, 21, 22, 24, 27, 39, 40], and the references therein, to cite but a few. In particular, the behavior of solutions of the nonlinear equation

$$\begin{gathered} D_0^{\alpha} y(t) = f(t,y), \quad 0<\alpha<1,\ t>0, \\ t^{1-\alpha} y(t)\big|_{t=0} = b \end{gathered} \tag{1.4}$$

has been considered by Furati and Tatar in [10]. They proved that solutions decay polynomially on their interval of existence provided that $f(t,y)$ satisfies the condition

$$|f(t,y)| \le t^{\mu} e^{-\sigma t} \varphi(t) |y|^m, \quad \mu \ge 0,\ m>1,\ \sigma>0, \tag{1.5}$$

where $\varphi(t)$ is a continuous function on $\mathbb{R}_+ := [0,\infty)$.

In 2012, Furati, Kassim and Tatar [7] studied the nonlinear fractional differential problem

$$\begin{gathered} D_0^{\alpha,\beta} y(t) = f(t,y), \quad 0<\alpha<1,\ t>0, \\ t^{(1-\alpha)(1-\beta)} y(t)\big|_{t=0} = b, \end{gathered} \tag{1.6}$$

where

$$D_0^{\alpha,\beta} = I_0^{\beta(1-\alpha)} D I_0^{(1-\beta)(1-\alpha)},$$

is the Hilfer fractional derivative of order $0<\alpha<1$ and type $0\le\beta\le1$. They showed that solutions of (1.6) decay as a power function under the same condition (1.5) on the function $f(t,y)$. Notice when $\beta=0$ in (1.6) we obtain the same derivative in (1.4).

In 2013, Płociniczak [35] considered the linear fractional differential equation

$$ {}^{c}D^{\alpha} y(t) = \lambda\, q(t) y(t), \qquad 0<\alpha<1, \quad t>0, \tag{1.7}$$

where ${}^{c}D^{\alpha}$ is the Caputo derivative of order α and $q(t) \sim C_q t^{\mu} > 0$, $\mu>0$. He proved that the solution of (1.7) for $\lambda>0$ obeys the asymptotic property

$$y(t) \sim C_1 \exp(\lambda^{1/\alpha} \int q^{1/\nu}(t)dt) \quad \text{as } t\to\infty, \quad \text{for some } C_1,$$

while for $\lambda<0$,

$$y(t) \sim \frac{y(0)}{1-\lambda\Gamma(1-\alpha) q(t) t^{\alpha}} \quad \text{as } t\to\infty.$$

In this article, we study the behavior of solutions of (1.1). We determine sufficient conditions on the nonlinear term which guarantee that solutions of (1.1) decay for all time in a weighted space of continuous functions. In particular, we prove that solutions decay like the power function $t^{\alpha-1}$. We mention here that the right hand side of (1.1) may contain several "fractional derivatives" but for simplicity we restrict ourselves to only one derivative. The presence of singular kernels in these derivatives is one of the main challenges we have to face.

This article is organized as follows. In the next section, we introduce some material needed in our study. In Section 3, we establish some inequalities involving some special classes of functions. Sections 4 and 5 are devoted to our results. In Section 6, we illustrate our findings by three examples.

2. Fractional calculus and preliminaries

In this section we present some definitions, lemmas, properties and notation related to our results. For more details, we refer the reader to [23, 34, 37]. For a finite interval $[a,b]$, let $C[a,b]$ and $C^n[a,b]$ denote the spaces of continuous and n-times continuously differentiable functions on $[a,b]$, respectively.

Definition 2.1. We consider the weighted spaces of continuous functions

$$C_\gamma[a,b] = \{f : (a,b] \to \mathbb{R} : (t-a)^\gamma f(t) \in C[a,b]\}, \quad 0 < \gamma < 1,$$
$$C[a,b] = C_0[a,b],$$

and

$$C_\gamma^n[a,b] = \{f \in C^{n-1}[a,b] : f^{(n)} \in C_\gamma[a,b]\}, \quad n \in \mathbb{N},$$
$$C_\gamma[a,b] = C_\gamma^0[a,b].$$

Remark 2.2. Note that $C_\gamma^n[a,b] \subset AC^n[a,b]$ for $n \geq 1$, where

$$AC^n[a,b] = \{f : [a,b] \to \mathbb{R} \text{ and } f^{(n-1)} \in AC[a,b]\},$$

and $AC[a,b]$ is the space of absolutely continuous functions on $[a,b]$.

Lemma 2.3 ([25])**.** *Let $\alpha > 0$ and $0 \leq \gamma < 1$. Then, I_a^α is bounded from $C_\gamma[a,b]$ into $C_\gamma[a,b]$.*

Lemma 2.4 ([17])**.** *Let g be a continuous function on $(a,b]$. Then, $g^{(n)} \in C_\gamma[a,b]$ if and only if $g \in C_\gamma^n[a,b]$, $0 \leq \gamma < 1$.*

For power functions we have the following property.

Property 2.5 ([23])**.** If $\alpha \geq 0$ and $\beta > 0$, then

$$D_a^\alpha (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \quad t > a.$$

A composition property between the fractional differentiation operator and the fractional integration operator is given next.

Property 2.6 ([23])**.** Let $0 < \beta < \alpha$ and $0 \leq \gamma < 1$. If $f \in C_\gamma[a,b]$, then the relation

$$D_a^\beta I_a^\alpha f(t) = I_a^{\alpha-\beta} f(t)$$

holds at any point $t \in (a,b]$. When $f \in C[a,b]$ this relation is valid at any point $t \in [a,b]$.

The following result provides another composition of the fractional integration operator I_a^α with the fractional differentiation operator D_a^α.

Lemma 2.7 ([23]). *Let $0 < \alpha < 1$, $0 \le \gamma < 1$. If $f \in C_\gamma[a,b]$ and $I_a^{1-\alpha} f \in C_\gamma^1[a,b]$, then the equality*

$$I_a^\alpha D_a^\alpha f(t) = f(t) - \frac{(I_a^{1-\alpha} f)(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1}, \tag{2.1}$$

holds at any point $t \in (a,b]$.

Lemma 2.8 ([17]). *Let $0 < \alpha < 1$ and $0 \le \gamma < 1$. If $f \in C_\gamma[a,b]$ and $I_a^{1-\alpha} f \in C_\gamma^1[a,b]$, then for $0 < \beta \le \alpha < 1$ we have*

$$D_a^\beta f(t) = I_a^{\alpha-\beta} D_a^\alpha f(t) + \frac{I_a^{1-\alpha} f(a)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1}, \quad t > a.$$

3. Useful inequalities

In this section we establish some inequalities involving special classes of functions. These inequalities are used in a crucial manner to prove our main results.

Remark 3.1. In the rest of the paper we use the following equivalency. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$p(\alpha - 1) + 1 > 0 \Longleftrightarrow q\alpha > 1, \quad \alpha > 0. \tag{3.1}$$

Lemma 3.2 ([30]). *If $\lambda, \nu, \omega > 0$, then for any $t > 0$, we have*

$$\int_0^t (t-s)^{\nu-1} s^{\lambda-1} e^{-\omega s} ds \le C t^{\nu-1},$$

where

$$C = \max\{1, 2^{1-\nu}\}\Gamma(\lambda)(1 + \lambda(\lambda+1)/\nu)\omega^{-\lambda} > 0.$$

Based on this result we prove here the following result.

Lemma 3.3. *Let $w > 0$ and υ, $\lambda > 1/q$, for some $q > 1$. Then, for any $t > 0$ and any nonnegative continuous function h defined on $\mathbb{R}_+$, we have*

$$\int_0^t (t-s)^{\upsilon-1} s^{\lambda-1} e^{-ws} h(s) ds \le C t^{\upsilon-1} \Big(\int_0^t h^q(s) ds \Big)^{1/q}, \tag{3.2}$$

where

$$C = \Big[\max\{1, 2^{1-\lambda_1}\}\Gamma(\lambda_2)\Big(1 + \frac{(\lambda_2)(\lambda_2+1)}{\lambda_1}\Big)(pw)^{-\lambda_2} \Big]^{1/p},$$

$1/p + 1/q = 1$, $\lambda_1 = p(\upsilon - 1) + 1$, and $\lambda_2 = p(\lambda - 1) + 1$.

Proof. Applying Hölder inequality to the left hand side of (3.2), for $t > 0$, we obtain

$$\begin{aligned} &\int_0^t (t-s)^{\upsilon-1} s^{\lambda-1} e^{-ws} h(s) ds \\ &\le \Big(\int_0^t (t-s)^{p(\upsilon-1)} s^{p(\lambda-1)} e^{-pws} \Big)^{1/p} \Big(\int_0^t h^q(s) ds \Big)^{1/q}. \end{aligned} \tag{3.3}$$

From the hypotheses stated in the lemma, we have

$$\lambda_1 = p(\upsilon - 1) + 1 > 0 \text{ and } \lambda_2 = p(\lambda - 1) + 1 > 0.$$

Applying Lemma 3.2 to (3.3) (with υ replaced by λ_1, λ replaced by λ_2 and w replaced by pw), gives the result. $\square$

Lemma 3.4 ([1, 26]). *Let $a > 0$ and $b > 0$. Then*

$$a^r + b^r \leq (a+b)^r \leq 2^{r-1}(a^r + b^r), \quad r \geq 1,$$
$$2^{r-1}(a^r + b^r) \leq (a+b)^r \leq a^r + b^r, \quad 0 \leq r \leq 1.$$

We recall now the Bihari inequality.

Theorem 3.5 ([33]). *Let u and f be nonnegative continuous functions defined on $\mathbb{R}_+$. Let $w(u)$ be a continuous nondecreasing function defined on $\mathbb{R}_+$ and $w(u) > 0$ on $(0, \infty)$. If*

$$u(t) \leq k + \int_0^t f(s)w(u(s))ds,$$

for $t \in \mathbb{R}_+$, where k is a nonnegative constant, then for $0 \leq t \leq t_1$,

$$u(t) \leq G^{-1}\Big(G(k) + \int_0^t f(s)ds\Big),$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{w(s)}, \quad r > 0,\ r_0 > 0,$$

G^{-1} is the inverse function of G, and $t_1 \in \mathbb{R}_+$ is chosen so that

$$G(k) + \int_0^t f(s)ds \in \mathrm{Dom}(G^{-1}),$$

for $0 \leq t \leq t_1$.

From Theorem 3.5 we have the following corollaries.

Corollary 3.6. *Let z and h be nonnegative continuous functions defined on $\mathbb{R}_+$. Let $w(z)$ be a continuous nondecreasing function defined on $\mathbb{R}_+$ and $w(z) > 0$ on $(0, \infty)$. If*

$$z(t) \leq K_1 + K_2\Big(\int_0^t h(s)w(z(s))ds\Big)^{1/q}, \quad q > 1,\ t > 0, \tag{3.4}$$

where K_i, $i = 1, 2$, are nonnegative constants, then for $0 \leq t \leq t_1$

$$z(t) \leq \Big[G^{-1}\Big(G(2^{q-1}K_1) + 2^{q-1}K_2 \int_0^t h(s)ds\Big)\Big]^{1/q},$$

where

$$G(x) = \int_{x_0}^x \frac{ds}{w(s^{1/q})}, \quad x > x_0 > 0,$$

and G^{-1} is the inverse function of G, and $t_1 \in \mathbb{R}_+$ is chosen so that

$$G(2^{q-1}K_1) + 2^{q-1}K_2 \int_0^t h(s)ds \in \mathrm{Dom}(G^{-1}),$$

for $0 \leq t \leq t_1$.

Proof. Raising both sides of (3.4) to the power q and using Lemma 3.4, we have

$$z^q(t) \leq B_1 + B_2 \int_0^t h(s)w(z(s))ds, \quad t > 0, \tag{3.5}$$

where $B_i = 2^{q-1}K_i$, $i = 1, 2$. Now, let $u(t) = z^q(t)$, then (3.5) can be written as

$$u(t) \leq B_1 + B_2 \int_0^t h(s)g(u(s))ds, \quad t > 0, \tag{3.6}$$

where

$$g(r) = w(r^{1/q}). \tag{3.7}$$

Since w is a continuous and nondecreasing functions, then g is a continuous and nondecreasing function. Applying Bihari's inequality (Theorem3.5) to (3.6), we obtain the result. □

Corollary 3.7. *Let z, h_i, w_i, $i = 1, 2$, and q be as in Corollary 3.6. If*

$$z(t) \le K_1 + K_2\Big[\Big(\int_0^t h_1(s)w_1(z(s))ds\Big)^{1/q} + \Big(\int_0^t h_2(s)w_2(z(s))ds\Big)^{1/q}\Big], \tag{3.8}$$

for $t > 0$, then, for $0 \le t \le t_1$,

$$z(t) \le \Big[G^{-1}\Big(G(2^{q-1}K_1^q\quad) + 2^{2(q-1)}K_2^q \int_0^t [h_1(s) + h_2(s)]ds\Big)\Big]^{1/q}, \tag{3.9}$$

where

$$G(x) = \int_{x_0}^x \frac{ds}{g(s)} = \int_{x_0}^x \frac{ds}{w_1(s^{1/q}) + w_2(s^{1/q})}, \quad x > x_0 > 0,$$

and G^{-1} is the inverse function of G, and $t_1 \in \mathbb{R}_+$ is chosen so that

$$G(2^{q-1}K_1^q) + 2^{2(q-1)}K_2^q \int_0^t [h_1(s) + h_2(s)]ds \in \operatorname{Dom}(G^{-1}),$$

for $0 \le t \le t_1$.

Proof. Raising both sides of (3.8) to the power q, we have

$$z^q(t) \le B_1 + B_2[\int_0^t h_1(s)w_1(z(s))ds + \int_0^t h_2(s)w_2(z(s))ds], \quad t > 0, \tag{3.10}$$

where

$$B_1 = 2^{q-1}K_1^q, \quad B_2 = 2^{2(q-1)}K_2^q.$$

Furthermore, we have

$$h_1(s)w_1(z(s)) + h_2(s)w_2(z(s)) \le [h_1(s) + h_2(s)][w_1(z(s)) + w_2(z(s))]. \tag{3.11}$$

Now, let $u(t) = z^q(t)$, then by (3.10) and (3.11) we can write

$$u(t) \le B_1 + B_2 \int_0^t [h_1(s) + h_2(s)]g(u(s))ds, \quad t > 0, \tag{3.12}$$

where

$$g(r) = w_1(r^{1/q}) + w_2(r^{1/q}). \tag{3.13}$$

Since w_i, $i = 1, 2$, are continuous and nondecreasing functions, then g is a continuous, nondecreasing function. Applying Bihari's inequality to (3.12), we obtain the result. □

4. Preliminaries

In this section we prove some lemmas which will be used to prove the main results. In the sequel, we consider the following assumptions:

(A1) The function $f:(0,\infty)\times\mathbb{R}^2\to\mathbb{R}$ is such that $f(\cdot,u(\cdot),v(\cdot))\in C_{1-\alpha}[0,\infty)$ for any u, $v\in C_{1-\alpha}[0,\infty)$.

(A2) There exist continuous functions h, φ_1, $\varphi_2:\mathbb{R}_+\to\mathbb{R}_+$, such that
$$|f(t,u,v)|\le t^{\gamma}e^{-\delta t}h(t)\varphi_1(t^{1-\alpha}|u|)\varphi_2(t^{1-(\alpha-\beta)}|v|), \tag{4.1}$$
where $h\in L_q(0,\infty)$ for some $q>\frac{1}{\alpha-\beta}$, $\gamma>\frac{1}{q}-1$, $\delta>0$, and φ_i, $i=1,2$, are nondecreasing functions.

(A3) There exist continuous functions h_i, $\varphi_i:\mathbb{R}_+\to\mathbb{R}_+$, such that
$$|f(t,u,v)|\le t^{\gamma_1}e^{-\delta_1 t}h_1(t)\varphi_1(t^{1-\alpha}|u|)+t^{\gamma_2}e^{-\delta_2 t}h_2(t)\varphi_2(t^{1-(\alpha-\beta)}|v|), \tag{4.2}$$
where $h_i\in L_q(0,\infty)$ for some $q>\frac{1}{\alpha-\beta}$, $\gamma_i>\frac{1}{q}-1$, $\delta_i>0$, and φ_i, $i=1,2$, are nondecreasing functions.

The following results provide useful estimates for the solutions of (1.1).

Lemma 4.1. *Assume that $y\in C_{1-\alpha}[0,\infty)$ is a global solution of* (1.1) *and f satisfies* (A1) *and* (A2). *Then*
$$\max\left\{t^{1-\alpha}|y(t)|,t^{1-(\alpha-\beta)}|D^{\beta}y(t)|\right\}\le z(t),\quad t>0,$$
where
$$z(t)=K_1+K_2\Big(\int_0^t h^q(s)\varphi_1^q(s^{1-\alpha}|y(s)|)\varphi_2^q(s^{1-(\alpha-\beta)}|D_0^{\beta}y(s)|)ds\Big)^{1/q}, \tag{4.3}$$
$$K_1=|b|\max\big\{\frac{1}{\Gamma(\alpha)},\frac{1}{\Gamma(\alpha-\beta)}\big\},\quad K_2=\max\big\{C_1,C_2\big\},$$
$$C_1=\frac{1}{\Gamma(\alpha)}\Big(\max\{1,2^{p(1-\alpha)}\}\Gamma(p\gamma+1)\Big(1+\frac{(p\gamma+1)(p\gamma+2)}{p(\alpha-1)+1}\Big)(p\delta)^{-(p\gamma+1)}\Big)^{1/p},$$
$$\begin{aligned}C_2=\frac{1}{\Gamma(\alpha-\beta)}\Big(&\max\{1,2^{p(1-(\alpha-\beta))}\}\Gamma(p\gamma+1)\\&\times\Big(1+\frac{(p\gamma+1)(p\gamma+2)}{p(\alpha-\beta-1)+1}\Big)(p\delta)^{-(p\gamma+1)}\Big)^{1/p}.\end{aligned}$$

Proof. Since $f\in C_{1-\alpha}[0,\infty)$, Equation (1.1) implies that $D_0^{\alpha}y=DI_0^{1-\alpha}y\in C_{1-\alpha}[0,\infty)$. By Lemma 2.4, we have $I_0^{1-\alpha}y\in C_{1-\alpha}^1[0,\infty)$. Applying I_0^{α} to (1.1) and using Lemma 2.7, having in mind that $y\in C_{1-\alpha}[0,\infty)$ and $I_0^{1-\alpha}y\in C_{1-\alpha}^1[0,\infty)$, we obtain
$$y(t)=\frac{b}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s),D_0^{\beta}y(s))ds,\quad t>0. \tag{4.4}$$
Next, multiplying both sides of (4.4) by $t^{1-\alpha}$ and using the assumption (4.1), we obtain
$$\begin{aligned}t^{1-\alpha}|y(t)|\le\frac{|b|}{\Gamma(\alpha)}+\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t&(t-s)^{\alpha-1}s^{\gamma}e^{-\delta s}h(s)\varphi_1(s^{1-\alpha}|y(s)|)\\&\times\varphi_2(s^{1-(\alpha-\beta)}|D_0^{\beta}y(s)|)ds,\quad t>0.\end{aligned} \tag{4.5}$$

In view of Lemma 3.3, we find

$$\begin{aligned} t^{1-\alpha}|y(t)| \le \frac{|b|}{\Gamma(\alpha)} + C_1\Big(\int_0^t h^q(s)\varphi_1^q(s^{1-\alpha}|y(s)|) \\ \times \varphi_2^q(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|)ds\Big)^{1/q}, \quad t>0. \end{aligned} \tag{4.6}$$

Since $0<\beta<\alpha<1$, by Lemma 2.8, we see that

$$\begin{aligned} D_0^\beta y(t) &= \frac{bt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} D_0^\alpha y(s)ds \\ &= \frac{bt^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} f(s,y(s),D_0^\beta y(s))ds, \end{aligned} \tag{4.7}$$

for $t>0$. Multiplying both sides of (4.7) by $t^{1-(\alpha-\beta)}$ and using the assumption (4.1), for $t>0$, we deduce

$$\begin{aligned} t^{1-(\alpha-\beta)}|D_0^\beta y(t)| \le \frac{|b|}{\Gamma(\alpha-\beta)} + \frac{t^{1-(\alpha-\beta)}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}s^\gamma e^{-\delta s}h(s) \\ \times \varphi_1(s^{1-\alpha}|y(s)|)\varphi_2(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|)ds. \end{aligned}$$

Again, by Lemma 3.3, we conclude that

$$\begin{aligned} t^{1-(\alpha-\beta)}|D_0^\beta y(t)| \le \frac{|b|}{\Gamma(\alpha-\beta)} + C_2\Big(\int_0^t h^q(s)\varphi_1^q(s^{1-\alpha}|y(s)|) \\ \times \varphi_2^q(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|)ds\Big)^{1/q}, \quad t>0. \end{aligned} \tag{4.8}$$

Therefore, the result follows from (4.3), (4.6) and (4.8). $\square$

Lemma 4.2. *Assume that $y \in C_{1-\alpha}[0,\infty)$ is a global solution of* (1.1) *and f satisfies* (A1) *and* (A3). *Then*

$$\max\big\{t^{1-\alpha}|y(t)|, t^{1-(\alpha-\beta)}|D^\beta y(t)|\big\} \le z(t), \quad t>0,$$

where, for $t>0$,

$$\begin{aligned} z(t) = K_1 + K_2\Big[\Big(\int_0^t h_1^q(s)\varphi_1^q(s^{1-\alpha}|y(s)|)ds\Big)^{1/q} \\ + \Big(\int_0^t h_2^q(s)\varphi_2^q(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|)ds\Big)^{1/q}\Big], \end{aligned} \tag{4.9}$$

$$K_1 = |b|\max\big\{\frac{1}{\Gamma(\alpha)}, \frac{1}{\Gamma(\alpha-\beta)}\big\}, \quad K_2 = \max\{C_3, C_3'\},$$

$$C_3 = \max\{C_1, C_2\}, \quad C_3' = \max\{C_1', C_2'\},$$

$$C_i = \frac{1}{\Gamma(\alpha)}\Big(\max\{1, 2^{p(1-\alpha)}\}\Gamma(p\gamma_i+1)\Big(1+\frac{p\gamma_i+1}{p(\alpha-1)+1}\Big)(p\delta_i)^{-(p\gamma_i+1)}\Big)^{1/p},$$

$$C_i' = \Big(\frac{\max\{1, 2^{p(1-(\alpha-\beta))}\}\Gamma(p\gamma_i+1)}{\Gamma^p(\alpha-\beta)}\Big(1+\frac{p\gamma_i+1}{p(\alpha-\beta-1)+1}\Big)(p\delta_i)^{-(p\gamma_i+1)}\Big)^{1/p},$$

for $i=1,2$.

Proof. Multiplying both sides of (4.4) by $t^{1-\alpha}$ and using the assumption (4.2), we obtain

$$t^{1-\alpha}|y(t)|$$

$$
\begin{aligned}
&\le \frac{|b|}{\Gamma(\alpha)} + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\gamma_1} e^{-\delta_1 s} h_1(s) \varphi_1(s^{1-\alpha}|y(s)|) ds \\
&\quad + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\gamma_2} e^{-\delta_2 s} h_2(s) \varphi_2(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|) ds, \quad t>0.
\end{aligned}
$$

Since $q > \frac{1}{\alpha-\beta}$, $\gamma_i > \frac{1}{q} - 1$, $\delta_i > 0$, then $p(\alpha-1)+1 > 0$, $p\gamma_i + 1 > 0$ and $p\delta_i > 0$, $i = 1, 2$, so we can apply Lemma 3.3, for $t > 0$, we obtain

$$
\begin{aligned}
t^{1-\alpha}|y(t)| \le \frac{|b|}{\Gamma(\alpha)} &+ C_3 \Big[\Big(\int_0^t h_1^q(s) \varphi_1^q(s^{1-\alpha}|y(s)|) ds \Big)^{1/q} \\
&+ \Big(\int_0^t h_2^q(s) \varphi_2^q(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|) ds \Big)^{1/q} \Big].
\end{aligned} \tag{4.10}
$$

Multiplying both sides of (4.7) by $t^{1-(\alpha-\beta)}$ and using (4.2), we obtain

$$
\begin{aligned}
&t^{1-(\alpha-\beta)}|D_0^\beta y(t)| \\
&\le \frac{|b|}{\Gamma(\alpha-\beta)} + \frac{t^{1-(\alpha-\beta)}}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^{\gamma_1} e^{-\delta_1 s} h_1(s) \varphi_1(s^{1-\alpha}|y(s)|) ds \\
&\quad + \frac{t^{1-(\alpha-\beta)}}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} s^{\gamma_2} e^{-\delta_2 s} h_2(s) \varphi_2\big(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|\big) ds, \quad t>0.
\end{aligned}
$$

By Lemma 3.3, for $t > 0$, we can write

$$
\begin{aligned}
&t^{1-(\alpha-\beta)}|D_0^\beta y(t)| \\
&\le \frac{|b|}{\Gamma(\alpha-\beta)} + C_3' \Big[\Big(\int_0^t h_1^q(s) \varphi_1^q(s^{1-\alpha}|y(s)|) ds \Big)^{1/q} \\
&\quad + \Big(\int_0^t h_2^q(s) \varphi_2^q\big(s^{1-(\alpha-\beta)}|D_0^\beta y(s)|\big) ds \Big)^{1/q} \Big].
\end{aligned} \tag{4.11}
$$

Therefore, the result follows from (4.9), (4.10) and (4.11). □

5. Power-type decay

We consider the space

$$
C_{1-\alpha}^\alpha[0,\infty) = \{ y \in C_{1-\alpha}[0,\infty) : D_0^\alpha y \in C_{1-\alpha}[0,\infty) \}. \tag{5.1}
$$

Theorem 5.1. *Suppose that f satisfies* (A1) *and* (A2)*, then, for any global solution $y \in C_{1-\alpha}[0,\infty)$ of* (1.1)*, there exists a positive constant C such that*

$$
|y(t)| \le C t^{\alpha-1} \quad \text{and} \quad |D_0^\beta y(t)| < C t^{\alpha-\beta-1}, \quad t>0,
$$

provided that

$$
\int_{x_0}^\infty \frac{ds}{\varphi_1^q(s^{1/q}) \varphi_2^q(s^{1/q})} = \infty, \quad x_0 > 0.
$$

Proof. From Lemma 4.1 we conclude

$$
\varphi_1(t^{1-\alpha}|y(t)|) \le \varphi_1(z(t)), \quad \varphi_2(t^{1-(\alpha-\beta)}|D^\beta y(t)|) \le \varphi_2(z(t)), \quad t>0, \tag{5.2}
$$

where $z(t)$ is as in (4.3). Using the inequalities in (5.2), from (4.3) it follows that

$$
z(t) \le K_1 + K_2 \Big(\int_0^t h^q(s) \varphi_1^q(z(s)) \varphi_2^q(z(s)) ds \Big)^{1/q}, \quad t>0.
$$

Therefore, by Corollary 3.6 with $w(t)=\varphi_1^q(t)\varphi_2^q(t)$, we deduce that

$$z(t)\le \Big[G^{-1}\Big(G(2^{q-1}K_1)+2^{q-1}K_2\int_0^t h^q(s)ds\Big)\Big]^{1/q},\quad t>0.$$

Since $h\in L_q(0,\infty)$, we have

$$z(t)\le C=\Big[G^{-1}\Big(G(2^{q-1}K_1)+2^{q-1}K_2\int_0^\infty h^q(s)ds\Big)\Big]^{1/q}<\infty.$$

Again, by Lemma 4.1,

$$|y(t)|\le Ct^{\alpha-1}\quad\text{and}\quad |D_0^\beta y(t)|<Ct^{\alpha-\beta-1},\quad t>0.$$

□

Theorem 5.2. *Suppose that f satisfies* (A1) *and* (A3). *Then, for each global solution $y\in C_{1-\alpha}[0,\infty)$ of* (1.1)*, there exists a positive constant C such that*

$$|y(t)|\le Ct^{\alpha-1}\quad\textit{and}\quad |D_0^\beta y(t)|<Ct^{\alpha-\beta-1},\quad t>0,$$

provided that

$$\int_{x_0}^\infty\frac{ds}{\varphi_1^q(s^{1/q})+\varphi_2^q(s^{1/q})}=\infty,\quad x_0>0.$$

Proof. By Lemma 4.2 we see that

$$\varphi_1(t^{1-\alpha}|y(t)|)\le\varphi_1(z(t)),\quad \varphi_2(\big(t^{1-(\alpha-\beta)}|D^\beta y(t)|\big)\le\varphi_2(z(t)),\quad t>0. \tag{5.3}$$

Taking into account (4.9) and (5.3), we have

$$z(t)\le K_1+K_2\Big[\Big(\int_0^t h_1^q(s)\varphi_1^q(z(s))ds\Big)^{1/q}+\Big(\int_0^t h_2^q(s)\varphi_2^q(z(s))ds\Big)^{1/q}\Big],\quad t>0.$$

Therefore, by Corollary 3.7 with $w_i(t)=\varphi_i^q(t)$, $i=1,2$, we find

$$z(t)\le\Big[G^{-1}\Big(G(2^{q-1}K_1^q)+2^{2(q-1)}K_2^q\int_0^t[h_1^q(s)+h_2^q(s)]ds\Big)\Big]^{1/q},\quad t>0.$$

Since $h_i\in L_q(0,\infty)$,

$$z(t)\le C=\Big[G^{-1}\Big(G(2^{q-1}K_1)+2^{q-1}K_2\int_0^\infty h^q(s)ds\Big)\Big]^{1/q}<\infty.$$

Thus

$$|y(t)|\le Ct^{\alpha-1}\quad\text{and}\quad |D_0^\beta y(t)|<Ct^{\alpha-\beta-1},\quad t>0.$$

□

6. Examples

In this section, we provide three examples, where we apply Theorems 5.1 and 5.2 to show that all global solutions decay like $t^{\alpha-1}$. Unlike the first example, the solutions of the second and third examples may not be available explicitly.

Example 1. Consider the problem

$$\begin{gathered} D_0^\alpha y(t) = -\lambda t^{\frac{\alpha-1}{q}}[E_{\alpha,\alpha}(-\lambda t^\alpha)]^{1/q}(y(t))^{1-1/q} \quad t>0, \\ I_0^{1-\alpha}y(t)|_{t=0} = 1, \quad 0<\alpha<1,\ q>1,\ \lambda>0, \end{gathered} \tag{6.1}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha,\beta>0.$$

We can rewrite the right-hand side of (6.1) as

$$\begin{aligned} &|-\lambda t^{\frac{\alpha-1}{q}}[E_{\alpha,\alpha}(-\lambda t^\alpha)]^{1/q}(y(t))^{1-1/q}| \\ &= \lambda t^{\frac{\alpha-1}{q}-(1-\alpha)(1-\frac{1}{q})}[E_{\alpha,\alpha}(-\lambda t^\alpha)]^{1/q}(t^{1-\alpha}y(t))^{1-\frac{1}{q}} \\ &= \lambda t^{\alpha-1}[E_{\alpha,\alpha}(-\lambda t^\alpha)]^{1/q}(t^{1-\alpha}y(t))^{1-1/q} \\ &= h(t)\varphi(t^{1-\alpha}y(t)), \end{aligned}$$

where $h(t) = \lambda t^{\alpha-1}[E_{\alpha,\alpha}(-\lambda t^\alpha)]^{1/q}$ and $\varphi(t) = t^{1-1/q}$. Notice that $h \in L_q(0,\infty)$ since [38, p 44, equation (1.8.33)]

$$\int_0^\infty t^{s-1}E_{\alpha,\beta}(-wt)dt = \frac{1}{w^s}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}, \quad \alpha,\beta,s,w>0, \tag{6.2}$$

and φ is a positive, continuous and nondecreasing function. Clearly, all conditions of Theorem 5.1 are satisfied. Therefore, any global solution $y \in C_{1-\alpha}[0,\infty)$ of (6.1) satisfies

$$|y(t)| \le Ct^{\alpha-1}, \quad t>0.$$

In fact the function

$$y(t) = t^{\alpha-1}[E_{\alpha,\alpha}(-\lambda t^\alpha)], \quad t>0,$$

is a global solution in $C_{1-\alpha}[0,\infty)$ of (6.1) and clearly

$$|y(t)| = |t^{\alpha-1}[E_{\alpha,\alpha}(-\lambda t^\alpha)]| \le Ct^{\alpha-1}, \quad t>0.$$

Example 2. Consider the problem

$$\begin{gathered} D_0^{1/2}y(t) = t^2e^{-2t}(\cos y^2)(y(t))^{1/5}(D_0^{1/3}y(t))^{1/3}, \quad t>0, \\ I_0^{1/2}y(t)\big|_{t=0} = b. \end{gathered} \tag{6.3}$$

Then the right-hand side satisfies

$$\begin{aligned} |f(t,y(t),D_0^{1/3}y(t))| &= |t^2e^{-2t}(\cos y^2)(y(t))^{1/5}\big(D_0^{1/3}y(t)\big)^{1/3}| \\ &\le t^\gamma e^{-t}h(t)\varphi_1(t^{1-1/2}y(t))\varphi_2\big(t^{1-(1/2-1/3)}D_0^{1/3}y(t)\big), \end{aligned}$$

where $\gamma = 73/45$, $h(t) = e^{-t}$, $\varphi_1(t) = t^{1/5}$ and $\varphi_2(t) = t^{1/3}$. All the conditions of Theorem 5.1 are satisfied. Therefore, any global solution $y \in C_{1/2}[0,\infty)$ of (6.3) satisfies

$$|y(t)| \le Ct^{\alpha-1} \quad \text{and} \quad |D_0^\beta y(t)| < Ct^{\alpha-\beta-1}, \quad \alpha=1/2, \beta=1/3.$$

Example 3. Consider the problem

$$\begin{gathered} D_0^{1/2}y(t) = t^2e^{-2t}(\cos y)\big(y(t)\big)^{1/3} + t^3e^{-4t}(\sin t^3)\big(D_0^{1/4}y(t)\big)^{1/3}, \quad t > 0, \\ I_0^{1/2}y(t)\big|_{t=0} = b. \end{gathered} \tag{6.4}$$

Note that this example is different from the previous ones. In fact, the right hand side of (6.4) is sum of two terms, similar to that of the assumption (A3). We can rewrite the right hand side of (6.4) as follows

$$\begin{aligned} &|t^2e^{-2t}(\cos y)(y(t))^{1/3} + t^3e^{-4t}(\sin t^3)\big(D_0^{1/4}y(t)\big)^{1/3}| \\ &\le t^{\gamma_1}e^{-t}h_1(t)\varphi_1\big(t^{1-1/2}y(t)\big) + t^{\gamma_2}e^{-2t}h_2(t)\varphi_2\big(t^{1-(1/2-1/4)}D_0^{1/4}y(t)\big), \end{aligned}$$

where $\gamma_1 = 11/6$, $\gamma_2 = 33/12$, $h_1(t) = e^{-t}$, $h_2(t) = e^{-2t}$ and $\varphi_1(t) = \varphi_2(t) = t^{1/3}$. All the conditions of Theorem 5.2 are satisfied and therefore any global solution $y \in C_{1/2}[0,\infty)$ of (6.4) satisfies

$$|y(t)| \le Ct^{\alpha-1}, \quad |D_0^{\beta}y(t)| < Ct^{\alpha-\beta-1}, \quad \alpha = 1/2, \beta = 1/4, \quad t > 0.$$

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