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NONEXISTENCE OF STABLE SOLUTIONS TO p-LAPLACE EQUATIONS WITH EXPONENTIAL NONLINEARITIES

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Abstract. In this note we prove the nonexistence of stable solutions to the p-Laplace equation $-\Delta_p u = e^u$ on the entire Euclidean space $\mathbb{R}^N$, where $p > 2$ and $N < \frac{p(p+3)}{p-1}$.

1. Introduction and statement of main results

We consider the p-Laplace equation

$$-\Delta_p u = e^u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $p > 2$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplace operator. We recall that $u \in C^1(\mathbb{R}^N)$ is said to be a weak solution of (1.1) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}(\nabla u, \nabla\varphi)dx = \int_{\mathbb{R}^N} e^u \varphi dx \tag{1.2}$$

for every $\varphi \in C_c^1(\mathbb{R}^N)$. This article concern the stable solutions of (1.1) in the following sense.

Definition 1.1. A weak solution u of (1.1) is stable if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}|\nabla\varphi|^2 dx + (p-2)\int_{\mathbb{R}^N} |\nabla u|^{p-4}(\nabla u, \nabla\varphi)^2 dx - \int_{\mathbb{R}^N} e^u\varphi^2 dx \geq 0$$

for every $\varphi \in C_c^1(\mathbb{R}^N)$.

Note that the above expression is nothing but the second that the variation of the energy functional associated with (1.1) is non-negative. Thus, if $u \in C^1(\mathbb{R}^N)$ is a local minimizer of the energy functional, then u is a stable solution of (1.1).

Remark 1.2. Let u be a stable solution of (1.1). Then

$$\int_{\mathbb{R}^N} e^u\varphi^2 dx \leq (p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-2}|\nabla\varphi|^2 dx \tag{1.3}$$

for every $\varphi \in C_c^1(\mathbb{R}^N)$.

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The nonexistence and stability of solutions to nonlinear elliptic partial differential equations have drawn much attention in the last decades. Readers can find recent developments on stable solutions in the monograph [6] by Dupaigne, and on related problems in [1, 3, 7, 13].

We should mention here the results in [8, 9] for Lane-Emden-Fowler equation $-\Delta u = |u|^{m-1}u$ where it is proved that there is no nontrivial stable solution if $1 < m < m_c(N)$, where $m_c(N)$ is explicitly given and is always greater than the Sobolev critical exponent. Later, these results were extended to quasilinear case $-\Delta_p u = |u|^{m-1}u$ in [4]. For more general nonlinearities, we mention paper [7] for semilinear equation $-\Delta u = f(u)$ and paper [2] for quasilinear equation $-\Delta_p u = f(u)$. In spite of dealing with general nonlinearity f, the nonexistence results in [2] can be applied only to (one-side) bounded solutions.

For the case of exponential nonlinearity, we refer to [5, 10] for a proof of nonexistence of stable solutions of the semilinear equation $-\Delta u = e^u$ in low dimensional Euclidean space. More precisely, the following theorem was proved in [10].

Theorem 1.3. *For $p = 2$ and $N \le 9$, there is no stable C^2-solution of* (1.1).

Recently, similar results were proved for the biharmonic equation $\Delta^2 u = e^u$ and, more generally, for the polyharmonic equation $(-\Delta)^m u = e^u$ in [11, 12]. The purpose of our paper is to come back to the second order elliptic equations and extend the results in [5, 10] to the p-Laplace equation $-\Delta_p u = e^u$. First of all, we prove the following a priori estimate for stable solutions.

Theorem 1.4. *Suppose that u is a stable solution of equation* (1.1)*. Then for any $\alpha \in (0, \frac{4}{p(p-1)})$, there exists $m = m(p, \alpha) > 0$ and a constant $C = C(p, \alpha) > 0$ such that for any function $\eta \in C_c^1(\mathbb{R}^N)$ with $0 \le \eta \le 1$ we have*

$$\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\eta^{pm}\,dx \le C\int_{\mathbb{R}^N} |\nabla\eta|^{p(p\alpha+1)}\,dx. \tag{1.4}$$

The method of proof is inspired by the techniques developed in [4, 5, 10]. Our main result is the following theorem, which is a generalization of the Theorem 1.3.

Theorem 1.5. *For $N < \frac{p(p+3)}{p-1}$, there is no stable C^1 solution of* (1.1).

Theorem 1.5 is sharp when $p = 2$ as already pointed out in [10]. However, the optimality of the dimension N in terms of p is still an interesting open question for $p > 2$.

Open problem. For $N \ge \frac{p(p+3)}{p-1}$, does equation (1.1) admit a stable C^1 solution?

As far as we know, there is no result on nonexistence of stable solutions for (1.1) on the case $p < 2$. Hence, this case should be also an interesting topic for future research.

2. Proofs

In the sequel, we denote by C a generic constant whose concrete values may change from line to line or even in the same line. If this constant depends on an arbitrary small number ε, then we will denote it by C_ε. We also use Young inequality in the form $ab \le \varepsilon a^p + C_\varepsilon b^q$ for $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

Proof of Theorem 1.4. We split the proof into two steps.

Step 1. For any $\varepsilon \in (0, p\alpha)$ and for any nonnegative function $\psi \in C_c^1(\mathbb{R}^N)$, there exists a constant $C_\varepsilon = C(p,\varepsilon) > 0$ such that

$$(p\alpha - \varepsilon)\int_{\mathbb{R}^N} |\nabla u|^p e^{p\alpha u}\psi^p\,dx \le C_\varepsilon \int_{\mathbb{R}^N} e^{p\alpha u}|\nabla \psi|^p\,dx + \int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx. \tag{2.1}$$

To prove this, using $\varphi = e^{p\alpha u}\psi^p$ as a test function in 1.2, since

$$\nabla\varphi = p\alpha e^{p\alpha u}\psi^p\nabla u + pe^{p\alpha u}\psi^{p-1}\nabla\psi,$$

we obtain

$$p\alpha\int_{\mathbb{R}^N} |\nabla u|^p e^{p\alpha u}\psi^p\,dx + p\int_{\mathbb{R}^N} |\nabla u|^{p-2}e^{p\alpha u}\psi^{p-1}(\nabla u, \nabla\psi)\,dx = \int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx.$$

Therefore,

$$\begin{aligned}
&p\alpha\int_{\mathbb{R}^N} |\nabla u|^p e^{p\alpha u}\psi^p\,dx\\
&\le p\int_{\mathbb{R}^N} |\nabla u|^{p-1}e^{p\alpha u}\psi^{p-1}|\nabla\psi|\,dx + \int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx\\
&\le \int_{\mathbb{R}^N} \varepsilon\Big(|\nabla u|^{p-1}e^{(p-1)\alpha u}\psi^{p-1}\Big)^{\frac{p}{p-1}} + C_\varepsilon\big(e^{\alpha u}|\nabla\psi|\big)^p\,dx + \int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx\\
&= \varepsilon\int_{\mathbb{R}^N} |\nabla u|^p e^{p\alpha u}\psi^p\,dx + C_\varepsilon\int_{\mathbb{R}^N} e^{p\alpha u}|\nabla\psi|^p\,dx + \int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx,
\end{aligned}$$

which implies (2.1).

Step 2. For any $\varepsilon \in (0, p\alpha)$, we set

$$\beta_\varepsilon = 1 - \Big(\frac{(p-1)p^2\alpha^2}{4} + \varepsilon\Big)\frac{1}{p\alpha - \varepsilon}$$

and we claim that there exists a constant $C_\varepsilon = C(p,\varepsilon) > 0$ such that

$$\beta_\varepsilon\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx \le C_\varepsilon\int_{\mathbb{R}^N} e^{p\alpha u}|\nabla\psi|^p\,dx. \tag{2.2}$$

To prove this, we use the stability assumption with $\varphi = e^{\frac{p\alpha u}{2}}\psi^{\frac{p}{2}}$. Since

$$\nabla\varphi = \frac{p\alpha}{2}e^{\frac{p\alpha u}{2}}\psi^{\frac{p}{2}}\nabla u + \frac{p}{2}e^{\frac{p\alpha u}{2}}\psi^{\frac{p-2}{2}}\nabla\psi,$$

using (1.3) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx &\le (p-1)\int_{\mathbb{R}^N} |\nabla u|^p\left(\frac{p\alpha}{2}\right)^2 e^{p\alpha u}\psi^p\,dx\\
&\quad + (p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-1}\frac{p^2\alpha}{2}e^{p\alpha u}\psi^{p-1}|\nabla\psi|\,dx\\
&\quad + (p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\left(\frac{p}{2}\right)^2 e^{p\alpha u}\psi^{p-2}|\nabla\psi|^2\,dx.
\end{aligned}$$

Now we use Young inequality to estimate the last two terms

$$\begin{aligned}
&(p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-1}\frac{p^2\alpha}{2}e^{p\alpha u}\psi^{p-1}|\nabla\psi|\,dx\\
&\le \int_{\mathbb{R}^N} \frac{\varepsilon}{2}\left(|\nabla u|^{p-1}e^{(p-1)\alpha u}\psi^{p-1}\right)^{\frac{p}{p-1}} + C_\varepsilon\left(e^{\alpha u}|\nabla\psi|\right)^p\,dx
\end{aligned}$$

$$= \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |\nabla u|^p e^{p\alpha u}\psi^p\,dx + C_\varepsilon \int_{\mathbb{R}^N} e^{p\alpha u}|\nabla\psi|^p\,dx,$$

and

$$\begin{aligned}
&(p-1)\int_{\mathbb{R}^N} |\nabla u|^{p-2}\Big(\frac{p}{2}\Big)^2 e^{p\alpha u}\psi^{p-2}|\nabla\psi|^2\,dx\\
&\le \int_{\mathbb{R}^N} \frac{\varepsilon}{2}\Big(|\nabla u|^{p-2}e^{(p-2)\alpha u}\psi^{p-2}\Big)^{\frac{p}{p-2}} + C_\varepsilon\left(e^{2\alpha u}|\nabla\psi|^2\right)^{\frac{p}{2}}\,dx\\
&= \frac{\varepsilon}{2}\int_{\mathbb{R}^N} |\nabla u|^p e^{p\alpha u}\psi^p\,dx + C_\varepsilon\int_{\mathbb{R}^N} e^{p\alpha u}|\nabla\psi|^p\,dx.
\end{aligned}$$

Plugging these two estimates into the previous one, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx\\
&\le \Big(\frac{(p-1)p^2\alpha^2}{4}+\varepsilon\Big)\int_{\mathbb{R}^N}|\nabla u|^p e^{p\alpha u}\psi^p\,dx + C_\varepsilon\int_{\mathbb{R}^N} e^{p\alpha u}|\nabla\psi|^p\,dx\\
&\le \Big(\frac{(p-1)p^2\alpha^2}{4}+\varepsilon\Big)\frac{1}{p\alpha-\varepsilon}\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\psi^p\,dx + C_\varepsilon\int_{\mathbb{R}^N} e^{p\alpha u}|\nabla\psi|^p\,dx.
\end{aligned}$$

We have used (2.1) in the last inequality. The claim (2.2) is now proved.

We are now in a position to prove the Theorem 1.4. Since $\lim_{\varepsilon\to 0}\beta_\varepsilon = 1 - \frac{\alpha p(p-1)}{4} > 0$, we can find some $\varepsilon\in(0,1)$ depending on p and α such that $\beta_\varepsilon > 0$. Next we choose some m large enough satisfying $(m-1)\frac{p\alpha+1}{\alpha} \ge pm$ and apply (2.2) for $\psi = \eta^m$ to obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\eta^{pm}\,dx &\le C\int_{\mathbb{R}^N} e^{p\alpha u}\eta^{p(m-1)}|\nabla\eta|^p\,dx\\
&\le \int_{\mathbb{R}^N} \varepsilon\Big(e^{p\alpha u}\eta^{p(m-1)}\Big)^{\frac{p\alpha+1}{p\alpha}} + C_\varepsilon(|\nabla\eta|^p)^{p\alpha+1}\,dx\\
&\le \varepsilon\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\eta^{pm}\,dx + C_\varepsilon\int_{\mathbb{R}^N}|\nabla\eta|^{p(p\alpha+1)}\,dx.
\end{aligned}$$

Hence, (1.4) follows. □

Proof of Theorem 1.5. By contradiction, we suppose that (1.1) admits a stable solution for $N < \frac{p(p+3)}{p-1}$. Since

$$\lim_{\alpha\to\frac{4}{p(p-1)}} N - p(p\alpha+1) = N - \frac{p(p+3)}{p-1} < 0,$$

we may find some $\alpha \in \Big(0, \frac{4}{p(p-1)}\Big)$ such that $N - p(p\alpha+1) < 0$. We then apply Theorem 1.4 for a test function $\eta_R \in C_c^1(\mathbb{R}^N)$ satisfying $0 \le \eta_R \le 1$ in $\mathbb{R}^N$, $\eta_R = 1$ in $B(0,R)$ and $\eta_R = 0$ in $\mathbb{R}^N\setminus B(0,2R)$ to obtain

$$\int_{B(0,R)} e^{(p\alpha+1)u}\,dx \le CR^{N-p(p\alpha+1)}.$$

Letting $R\to\infty$ in the last inequality we obtain $\int_{\mathbb{R}^N} e^{(p\alpha+1)u}\,dx = 0$, a contradiction. This completes the proof. □

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