Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 35, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF TWO POSITIVE SOLUTIONS FOR INDEFINITE KIRCHHOFF EQUATIONS IN $\mathbb{R}^3$

LING DING, YI-JIE MENG, SHI-WU XIAO, JIN-LING ZHANG

ABSTRACT. In this article we study the Kirchhoff type equation
$$-\Big(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+u=k(x)f(u)+\lambda h(x)u,\quad x\in\mathbb{R}^3,$$
$$u\in H^1(\mathbb{R}^3),$$
involving a linear part $-\Delta u+u-\lambda h(x)u$ which is coercive if $0<\lambda<\lambda_1(h)$ and is noncoercive if $\lambda>\lambda_1(h)$, a nonlocal nonlinear term $-b\int_{\mathbb{R}^3}|\nabla u|^2dx\Delta u$ and a sign-changing nonlinearity of the form $k(x)f(s)$, where $b>0$, $\lambda>0$ is a real parameter and $\lambda_1(h)$ is the first eigenvalue of $-\Delta u+u=\lambda h(x)u$. Under suitable assumptions on f and h, we obtain positives solution for $\lambda\in(0,\lambda_1(h))$ and two positive solutions with a condition on k.

1. Introduction and statement of main results

In this paper, we consider the Kirchhoff equation
$$\begin{gathered}-\Big(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\Delta u+u=k(x)f(u)+\lambda h(x)u,\quad x\in\mathbb{R}^3,\\ u\in H^1(\mathbb{R}^3),\end{gathered}\tag{1.1}$$
where b is a positive constant, $\lambda>0$ is a real parameter, $k(x)$ is sign changing in $\mathbb{R}^3$ which is why we call problem (1.1) indefinite Kirchhoff equation, $f:\mathbb{R}^+\to\mathbb{R}$ is a continuous function and $h(x)$ is a positive function.

When Ω is a smooth bounded domain in $\mathbb{R}^N$, the problem
$$\begin{gathered}-\Big(a+b\int_{\Omega}|\nabla u|^2dx\Big)\Delta u=g(x,u),\quad x\in\Omega,\\ u=0,\quad x\in\partial\Omega,\end{gathered}\tag{1.2}$$
is related to the stationary analogue of the Kirchhoff equation which was proposed by Kirchhoff in 1883 (see [17]) as an generalization of the well-known d'Alembert's wave equation
$$\rho\frac{\partial^2u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2u}{\partial x^2}=g(x,u)$$

2010 *Mathematics Subject Classification.* 35J60, 35B38, 35A15.
Key words and phrases. Indefinite Kirchhoff equation; concentration compactness lemma; (PS) condition; Ekeland's variational principle.

Submitted August 20, 2015. Published January 25, 2016.

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension.

In [3], it was pointed out that the problem (1.2) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effect also finds its applications in biological systems. A parabolic version of equation (1.1) can, in theory, be used to describe the growth and movement of a particular species. The movement, modeled by the integral term, is assumed to be dependent on the energy of the entire system with u being its population density. Alternatively, the movement of a particular species may be subject to the total population density within the domain (for instance, the spreading of bacteria) which gives rise to equations of the type $u_t - a(\int_\Omega u dx)\Delta u = h$. Some early classical studies of Kirchhoff's equation were those of Bernstein [9] and Pohožaev [22]. However, equation (1.2) received great attention only after that Lions [18] proposed an abstract framework for the problem. Some interesting results for problem (1.2) can be found in [6, 10, 5] and the references therein.

Some interesting studies by variational methods can be found in [3] and [15]-[12] for Kirchhoff-type problem (1.2) in a bounded domain Ω of $\mathbb{R}^N$. Very recently, some authors had studied the multiplicity of solutions for the Kirchhoff equation on the whole space $\mathbb{R}^N$. Jin and Wu [16] obtained the existence of infinitely many radial solutions for problem (1) in $\mathbb{R}^N$ using the Fountain Theorem. Wu [24] obtained four new existence results for nontrivial solutions and a sequence of high energy solutions for (1.1) in $\mathbb{R}^N$ which was obtained by using the Symmetric Mountain Pass Theorem. Azzollini, d'Avenia and Pomponio [7] obtained a multiplicity result concerning the critical points of a class of functionals involving local and nonlocal nonlinearities, then they apply their result to the nonlinear elliptic Kirchhoff equation (1.1) in $\mathbb{R}^N$ assuming that the local nonlinearity satisfies the general hypotheses introduced by Berestycki and Lions [8]. He and Zou [14] study the existence, multiplicity and concentration behavior of positive solutions for the nonlinear Kirchhoff type problem. They relate the number of solutions with the topology of the set. Alves and Figueiredo in [4] study a periodic Kirchhoff equation in $\mathbb{R}^N$, they get the nontrivial solution when the nonlinearity is in subcritical case and critical case. Liu and He [20] get multiplicity of high energy solutions for superlinear Kirchhoff equations in $\mathbb{R}^3$. Recently, Chen in [11] obtained the existence result of a positive solution for any $\lambda \in (0, \lambda_1(h))$ and the multiplicity result of two positive solutions for any $\lambda \in (\lambda_1(h), \lambda_1(h) + \widetilde{\delta})$ for problem (1.1) with the indefinite nonlinearity $k(x)f(s) = k(x)|s|^{p-2}s$ $(4 < p < 6)$. Especially, inspired by [11, 13], we shall consider the general $f(s)$ instead of power nonlinearity $f(s) = |s|^{p-2}s$ in the indefinite nonlinearity $k(x)f(s)$ like as problem (1.1).

In this article, we shall prove that (1.1) has multiple positive solutions for suitable range λ. We assume that functions k, f, h satisfy the following hypotheses:

- (H1) $k \in C(\mathbb{R}^3)$ and $k(x)$ changes sign, i.e., $\Omega^+ \neq \emptyset$, $\Omega^- \neq \emptyset$, where $\Omega^+ = \{x \in \mathbb{R}^3 | k(x) > 0\}$, $\Omega^- = \{x \in \mathbb{R}^3 | k(x) < 0\}$. Moreover, let $\Omega^0 = \{x \in \mathbb{R}^3 | k(x) = 0\}$.
- (H2) There exist positive constants R_0, K_0 and M such that $k(x) < -K_0$ and $|k(x)| \leq M$ if $|x| > R_0$.
- (H3) $f \in C(\mathbb{R}^+, \mathbb{R})$, $f(s) > 0$ for any $s > 0$.

(H4) $\lim_{s\to 0}\frac{f(s)}{s^{p-1}}=1$, $4<p<6$
(H5) $|f(s)|=|s|^{q-1}+O(|s|^{\alpha})$ as $|s|\to\infty$ for any $4<q<6$ and some $\alpha\in[0,1)$.
(H6) $h\in L^{3/2}(\mathbb{R}^3)$, $h(x)\geq 0$ for any $x\in\mathbb{R}^3$ and $h\not\equiv 0$.

Furthermore, without loss of generality, we will assume that $f(s)$ is defined for all $s\in\mathbb{R}$ as an odd function.

Before stating our main results, we give some notations and remarks. For any $1\leq t\leq+\infty$, we denote by $\|\cdot\|_t$ the usual norm of the Lebesgue space $L^t(\mathbb{R}^3)$. Define the function space

$$H^1(\mathbb{R}^3):=\{u\in L^2(\mathbb{R}^3):\nabla u\in L^2(\mathbb{R}^3)\}$$

with the standard product and norm

$$(u,v)=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla v+uv)dx,\quad \|u\|:=\Big(\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx\Big)^{1/2}.$$

The functional corresponding to problem (1.1) is defined in $H^1(\mathbb{R}^3)$ by

$$\begin{aligned}I_\lambda(u)&=\frac12\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2\\&\quad-\int_{\mathbb{R}^3}k(x)F(u)dx-\frac{\lambda}{2}\int_{\mathbb{R}^3}h(x)u^2dx,\end{aligned}$$

where $F(s)=\int_0^s f(\sigma)d\sigma$. Clearly, I_λ is well defined and is of class C^1 in $H^1(\mathbb{R}^3)$ by (H1)–(H3) and (H5)–(H6). Moreover,

$$\begin{aligned}\langle I_\lambda'(u),\varphi\rangle&=\int_{\mathbb{R}^3}(\nabla u\cdot\nabla\varphi+u\varphi)dx+b\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)\int_{\mathbb{R}^3}\nabla u\cdot\nabla\varphi dx\\&\quad-\int_{\mathbb{R}^3}k(x)f(u)\varphi dx-\lambda\int_{\mathbb{R}^3}h(x)u\varphi dx.\end{aligned}$$

There is one to one correspondence between the solution of problem (1.1) and the critical point of the functional I_λ. So in order to seek for the positive solution of (1.1), we only need to study the existence of the positive critical point of I_λ. Furthermore, if (H6) holds, then for every $u\in H^1(\mathbb{R}^3)$, there exists a unique $w\in H^1(\mathbb{R}^3)$ such that

$$-\Delta w+w=h(x)u.$$

Moreover, the operator $\mathcal{F}_h:H^1(\mathbb{R}^3)\to H^1(\mathbb{R}^3)$ defined by $\mathcal{F}_h(u)=w$ is positive and compact. Using this and the spectral theory of compact symmetric operator on Hilbert space, there exists a sequence of eigenvalues $\lambda_n(h)$ going to infinity for problem

$$-\Delta u+u=\lambda h(x)u,\ u\in H^1(\mathbb{R}^3)$$

with $0<\lambda_1(h)<\lambda_2(h)\leq\dots\leq\lambda_n(h)\leq\dots$ and each eigenvalue being of finite multiplicity. The first eigenvalue $\lambda_1(h)$ is simple, has a positive eigenfunction. The associated normalized eigenfunctions to sequence of eigenvalues are denoted by $e_1,e_2,\dots$ with $\|e_i\|=1$. In addition, we have the following variational characterization of $\lambda_n(h)$:

$$\lambda_1(h)=\inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}}\frac{\|u\|^2}{\int_{\mathbb{R}^3}h(x)u^2dx},\quad \lambda_n(h)=\inf_{u\in S_{n-1}^{\perp}\setminus\{0\})}\frac{\|u\|^2}{\int_{\mathbb{R}^3}h(x)u^2dx},\tag{1.3}$$

where $S_{n-1}^{\perp}=\{\mathrm{span}\{e_1,e_2,\dots,e_{n-1}\}\}^{\perp}$.

Let us consider the closed subspace of $H^1(\mathbb{R}^3)$ defined by

$$H^1\Big(\frac{1}{\sqrt{1+a}}\Omega^0\Big) = \big\{u \in H^1(\mathbb{R}^3)|u(x) = 0 \text{ a. e. in } \mathbb{R}^3 \setminus \frac{1}{\sqrt{1+a}}\Omega^0\big\}$$

for some $a > 0$. Let $u(x) = w(x/\sqrt{1+a})$, then the equation

$$-(1+a)\Delta u + u = \lambda h(x)u,\ x \in \Omega^0$$

becomes

$$-\Delta w + w = \lambda h(\sqrt{1+a}x)w,\ x \in \frac{1}{\sqrt{1+a}}\Omega^0,$$

which has a sequence of eigenvalues $\lambda_n(a,h)$ with $0 < \lambda_1(a,h) < \lambda_2(a,h) \le \dots \le \lambda_n(a,h) \le \dots$, each eigenvalue being of finite multiplicity and

$$\lambda_1(a,h) = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}[(1+a)|\nabla u|^2 + |u|^2]dx}{\int_{\mathbb{R}^3} h(x)u^2dx}.$$

Clearly, we have

$$\lambda_1(a,h) > \lambda_1(h).$$

Especially, $\lambda_1(a,h) = \lambda_1(h)$ if $a = 0$. Let $\delta_* = \lambda_1(a,h) - \lambda_1(h)$. If $\overline{\delta} \in [0,\delta_*)$, then

$$\lambda \notin \sigma(-\Delta, \frac{1}{\sqrt{1+a}}\Omega^0, h(\sqrt{1+a}x)) \quad \text{if } \lambda \in (0, \lambda_1(h) + \overline{\delta}),$$

where $\sigma(-\Delta, \frac{1}{\sqrt{1+a}}\Omega^0, h(\sqrt{1+a}x))$ denotes by the collection of eigenvalues of $-(1+a)\Delta + Id$ in $H_0^1(\frac{1}{\sqrt{1+a}}\Omega^0)$. If the Lebesgue measure of Ω^0 is zero, i.e., $|\Omega^0| = 0$, and $a = 0$, then $\sigma(-\Delta,\Omega^0,h(x)) = \emptyset$. If $|\Omega^0| \ne 0$ and $H^1(\frac{1}{\sqrt{1+a}}\Omega^0) \ne \{0\}$, it follows that $\sigma(-\Delta, \frac{1}{\sqrt{1+a}}\Omega^0, h(\sqrt{1+a}x))$ is discrete set and the equation

$$\int_{\Omega^0}[(1+a)\nabla u\cdot\nabla\varphi + u\varphi]dx = \lambda\int_{\Omega^0} h(x)u\varphi dx, \quad \forall\varphi \in H^1(\Omega^0)$$

has a nontrivial solution $u \in H_0^1(\Omega^0)$ if and only if $\lambda \in \sigma(-\Delta, \frac{1}{\sqrt{1+a}}\Omega^0, h(\sqrt{1+a}x))$.

Our main result is as follows.

Theorem 1.1. *Suppose that* (H1)–(H6) *hold. Then*

1. *for $0 < \lambda \le \lambda_1(h)$, problem* (1.1) *has at least one positive solution in $H^1(\mathbb{R}^3)$;*
2. *there exists $\widetilde{\delta} > 0$, for $\lambda_1(h) < \lambda < \lambda_1(h) + \widetilde{\delta}$, problem* (1.1) *has at least two positive solutions in $H^1(\mathbb{R}^3)$.*

Remark 1.2. Theorem 1.1 generalizes [11, Theorem 1.1] with $f(s) = |s|^{p-2}s$ to general form $f(s)$ satisfying (H3)-(H5). Furthermore, if $|\Omega^0| \ne 0$, Theorem 1.1 still holds in this paper, which is not considered in [11], and conditions of (H1) and (H2) are weaker than the corresponding ones in [11] because the existence of limit of $k(x)$ at $|x| \to \infty$ is not necessary. Moreover, it is not difficult to find some functions f satisfying (H3)-(H5). The typical example is that $f(s) = |s|^{p-2}s$. More generally, taking $\psi \in C_0^\infty(\mathbb{R}^3,[0,1])$ such that $\psi(x) = 1$ if $|x| < 1$ and $\psi(x) = 0$ if $|x| > 2$. Let

$$f(s) = \psi(x)s^{p-1} + (1-\psi(x))\big[s^{q-1} + P(x)/\widetilde{P}(x)s^\alpha\big] \quad \text{for } s > 0,$$

where $P(x)$ and $\widetilde{P}(x)$ are two polynomials with the same degree. Clearly, f satisfies (H3)-(H5).

Remark 1.3. For elliptic equations with indefinite nonlinearity, Alama and Tarantello [2] also have studied the existence of multiple positive solutions of

$$-\Delta u - \widetilde{\lambda} u = W(x)f(u), \quad u \in H_0^1(\Omega)$$

under the suitable assumptions f behaving like $|t|^{p-1}t(p \in (2, 2N/(N-2)))$ near zero with $\int_\Omega W(x)\widetilde{e}_1^p dx < 0$, where Ω is a smooth bounded domain of $\mathbb{R}^N (N \geq 3)$, W is sign changing on Ω and $\widetilde{e}_1$ is the positive eigenfunction corresponding to the first eigenvalue $\widetilde{\lambda}_1$ of the problem $-\Delta u = \widetilde{\lambda} u,\ u \in H_0^1(\Omega)$. Costa and Tehrani [13] obtained existence results of positive solutions for

$$-\Delta u - \widehat{\lambda} h(x) u = A(x)g(u), \quad u \in D^{1,2}(\mathbb{R}^N)$$

for suitable h, sign changing A and g with $\int_{\mathbb{R}^N} A(x)\overline{e}_1^p dx < 0$, where $\overline{e}_1$ is the positive eigenfunction corresponding to the first eigenvalue $\overline{\lambda}_1$ of problem $-\Delta u = \overline{\lambda} h(x) u,\ u \in D^{1,2}(\mathbb{R}^N)$ and A has "thick" zero set. But for Kirchhoff equations with indefinite nonlinerity like as (1.1), this kind of condition such as $\int_{\mathbb{R}^N} a(x)\overline{e}_1^p dx < 0$ and $\int_\Omega W(x)\widetilde{e}_1^p dx < 0$ and so on is not necessary, because the nonlocal nonlinear term $-b\int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u$ dominated over indefinite nonlinear term $k(x)f(s)$(see [11]). Furthermore, this yields that the condition: A has "thick" zero set in [13] or $|\Omega^0| = 0$ like as in [11] is also not necessary.

To obtain our result, we have to overcome various difficulties. First of all, since the equation is considered in the whole space $\mathbb{R}^3$ and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)(2 \leq s < 6)$ is no longer compact, the concentration-compactness lemma in [19] is applied to restore compactness properties to prove that I_λ satisfies (PS) condition by constructing sequences of "almost critical points" at those energy levels where compactness is available. On the other hand, because of the general term f in indefinite nonlinearity $k(x)f(s)$, we use the concentration-compactness lemma in [19] and not use Brezis-Lieb Lemma to prove (PS) condition like as in [11]. When $\lambda \in (0, \lambda_1(h))$, the linear part $-\Delta u + u - \lambda h(x) u$ of problem (1.1) is coercive, we can use standard variational techniques to find that zero is a local minimizer of the corresponding functional I_λ. But when $\lambda \in (\lambda_1(h), \lambda_1(h) + \widetilde{\delta})$, the linear part $-\Delta u + u - \lambda h(x) u$ of problem (1.1) is not coercive, this case with indefinite nonlinearity $k(x)f(s)$ involving general f and the nonlocal nonlinear term $-b\int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u$ makes us to face more difficult than the case of $\lambda \in (0, \lambda_1(h))$ such as the proofs of the boundedness of (PS) sequence and the mountain pass geometry of I_λ. To overcome these difficulties, we need more analysis technical to delicately analyze the behavior of the nonlocal nonlinear term $-b\int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u$ and the indefinite nonlinear term $k(x)f(s)$.

This paper is organized as follows. In section 2, we prove a (PS) condition. In section 3, we obtain the proof of our main result. In the following discussion, we denote various positive constants by C or $C_i (i = 1, 2, 3, \dots)$ for convenience.

2. Palais-Smale condition

In this section, we shall prove that the functional I_λ satisfies the (PS) condition, that is, any $(PS)_c$ sequence has a convergent subsequence in $H^1(\mathbb{R}^3)$, where $(PS)_c$ sequence for the functional I_λ is referred to a sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ such that $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$ for $c \in \mathbb{R}$. We need the following Lemmas.

Lemma 2.1 ([25]). *Suppose that* (H6) *holds. Then the functional defined by* $u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)u^2 dx$ *is weakly continuous.*

Lemma 2.2. *Suppose that* (H1)–(H6) *hold. Then for every* $c \in \mathbb{R}$, *the* $(PS)_c$ *sequence is bounded in* $H^1(\mathbb{R}^3)$ *if* $\lambda \in (0, \lambda_1(h) + \overline{\delta})$.

Proof. Let $\{u_n\} \subset H^1(\mathbb{R}^3)$ be a $(PS)_c$ sequence for I_λ at the level c, i. e.,

$$\begin{aligned} I_\lambda(u_n) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + |u_n|^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2 dx\Big)^2 \\ - \int_{\mathbb{R}^3} k(x)F(u_n)dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} h(x)u_n^2 dx \to c \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \langle I'_\lambda(u_n), \varphi\rangle = \int_{\mathbb{R}^3}(\nabla u_n \cdot \nabla\varphi + u_n\varphi)dx + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\varphi dx \\ - \int_{\mathbb{R}^3} k(x)f(u_n)\varphi dx - \lambda\int_{\mathbb{R}^3} h(x)u_n\varphi dx = o(1)\|\varphi\| \end{aligned} \tag{2.2}$$

for any $\varphi \in H^1(\mathbb{R}^3)$ as $n \to \infty$. Arguing by contradiction, we assume that $t_n = \|u_n\|$ and $t_n \to \infty$ as $n \to \infty$. Denote $v_n := u_n/t_n$. Then, we have that $\|v_n\| = 1$ for each n. Going to a subsequence, if necessary, we may assume that there is $v \in H^1(\mathbb{R}^3)$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{gathered} v_n \rightharpoonup v \quad \text{in } H^1(\mathbb{R}^3), \\ v_n(x) \to v(x) \quad \text{a. e. in } \mathbb{R}^3, \\ v_n \to v \quad \text{in } L^t(\Omega) \text{ for } 2 \le t < 6, \\ |v_n(x)| \le w(x) \quad \text{for some } w \in L^t(\Omega). \end{gathered} \tag{2.3}$$

Hence, for any $\varphi \in H^1(\mathbb{R}^3)$, we have that

$$\int_{\mathbb{R}^3}\nabla v_n\cdot\nabla\varphi dx \to \int_{\mathbb{R}^3}\nabla v\cdot\nabla\varphi dx, \quad \int_{\mathbb{R}^3} v_n\varphi dx \to \int_{\mathbb{R}^3} v\varphi dx. \tag{2.4}$$

Step I: We claim $v(x) = 0$ a. e. in $\mathbb{R}^3$. In fact, since $u_n = t_nv_n$, (2.2) becomes

$$\begin{aligned} &\int_{\mathbb{R}^3}(\nabla v_n\cdot\nabla\varphi + v_n\varphi)dx + bt_n^2\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\cdot\nabla\varphi dx \\ &- |t_n|^{q-2}A_n(\varphi) - \lambda\int_{\mathbb{R}^3} h(x)v_n\varphi dx \\ &= \frac{o(1)\|\varphi\|}{t_n} = o(1), \end{aligned} \tag{2.5}$$

where $A_n(\varphi) = \int_{\mathbb{R}^3} h_n(x)dx$ and $h_n(x) := k(x)|v_n|^{q-2}v_n\frac{f(t_nv_n)}{|t_nv_n|^{q-2}t_nv_n}\varphi$. Note that

$$|t_n|^{q-2}A_n(\varphi) = \int_{\mathbb{R}^3} k(x)\frac{f(t_nv_n)}{t_nv_n}v_n\varphi dx.$$

On the set $\{x|v(x) \neq 0\}$, we have $|t_nv_n| \to +\infty$, and then, (2.3) and (H5) imply

$$\begin{aligned} h_n(x) &= k(x)|v_n|^{q-2}v_n\frac{f(t_nv_n)}{|t_nv_n|^{q-2}t_nv_n}\varphi \\ &\to k(x)|v|^{q-2}v\varphi; \end{aligned}$$

On the set $\{x : v(x) = 0\}$, we have $v_n(x) \to 0$, so, by (H1)–(H2), $\varphi \in H^1(\mathbb{R}^3)$, (H3) and (H5), we obtain

$$|h_n(x)| = \big|k(x)\frac{f(t_n v_n)}{t_n^{q-1}}\varphi\big| \le \frac{C_1(1 + |t_n|^{q-1}|v_n(x)|^{q-1})}{|t_n|^{q-1}} \to 0.$$

This with (2.3) involving $t = q - 1$ yield that

$$|h_n(x)| \le C_1(1 + |w(x)|^{q-1}) \in L^1(\Omega)$$

where $\Omega = \operatorname{supp}(\varphi)$. From the discussion above, by the Lebesgue dominated convergence theorem, we conclude

$$\begin{aligned} A_n(\varphi) := \int_{\mathbb{R}^3} h_n(x)dx &= \int_{\{x|v(x)\neq 0\}} h_n(x)dx + \int_{\{x|v(x)=0\}} h_n(x)dx \\ &\to \int_{\mathbb{R}^3} k(x)|v|^{q-2}v\varphi dx \end{aligned} \tag{2.6}$$

as $n \to \infty$. Divided (2.5) by t_n^{q-2} and passing to limit, together with $q \in (4, 6)$, $t_n \to \infty$ and (2.6), we obtain

$$A_n(\varphi) \to \int_{\mathbb{R}^3} k(x)|v|^{q-2}v\varphi dx = 0. \tag{2.7}$$

Since $\varphi \in H^1(\mathbb{R}^3)$ is arbitrary, (2.7) implies

$$v(x) = 0 \text{ if } x \in \Omega^+ \cup \Omega^-. \tag{2.8}$$

Taking any $\varphi \in C_0^\infty(\mathbb{R}^3)$, passing to limit in (2.5), by (2.8), (2.7) and the definition of Ω^0 in (H1), we obtain

$$(1 + b\|\nabla u_n\|_2^2)\int_{\mathbb{R}^3} \nabla v_n \cdot \nabla\varphi dx + \int_{\mathbb{R}^3} v_n\varphi dx - \lambda\int_{\mathbb{R}^3} h(x)v_n\varphi dx \to 0 \tag{2.9}$$

as $n \to \infty$. If $\{\|\nabla u_n\|_2^2\}$ is bounded, then there exist a convergent subsequence (still denoted by $\|\nabla u_n\|_2^2$) and some $a > 0$ such that $\|\nabla u_n\|_2^2 \to a/b$ as $n \to \infty$. Together with (2.4), Lemma 2.1 and (2.8), (2.9) can become to

$$\int_{\Omega_0} (1 + a)\nabla v \cdot \nabla\varphi dx + \int_{\Omega_0} v\varphi dx - \lambda\int_{\Omega_0} h(x)v\varphi dx = 0$$

as $n \to \infty$. Since $\lambda \in (0, \lambda_1(h) + \overline{\delta})$, it follows that $v = 0$ a.e. on Ω_0. If $\|\nabla u_n\|_2^2 \to \infty$, divided (2.9) by $1 + b\|\nabla u_n\|_2^2$, passing to limit, we obtain

$$\int_{\Omega_0} \nabla v \cdot \nabla\varphi dx = 0.$$

Together with $v \in H^1(\mathbb{R}^3)$ and any $\varphi \in C_0^\infty(\mathbb{R}^3)$, we have $v = 0$ a.e. on Ω_0. Thus, $v = 0$ a.e. in $\mathbb{R}^3$. The proof of the claim is completed.

Step II: We shall prove that u_n is bounded in $H^1(\mathbb{R}^3)$. Indeed, Suppose that $\int_{\mathbb{R}^3} |\nabla v_n|^2dx \to \beta \ge 0$ as $n \to \infty$. If $\beta = 0$, since $\|v_n\| = 1$, then $\int_{\mathbb{R}^3} |v_n|^2dx \to 1$ as $n \to \infty$. This contradicts to $v_n \to 0$ a. e. in $\mathbb{R}^3$ which follows from Step I. Therefore, we conclude that $\|u_n\|$ is bounded. If $\beta > 0$, we need the following arguments.

Divided (2.1) by $t_n^2 = \|u_n\|^2$ and (2.2) by $t_n = \|u_n\|$, we obtain

$$\frac{1}{2} + \frac{b}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2dx \int_{\mathbb{R}^3} |\nabla v_n|^2dx - \int_{\mathbb{R}^3} k(x)\frac{F(u_n)}{t_n^2}dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} h(x)v_n^2dx \to 0 \tag{2.10}$$

and

$$\begin{aligned}\frac{1}{t_n}\langle I'_\lambda(u_n),\varphi\rangle &= \int_{\mathbb{R}^3}(\nabla v_n\nabla\varphi + v_n\varphi)dx + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}\nabla v_n\nabla\varphi dx\\ &\quad -\int_{\mathbb{R}^3}k(x)\frac{f(u_n)}{t_n}\varphi dx - \lambda\int_{\mathbb{R}^3}h(x)v_n\varphi dx \to 0\end{aligned}\tag{2.11}$$

as $n\to\infty$. Moreover, if we localize and take $\varphi = v\xi$ in (2.11) with $\xi\in C_0^\infty(\mathbb{R}^3)$, since $\langle I'_\lambda(u_n), v\xi\rangle = \langle I'_\lambda(u_n), v_n\xi\rangle - \langle I'_\lambda(u_n),(v_n-v)\xi\rangle$ and (2.3), passing to limit, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}(|\nabla v_n|^2\xi + v_n^2\xi + v_n\nabla v_n\nabla\xi)dx + b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla v_n|^2\xi dx\\ &+ b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}v_n\nabla v_n\nabla\xi dx - \int_{\mathbb{R}^3}k(x)\frac{f(u_n)u_n}{t_n^2}\xi dx\\ &-\lambda\int_{\mathbb{R}^3}h(x)v_n^2\xi dx\to 0.\end{aligned}\tag{2.12}$$

Since $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, $v=0$ a.e. in $\mathbb{R}^3$, (2.3) and Lemma 2.1, we obtain

$$\int_{\mathbb{R}^3}v_n\nabla v_n\nabla\xi dx\to 0,\quad \int_{\mathbb{R}^3}|v_n|^2\xi dx\to 0,\tag{2.13}$$

$$\int_{\mathbb{R}^3}h(x)v_n^2\xi dx\to 0,\quad \lim_{n\to\infty}\int_{\mathbb{R}^3}h(x)v_n^2dx = 0.\tag{2.14}$$

as $n\to\infty$. Inserting (2.14) into (2.10), we have

$$\int_{\mathbb{R}^3}k(x)\frac{F(u_n)}{t_n^2}dx = \frac{1}{2}+\frac{b}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla v_n|^2dx + o(1).\tag{2.15}$$

Inserting (2.13) and (2.14) into (2.12), we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}k(x)\frac{f(u_n)u_n}{t_n^2}\xi dx &= \Big(1+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}|\nabla v_n|^2\xi dx\\ &\quad + b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}v_n\nabla v_n\nabla\xi dx + o(1).\end{aligned}\tag{2.16}$$

Now, we claim that

$$\int_{\mathbb{R}^3}k(x)\frac{qF(u_n)}{t_n^2}\xi dx = \int_{\mathbb{R}^3}k(x)\frac{f(u_n)u_n}{t_n^2}\xi dx + o(1).\tag{2.17}$$

Indeed, by (H3), we know that $|qF(s)-f(s)s|\le C_2$ for $|s|\le M$. From $t_n\to\infty$ and (H1)–(H2), we clearly have

$$\int_{[|u_n|\le M]}|k(x)|\Big|\frac{qF(u_n)-f(u_n)u_n}{t_n^2}\Big||\xi|dx = o(1).$$

Also, (H5) implies that $|qF(s)-f(s)s| = O(|s|^{\alpha+1})$ as $|s|\to\infty$, where $1\le\alpha+1<2$, together with (H1) and (H2), we have

$$\int_{[|u_n|\ge M]}|k(x)|\Big|\frac{qF(u_n)-f(u_n)u_n}{t_n^2}\Big||\xi|dx \le C_3\int_{[supp\xi]\cap[|u_n|\ge M]}\frac{|u_n|^{\alpha+1}}{t_n^2}dx\to 0$$

as $n\to\infty$. Therefore, claim (2.17) is proved.

Choosing $\xi \in C_0^\infty(\mathbb{R}^3)$ such that $\xi \in [0,1]$, $\xi(x)=1$ if $|x|<R_0$ and $\xi(x)=0$ if $|x|>2R_0$, from (2.17), (2.15), (2.16) and (2.13), we obtain

$$
\begin{aligned}
&\liminf_{n\to\infty}\int_{\mathbb{R}^3} k(x)\frac{F(u_n)}{t_n^2}(1-\xi)dx\\
&=\liminf_{n\to\infty}\Big[\int_{\mathbb{R}^3} k(x)\frac{F(u_n)}{t_n^2}dx-\int_{\mathbb{R}^3} k(x)\frac{F(u_n)}{t_n^2}\xi dx\Big]\\
&=\liminf_{n\to\infty}\Big[\int_{\mathbb{R}^3} k(x)\frac{F(u_n)}{t_n^2}dx-\frac{1}{q}\int_{\mathbb{R}^3} k(x)\frac{f(u_n)u_n}{t_n^2}\xi dx\Big]\\
&=\liminf_{n\to\infty}\Big[\frac{1}{2}+\frac{b}{4}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla v_n|^2dx\\
&\quad -\frac{1}{q}\Big(1+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}|\nabla v_n|^2\xi dx-\frac{b}{q}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\xi dx\Big]\\
&\ge\liminf_{n\to\infty}\Big[\frac{1}{2}-\frac{1}{q}+b\big(\frac{1}{4}-\frac{1}{q}\big)\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla v_n|^2dx\\
&\quad -\frac{b}{q}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\xi dx\Big]\\
&\ge\frac{1}{2}-\frac{1}{q}+b\liminf_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big[\big(\frac{1}{4}-\frac{1}{q}\big)\int_{\mathbb{R}^3}|\nabla v_n|^2dx-\frac{1}{q}\int_{\mathbb{R}^3} v_n\nabla v_n\nabla\xi dx\Big]\\
&\ge\frac{1}{2}-\frac{1}{q}>0
\end{aligned}
\tag{2.18}
$$

because $\int_{\mathbb{R}^3}|\nabla v_n|^2dx\to\beta>0$ as $n\to\infty$ and $q>4$. Moreover, since f is an odd function, by the definition ξ and (H2), we have

$$
\liminf_{n\to\infty}\int_{\mathbb{R}^3} k(x)\frac{F(u_n)}{t_n^2}(1-\xi)dx=\liminf_{n\to\infty}\int_{|x|\ge R_0} k(x)\frac{F(u_n)}{t_n^2}(1-\xi)dx\le 0,
$$

which contradicts to (2.18). Thus, we conclude that $t_n=\|u_n\|$ is bounded. $\square$

Lemma 2.3 ([19]). *Let $\{\rho_n\}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying $\rho_n\ge 0$ and $\int_{\mathbb{R}^N}\rho_n dx=\overline{\lambda}>0$. Then, there exists a subsequence $\{\rho_{n_k}\}$ for which one of the three possibilities holds:*

Vanishing: $\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{y+B_R}\rho_{n_k}dx=0$ *for all $R>0$;*

Dichotomy: There exists $0<\alpha<\overline{\lambda}$ such that, for any given $\varepsilon>0$, there is a sequence $\{y_n\}\subset\mathbb{R}^N$, a number $R>0$ and a sequence $\{R_n\}\subset\mathbb{R}_+$, with $R<R_1$, $R_n<R_{n+1}\to+\infty$, such that, if we set $\rho_n^1=\rho_n\chi_{[|x-y_n|\le R]}$ and $\rho_n^2=\rho_n\chi_{[|x-y_n|\ge R_n]}$, then we have

$$
\|\rho_n-\rho_n^1-\rho_n^2\|_1\le\varepsilon,\quad \Big|\int_{\mathbb{R}^N}\rho_n^1dx-\alpha\Big|\le\varepsilon,\quad \Big|\int_{\mathbb{R}^N}\rho_n^2dx-(\overline{\lambda}-\alpha)\Big|\le\varepsilon;
$$

Compactness: There exists $y_k\in\mathbb{R}^N$ such that $\rho_{n_k}(\cdot+y_k)$ is tight, i.e. for each varepsilon >0 there exists $R>0$ such that

$$
\int_{y_k+B_R}\rho_{n_k}dx\ge\overline{\lambda}-\varepsilon.
$$

Lemma 2.4. *Suppose that* (H1)–(H6) *hold, then I_λ satisfies the (PS) condition if $\lambda\in(0,\lambda_1(h)+\overline{\delta})$.*

Proof. By Lemma 2.2, we know that a $(PS)_c$ sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Without loss of generality, we may assume that

$$C_0 \geq \|u_n\|^2 \geq C_4 > 0. \tag{2.19}$$

Therefore, by considering the sequence of $L^1(\mathbb{R}^3)$ functions

$$\rho_n = |\nabla u_n|^2 + |u_n|^2,$$

we have(by passing to a subsequence, if necessary) that

$$\int_{\mathbb{R}^3} \rho_n dx \to \overline{\lambda} > 0.$$

We shall use the concentration-compactness (Lemma 2.3) to show that $\{u_n\}$ has a convergent subsequence in $H^1(\mathbb{R}^3)$. In fact, we will rule out vanishing and dichotomy for the L^1 sequence of $\{\rho_n\}$.

Vanishing: In our situation vanishing can not occur. Indeed, if vanishing happens, i.e., $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{y+B_t} \rho_n dx = 0$ for all $t > 0$, then we have $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$, therefore, $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^3)$ for any $2 \leq s \leq 6$. Together with (2.1) and (2.2), we have

$$\frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} k(x)F(u_n)dx \to c$$

and

$$\begin{aligned} &\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 \\ &- \int_{|x|\leq R_0} k(x)f(u_n)u_n dx - \int_{|x|>R_0} k(x)f(u_n)u_n dx \to 0 \end{aligned} \tag{2.20}$$

as $n \to \infty$. From (H3)-(H5), there exist $C_i (i = 5,\dots,12)$ and $\delta_0 > 0$ such that

$$\begin{gathered} |f(s)| \leq C_5|s|^{p-1} + C_6|s|^{q-1}, \quad |F(s)| \leq C_7|s|^p + C_8|s|^q, \\ C_9|s|^p \leq |F(s)|, \quad C_{10}|s|^p \leq f(s)s \quad \text{if } |s| \leq \delta_0, \\ C_{11}|s|^q \leq |F(s)|, \quad C_{12}|s|^q \leq f(s)s \quad \text{if } |s| \geq \delta_0. \end{gathered} \tag{2.21}$$

Since $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^3)$ for any $2 \leq s \leq 6$, by (H1)–(H2) and (2.21), we obtain

$$\big|\int_{[|x|\leq R_0]} k(x)f(u_n)u_n dx\big| + \big|\int_{[|x|\leq R_0]} k(x)F(u_n)dx\big| = o(1),$$

where R_0 is given in (H2). Then (2.20) yields

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 - \int_{|x|>R_0} k(x)f(u_n)u_n dx \to 0$$

as $n \to \infty$. By (H2), we deduce that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx = o(1), \quad \int_{\mathbb{R}^3} k(x)f(u_n)u_n dx = o(1).$$

Hence, $\|u_n\| \to 0$ as $n \to \infty$, which is a contradiction.

Dichotomy: If dichotomy occurs, then there exists $0 < \alpha < \overline{\lambda}$ such that, for any given $\varepsilon > 0$ and $\overline{R} \geq 1$, there are sequences $\{y_n\} \subset \mathbb{R}^3$, $\{R_n\} \subset \mathbb{R}_+$ and $\widehat{R} > \overline{R}$ satisfying $R_0 < \widehat{R} < \frac{1}{2}R_1$, $R_n < R_{n+1} \to +\infty$ and

$$\alpha - \varepsilon < \int_{|x-y_n|\leq \frac{1}{2}\widehat{R}} (|\nabla u_n|^2 + |u_n|^2)dx < \alpha + \varepsilon,$$

$$\overline{\lambda} - \alpha - \varepsilon < \int_{|x-y_n|\geq 3R_n} (|\nabla u_n|^2 + |u_n|^2)dx < \overline{\lambda} - \alpha + \varepsilon. \tag{2.22}$$

In particular, we have

$$\int_{\frac{1}{2}\widehat{R}<|x-y_n|\leq 3R_n} (|\nabla u_n|^2 + |u_n|^2)dx < 2\varepsilon. \tag{2.23}$$

Note that we also have

$$\int_{\widehat{R}<|x-y_n|\leq 2R_n} |u_n|^6 dx < C_{13}\varepsilon^3. \tag{2.24}$$

In fact, if $\xi_n \in C_0^\infty(\mathbb{R}^3)$ is such that $\xi_n(x) = 0$ if $|x| \leq \frac{1}{2}\widehat{R}$ and $|x| \geq 3R_n$, $\xi_n = 1$ if $\widehat{R} \leq |x| \leq 2R_n$ and $|\nabla \xi_n| \leq \frac{1}{R_n - \widehat{R}}$, then $\xi_n u_n \in H^1(\mathbb{R}^3)$, by the Sobolev's inequality, (2.19) and (2.23), we have

$$\begin{aligned}
&\int_{\widehat{R}<|x-y_n|\leq 2R_n} |u_n|^6 dx \\
&\leq \int_{\frac{1}{2}\widehat{R}<|x-y_n|\leq 3R_n} |\xi_n u_n|^6 dx \\
&\leq \int_{\mathbb{R}^3} |\xi_n u_n|^6 dx \\
&\leq C_{14}\Big(\int_{\mathbb{R}^3} (|\nabla(\xi_n u_n)|^2 + |\xi_n u_n|^2)dx\Big)^3 \\
&\leq C_{15}\Big(\int_{\mathbb{R}^3} |\nabla \xi_n|^2 u_n^2 dx + \int_{\frac{1}{2}\widehat{R}<|x-y_n|\leq 3R_n} (|\nabla u_n|^2 + |u_n|^2)dx\Big)^3 \\
&\leq C_{13}\varepsilon^3
\end{aligned}$$

as $n \to \infty$.

Next, let us take $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $\zeta(x) = 1$ for $|x| \leq 1$, $\zeta(x) = 0$ for $|x| \geq 2$, as well as $\eta(x) = 1 - \zeta(x)$, and set

$$u_n^1 = \zeta\Big(\frac{\cdot - y_n}{\widehat{R}}\Big)u_n := \zeta_n u_n, \quad u_n^2 = \eta\Big(\frac{\cdot - y_n}{R_n}\Big)u_n := \eta_n u_n.$$

Clearly, $u_n^1 u_n^2 = 0$.

Case 1: If $\{y_n\}$ is bounded. Then, the support of the sequence $\{u_n^2\}$ approaches infinity, let $R_n \to \infty$, then $u_n^2 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. By (2.24), then we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3} h(x)u_n^2(u_n^2 - u_n)dx\Big| &\leq \int_{R_n\leq|x-y_n|\leq 2R_n} |h(x)||u_n^2||u_n^2 - u_n|dx \\
&\leq \int_{R_n\leq|x-y_n|\leq 2R_n} |h(x)||\zeta_n\eta_n||u_n|^2 dx \\
&\leq C_{16}\|h\|_{3/2}\varepsilon := O(\varepsilon)
\end{aligned}$$

In particular, letting $\mu(\varepsilon)$ denote a function which goes to zero as $\varepsilon \to 0$, we have

$$\int_{\mathbb{R}^3} h(x)u_n^2 u_n dx = \int_{\mathbb{R}^3} h(x)|u_n^2|^2 dx + \mu(\varepsilon). \tag{2.25}$$

Similarly, using (2.23) and argue as above, it is easy to see that

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla u_n^2 dx = \int_{\mathbb{R}^3} |\nabla u_n^2|^2 dx + \mu(\varepsilon), \quad \int_{\mathbb{R}^3} u_n u_n^2 dx = \int_{\mathbb{R}^3} |u_n^2|^2 dx + \mu(\varepsilon). \tag{2.26}$$

Furthermore, combining (H1)–(H2), (2.23), (2.24), (2.21) and Sobolev's inequality, we obtain

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3} k(x)(f(u_n)-f(u_n^2))u_n^2dx\Big| \\
&\leq C_{17}\int_{R_n\leq|x-y_n|\leq 2R_n}|u_n|^pdx + C_{18}\int_{R_n\leq|x-y_n|\leq 2R_n}|u_n|^qdx \\
&\leq C_{19}\Big(\int_{R_n\leq|x-y_n|\leq 2R_n}|u_n|^6dx\Big)^{p/6} + C_{20}\Big(\int_{R_n\leq|x-y_n|\leq 2R_n}|u_n|^6dx\Big)^{q/6} \\
&\leq C_{21}\varepsilon^{p/2} + C_{22}\varepsilon^{q/2} = \mu(\varepsilon).
\end{aligned}
$$

This yields

$$
\int_{\mathbb{R}^3} k(x)f(u_n)u_n^2dx = \int_{\mathbb{R}^3} k(x)f(u_n^2)u_n^2dx + \mu(\varepsilon). \tag{2.27}
$$

By (2.25)-(2.27), we obtain

$$
\begin{aligned}
&o(1) \\
&= \langle I'(u_n), u_n^2\rangle \\
&= \int_{\mathbb{R}^3}(\nabla u_n\nabla u_n^2 + u_nu_n^2)dx + b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}\nabla(u_n-u_n^2+u_n^2)\nabla u_n^2dx \\
&\quad -\int_{\mathbb{R}^3}k(x)f(u_n)u_n^2dx - \lambda\int_{\mathbb{R}^3}h(x)u_nu_n^2dx \\
&= \int_{\mathbb{R}^3}(|\nabla u_n^2|^2+|u_n^2|^2)dx + b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla u_n^2|^2dx \\
&\quad -\int_{\mathbb{R}^3}k(x)f(u_n^2)u_n^2dx - \lambda\int_{\mathbb{R}^3}h(x)|u_n^2|^2dx + \mu(\varepsilon).
\end{aligned}
$$

Since $R_n\to\infty$, $\{y_n\}$ is bounded, $u_n^2\rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3}h(x)|u_n^2|^2dx = o(1)$ as $n\to\infty$, we deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(|\nabla u_n^2|^2+|u_n^2|^2)dx + b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla u_n^2|^2dx - \int_{\mathbb{R}^3}k(x)f(u_n^2)u_n^2dx \\
&= o(1)+\mu(\varepsilon),
\end{aligned}
$$

which implies that

$$
\begin{gathered}
\int_{\mathbb{R}^3}(|\nabla u_n^2|^2+|u_n^2|^2)dx = o(1)+\mu(\varepsilon), \\
b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla u_n^2|^2dx = o(1)+\mu(\varepsilon), \\
-\int_{\mathbb{R}^3}k(x)f(u_n^2)u_n^2dx = o(1)+\mu(\varepsilon)
\end{gathered}
$$

because (H2) as $n\to\infty$. Then we have a contradiction, indeed, by (2.22), we have

$$
\int_{\mathbb{R}^3}(|\nabla u_n^2|^2+|u_n^2|^2)dx \geq \int_{|x-y_n|\geq 3R_n}(|\nabla u_n^2|^2+|u_n^2|^2)dx > \overline{\lambda}-\alpha-\varepsilon.
$$

Case 2: If $\{y_n\}$ is not bounded. Then, passing to a subsequence if necessary, we can assume that $|y_n|\to\infty$. In this case, the support of the sequence $\{u_n^1\}$ approcahes infinity, we can apply the same arguments above to u_n^1 to get a contradiction.

Compactness: Since we have ruled out vanishing and dichotomy, it follows that compactness necessarily take place, i.e. there exists $y_n \in \mathbb{R}^3$ such that for each $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{y_n+B_R} (|\nabla u_n|^2 + |u_n|^2)dx \geq \overline{\lambda} - \varepsilon.$$

In particular, we have

$$\int_{|x-y_n|\geq R} (|\nabla u_n|^2 + |u_n|^2)dx < \varepsilon. \tag{2.28}$$

By (2.28) and a similar method as above, we claim that $\{y_n\}$ must remain bounded. In fact, if not, then (2.28) implies that $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$, together with (H1)–(H6), we have

$$\begin{aligned}
o(1) &= \langle I'_\lambda(u_n), u_n\rangle \\
&= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 - \int_{\mathbb{R}^3} k(x)f(u_n)u_n dx + o(1) \\
&= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 - \int_{|x|\leq R_0} k(x)f(u_n)u_n dx \\
&\quad - \int_{|x|>R_0} k(x)f(u_n)u_n dx + o(1) \\
&= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + b\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 \\
&\quad - \int_{|x|>R_0} k(x)f(u_n)u_n dx + o(1) + \mu(\varepsilon).
\end{aligned}$$

This yields

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx = o(1) + \mu(\varepsilon), \quad \Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 = o(1) + \mu(\varepsilon),$$
$$\int_{\mathbb{R}^3} k(x)f(u_n)u_n dx = o(1) + \mu(\varepsilon).$$

Therefore,

$$\begin{aligned}
c + o(1) &= I_\lambda(u_n) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u_n|^2 + |u_n|^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\Big)^2 \\
&\quad - \int_{\mathbb{R}^3} k(x)F(u_n)dx - \frac{\lambda}{2}\int_{\mathbb{R}^3} h(x)u_n^2 dx \\
&= o(1) + \mu(\varepsilon).
\end{aligned}$$

This is a contradiction. Thus, $\{y_n\}$ is bounded in $\mathbb{R}^3$. The boundedness of $\{y_n\}$ and (2.28) imply

$$\int_{|x|\geq R} (|\nabla u_n|^2 + |u_n|^2)dx < \varepsilon. \tag{2.29}$$

Since $\{u_n\}$ is bounded, we have $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$ and $u_n \to u$ strongly in $L^t(\Omega)$ for any $2 \leq t < 6$, where Ω is bounded. From (2.29), we obtain

$$u_n \to u \quad \text{strongly in } L^t(\mathbb{R}^3) \text{ for } 2 \leq t < 6. \tag{2.30}$$

Equations (2.19), (2.30), (H1)–(H6) imply

$$\int_{\mathbb{R}^3} k(x)(f(u_n)-f(u))(u_n-u)dx \to 0, \quad \int_{\mathbb{R}^3} h(x)(u_n-u)^2dx \to 0,$$
$$\int_{\mathbb{R}^3} k(x)f(u_n)(u_n-u)dx \to 0, \quad \int_{\mathbb{R}^3} h(x)u_n(u_n-u)dx \to 0$$

as $n\to\infty$. This yields

$$\begin{aligned}
o(1) &= \langle I'_\lambda(u_n), u_n-u\rangle \\
&= \Big(1+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}\nabla u_n\nabla(u_n-u)dx+\int_{\mathbb{R}^3}u_n(u_n-u)dx \\
&\quad -\int_{\mathbb{R}^3}k(x)f(u_n)(u_n-u)dx-\lambda\int_{\mathbb{R}^3}h(x)u_n(u_n-u)dx \\
&= \Big(1+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}\nabla u_n\nabla(u_n-u)dx+o(1).
\end{aligned}$$

From the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^3)$, we obtain

$$\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u)dx\to 0$$

as $n\to\infty$. Moreover, we have

$$\begin{aligned}
&\langle I'_\lambda(u_n)-I'_\lambda(u), u_n-u\rangle \\
&= \Big(1+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}|\nabla(u_n-u)|^2dx+\int_{\mathbb{R}^3}|u_n-u|^2dx \\
&\quad +b\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx-\int_{\mathbb{R}^3}|\nabla u_n|^2dx\Big)\int_{\mathbb{R}^3}\nabla u\nabla(u_n-u))dx \\
&\quad -\int_{\mathbb{R}^3}k(x)(f(u_n)-f(u))(u_n-u)dx-\lambda\int_{\mathbb{R}^3}h(x)(u_n-u)^2dx \\
&\geq \int_{\mathbb{R}^3}(|\nabla(u_n-u)|^2+|u_n-u|^2)dx+o(1).
\end{aligned}$$

This yields

$$\|u_n-u\|^2\leq\langle I'_\lambda(u_n)-I'_\lambda(u), u_n-u\rangle+o(1)\to 0$$

as $n\to\infty$. Thus, $u_n\to u$ strongly in $H^1(\mathbb{R}^3)$. □

3. Existence of positive solutions

In this section, we shall prove our main result. Firstly, we obtain the local minimum of I_λ for $\lambda\in(0,\lambda_1(h))$ and prove I_λ has the mountain pass structure. Then we prove the existence and multiplicity of positive solution for (1.1) by the mountain pass theorem and Ekeland's variational principle, respectively.

Lemma 3.1. *Suppose that* (H1)–(H6) *hold.*

(a) *If $\lambda\in(0,\lambda_1(h))$, then $u=0$ is a local minimum of I_λ;*
(b) *There exist $\widetilde{\delta}$, ρ and α such that, for any $\lambda\in[\lambda_1(h),\lambda_1(h)+\widetilde{\delta})$, $I_\lambda(u)\geq\alpha>0$ if $\|u\|=\rho$;*
(c) *There exists $w\in H^1(\mathbb{R}^3)$ with $\|w\|>\rho$ such that $I_\lambda(w)<0$ for any $\lambda>0$.*

Proof. (a) By (1.3), (H1)–(H2), (2.21) and the Sobolev inequality, we have

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 - \int_{\mathbb{R}^3}k(x)F(u)dx \\
&\quad - \frac{\lambda}{2}\int_{\mathbb{R}^3}h(x)u^2dx \\
&\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1(h)}\Big)\|u\|^2 - \int_{|x|>R_0}k(x)F(u)dx - \int_{|x|\le R_0}k(x)F(u)dx \\
&\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1(h)}\Big)\|u\|^2 - \int_{|x|\le R_0}k(x)F(u)dx \\
&\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1(h)}\Big)\|u\|^2 - C_{23}\int_{\mathbb{R}^3}|F(u)|dx \\
&\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1(h)}\Big)\|u\|^2 - C_{24}\|u\|_p^p - C_{24}\|u\|_q^q \\
&\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_1(h)}\Big)\|u\|^2 - C_{25}\|u\|^p - C_{26}\|u\|^q \\
&\ge C_{27}\|u\|^2
\end{aligned}$$

for $\|u\|$ suitable small. Hence $u=0$ is a local minimizer of I_λ. Thus, (a) holds.

(b) For any $u\in H^1(\mathbb{R}^3)$, we decompose u as $u=te_1+v$, where $t\in\mathbb{R}$ and $v\in\{span\{e_1\}\}^\perp$. Clearly, we have

$$\|u\|^2=t^2+\|v\|^2,\ \lambda_1(h)\int_{\mathbb{R}^3}h(x)e_1^2dx=\|e_1\|^2=1,$$
$$\lambda_2(h)\int_{\mathbb{R}^3}h(x)|v|^2dx\le\|v\|^2,\quad \lambda_1(h)\int_{\mathbb{R}^3}h(x)e_1vdx=\int_{\mathbb{R}^3}(\nabla e_1\nabla v+e_1v)=0.$$

Using this decomposition, we also know that

$$\int_{\mathbb{R}^3}h(x)u^2dx=t^2\int_{\mathbb{R}^3}h(x)e_1^2dx+\int_{\mathbb{R}^3}h(x)v^2dx,$$

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^2 \\
&= \Big(t^2\int_{\mathbb{R}^3}|\nabla e_1|^2dx+2t\int_{\mathbb{R}^3}\nabla e_1\nabla vdx+\int_{\mathbb{R}^3}|\nabla v|^2dx\Big)^2 \\
&= t^4\Big(\int_{\mathbb{R}^3}|\nabla e_1|^2dx\Big)^2+4t^3\int_{\mathbb{R}^3}|\nabla e_1|^2dx\int_{\mathbb{R}^3}\nabla e_1\nabla vdx \\
&\quad + 2t^2\int_{\mathbb{R}^3}|\nabla e_1|^2dx\int_{\mathbb{R}^3}|\nabla v|^2dx+4t^2\Big(\int_{\mathbb{R}^3}\nabla e_1\nabla vdx\Big)^2 \\
&\quad + 4t\int_{\mathbb{R}^3}\nabla e_1\nabla vdx\int_{\mathbb{R}^3}|\nabla v|^2dx+\Big(\int_{\mathbb{R}^3}|\nabla v|^2dx\Big)^2 \\
&\le t^4\Big(\int_{\mathbb{R}^3}|\nabla e_1|^2dx\Big)^2+\Big(\int_{\mathbb{R}^3}|\nabla v|^2dx\Big)^2+C_{28}|t|^3\|v\|+C_{29}|t|^2\|v\|^2+C_{30}|t|\|v\|^3,
\end{aligned}$$

and

$$-\int_{\mathbb{R}^3}k(x)F(u)dx$$

$$
\begin{aligned}
&= -\int_{|x|\le R_0} k(x)F(u)dx - \int_{|x|>R_0} k(x)F(u)dx \\
&= -\frac{1}{p}\int_{|x|\le R_0} k(x)|te_1|^p dx - \int_{|x|\le R_0} k(x)\big[F(te_1) - \frac{1}{p}|te_1|^p\big]dx \\
&\quad - \int_{|x|\le R_0} k(x)[F(te_1+v) - F(te_1)]dx - \int_{|x|>R_0} k(x)F(te_1+v)dx \\
&\ge -\frac{1}{p}\int_{|x|\le R_0} k(x)|te_1|^p dx - \int_{|x|\le R_0} k(x)\big[F(te_1) - \frac{1}{p}|te_1|^p\big]dx \\
&- C_{31}\int_{|x|\le R_0} k(x)f(te_1+\theta v)v dx - \int_{|x|>R_0} k(x)F(te_1+v)dx \\
&\ge -C_{32}|t|^p - \int_{|x|\le R_0} k(x)\big[F(te_1) - \frac{1}{p}|te_1|^p\big]dx + K_0\int_{|x|>R_0} F(te_1+v)dx \\
&- C_{33}\Big[\int_{|x|\le R_0} k(x)|te_1+\theta v|^{p-1} v dx + \int_{|x|\le R_0} k(x)|te_1+\theta v|^{q-1} v dx\Big] \\
&\ge -C_{32}|t|^p - \int_{|x|\le R_0} k(x)\big[F(te_1) - \frac{1}{p}|te_1|^p\big]dx + K_0\int_{|x|>R_0} F(te_1+v)dx \\
&- C_{34}\Big[|t|^{p-1}\Big(\int_{|x|\le R_0} |v|^p dx\Big)^{1/p} + |t|^{q-1}\Big(\int_{|x|\le R_0} |v|^q dx\Big)^{1/q} \\
&\quad + \int_{|x|\le R_0} |v|^p dx + \int_{|x|\le R_0} |v|^q dx\Big] \\
&\ge -C_{32}|t|^p - \int_{|x|\le R_0} k(x)\big[F(te_1) - \frac{1}{p}|te_1|^p\big]dx + K_0\int_{|x|>R_0} F(te_1+v)dx \\
&- C_{35}[|t|^{p-1}\|v\| + |t|^{q-1}\|v\| + \|v\|^p + \|v\|^q] \\
&\ge -C_{32}|t|^p + o(1)|t|^p - C_{36}\left[|t|^{p-1}\|v\| + \|v\|^p\right],
\end{aligned}
$$

because (H1)–(H3), (2.21), (H5) and the odd nature of f as $|t|$ and $\|u\|$ small enough, where $\theta \in [0,1]$. Then

$$
\begin{aligned}
I_{\lambda_1(h)}(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^2 - \int_{\mathbb{R}^3} k(x)F(u)dx \\
&\quad - \frac{\lambda_1(h)}{2}\int_{\mathbb{R}^3} h(x)u^2 dx \\
&\ge \frac{1}{2}\Big(1 - \frac{\lambda_1(h)}{\lambda_2(h)}\Big)\|v\|^2 + \frac{b}{4}\theta_0 t^4 + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |\nabla v|^2 dx\Big)^2 - C_{28}|t|^3\|v\| \\
&\quad - C_{29}|t|^2\|v\|^2 - C_{30}|t|\|v\|^3 - C_{32}|t|^p + o(1)|t|^p \\
&\quad - C_{36}[|t|^{p-1}\|v\| + \|v\|^p],
\end{aligned} \tag{3.1}
$$

as $|t|$ and $\|u\|$ small enough. Furthermore, by the Young inequality,

$$
|t|^2\|v\|^2 \le \frac{2}{p}|t|^p + \frac{p-2}{p}\|v\|^{\frac{2p}{p-2}}, \tag{3.2}
$$

$$
|t|\|v\|^3 \le \frac{1}{p}|t|^p + \frac{p-1}{p}\|v\|^{\frac{3p}{p-1}}, \tag{3.3}
$$

$$
|t|^3\|v\| \le \frac{q_0-1}{q_0}|t|^{\frac{3q_0}{q_0-1}} + \frac{1}{q_0}\|v\|^{q_0} \tag{3.4}
$$

for some q_0 with $q_0 \in (2,4)$. Inserting (3.2)-(3.4) to (3.1), letting $\|v\|$ and $|t|$ small enough, we obtain

$$I_{\lambda_1(h)}(u) \geq C_{37}\|v\|^2 + C_{38}|t|^4$$

as $|t|$ and $\|u\|$ small enough, because $p > 4$, $4 > q_0 > 2$, $\frac{3q_0}{q_0-1} > 4$, $\frac{2p}{p-2} > 2$ and $\frac{3p}{p-1} > 2$. Therefore, there exists $\widetilde{\rho} > 0$ and $\widetilde{\alpha} > 0$ such that

$$I_{\lambda_1(h)}(u) \geq \widetilde{\alpha}\|u\|^4 \quad \text{for } \|u\| \leq \widetilde{\rho}. \tag{3.5}$$

Taking

$$\widetilde{\delta} = \min\Big\{\frac{\lambda_1(h)}{2}\widetilde{\alpha}\widetilde{\rho}, \overline{\delta}, \lambda_2(h) - \lambda_1(h)\Big\}.$$

Note that, for any $\lambda \in [\lambda_1(h), \lambda_1(h) + \widetilde{\delta})$, we obtain

$$\begin{aligned}
I_\lambda(u) &= I_{\lambda_1(h)}(u) + \frac{1}{2}(\lambda_1(h) - \lambda)\int_{\mathbb{R}^3} h(x)u^2dx \\
&\geq \widetilde{\alpha}\|u\|^4 - \frac{\lambda - \lambda_1(h)}{2\lambda_1(h)}\|u\|^2 \\
&\geq \|u\|^2\Big(\widetilde{\alpha}\|u\|^2 - \frac{\lambda - \lambda_1(h)}{2\lambda_1(h)}\Big) \\
&\geq \|u\|^2\Big(\frac{\widetilde{\alpha}\widetilde{\rho}}{2} - \frac{\lambda - \lambda_1(h)}{2\lambda_1(h)}\Big) \\
&\geq \|u\|^2\Big(\frac{\widetilde{\alpha}\widetilde{\rho}}{2} - \frac{1}{4}\widetilde{\alpha}\widetilde{\rho}\Big) \\
&= \frac{1}{4}\widetilde{\alpha}\widetilde{\rho}\|u\|^2
\end{aligned}$$

for $\frac{\widetilde{\rho}}{2} \leq \|u\| \leq \widetilde{\rho}$. Choosing $\rho \in [\frac{\widetilde{\rho}}{2}, \widetilde{\rho}]$ and $\alpha = \frac{1}{8}\widetilde{\alpha}\widetilde{\rho}^2$, we obtain (b).

(c) Choose $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \Omega^+$ such that $\psi(x) \geq 0$ for all $x \in \Omega^+$ and $\psi = t_0e_1 + v$ with $t_0 \neq 0$. Then for any $s > 0$ large such $\|s\psi\| \geq \max\{\delta_0, \rho\}$, by (**??**), we have

$$\begin{aligned}
&I_\lambda(s\psi) \\
&= \frac{s^2}{2}\|\psi\|^2 + \frac{bs^4}{4}\Big(\int_{\mathbb{R}^3}|\nabla\psi|^2dx\Big)^2 - \int_{\Omega^+}k(x)F(s\psi)dx - \frac{\lambda s^2}{2}\int_{\mathbb{R}^3}h(x)\psi^2dx \\
&\leq \frac{s^2}{2}\|\psi\|^2 + \frac{bs^4}{4}\Big(\int_{\mathbb{R}^3}|\nabla\psi|^2dx\Big)^2 - Cs^q\int_{\Omega^+}k(x)|\psi|^qdx - \frac{\lambda s^2}{2}\int_{\mathbb{R}^3}h(x)\psi^2dx \\
&\to -\infty
\end{aligned}$$

as $s \to +\infty$, because $q > 4$. From the choice of ψ, take $w = s\psi$ with s large enough, then $I_\lambda(w) < 0$, (c) is proved. □

Lemma 3.2. *Suppose that* (H1)–(H6) *hold. Then problem* (1.1) *has at least one positive solution* u_λ *with* $I_\lambda(u_\lambda) > 0$ *for* $0 < \lambda < \lambda_1(h) + \widetilde{\delta}$.

Proof. From Lemma 3.1 and the Mountain Pass Theorem, then there exists a $(PS)_{c_\lambda}$ sequence $\{u_n\}$ such that $I_\lambda(u_n) \to c_\lambda > 0$ and $I'_\lambda(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$, where

$$c_\lambda = \inf_{g\in\Gamma}\max_{u\in g[0,1]} I_\lambda(u) \quad \text{with } \Gamma = \{g \in C([0,1], H^1(\mathbb{R}^3)) : g(0) = 0, g(1) = w\}.$$

Then by Lemma 2.4, we know that I_λ satisfies (PS) condition. Thus, the mountain pass theorem implies c_λ is a critical value of I_λ, $c_\lambda > 0$ and u_λ is a critical point of I_λ. Since $I_\lambda(u) = I_\lambda(|u|)$ for any $u \in H^1(\mathbb{R}^3)$, by using an idea from [1, 2], for every $n \in \mathbb{N}$, there exists $g_n(t) \in \Gamma$ with $g_n(s) \geq 0$ for all $s \in [0,1]$ such that

$$c_\lambda \leq \max_{s\in[0,1]} I_\lambda(g_n(s)) < c_\lambda + \frac{1}{n}.$$

By using Ekeland's variational principle in [25], there exists $g_n^* \in \Gamma$ satisfying the following properties:

$$\begin{gathered} c_\lambda \leq \max_{s\in[0,1]} I_\lambda(g_n^*(s)) \leq \max_{s\in[0,1]} I_\lambda(g_n(s)) < c_\lambda + \frac{1}{n}, \\ \max_{s\in[0,1]} \|g_n - g_n^*\| \leq \frac{1}{\sqrt{n}}, \quad \|I'_\lambda(w_n)\| \leq \frac{1}{n}, \end{gathered} \tag{3.6}$$

and there exists $s_n \in [0,1]$ such that $z_n = g_n^*(s_n)$ satisfying

$$I_\lambda(z_n) = \max_{s\in[0,1]} I_\lambda(g_n^*(s)), \quad \|I'_\lambda(z_n)\| \leq \frac{1}{\sqrt{n}}. \tag{3.7}$$

This inequality implies that $\{z_n\} \subset H^1(\mathbb{R}^3)$ is a $(PS)_{c_\lambda}$ sequence, by Lemma 2.4, there exists a convergent subsequence (still denoted by $\{z_n\}$ and $z \in H^1(\mathbb{R}^3)$ satisfying $z_n \to z$ as $n \to \infty$. Thus $g_n(s_n) \to z$ in $H^1(\mathbb{R}^3)$ as $n \to \infty$ by (3.6) and (3.7). It follows that $z \geq 0$ from $g_n(s) \geq 0$ a. e. in $\mathbb{R}^3$ with $I_\lambda(z) > 0$. Let $u_\lambda = z \geq 0$. By (2.19), we know that $0 < C \leq \|u_\lambda\| \leq C_0$, which implies that there exists $M_0 > 0$ such that $u_\lambda \leq M_0$ a. e. in $\mathbb{R}^3$. Moreover, by (H1)–(H5), there exists a constant $C(M_0) > 0$ depending M_0 such that

$$|k(x)f(u_\lambda)| \leq C(M_0)u_\lambda.$$

Together with (H6) and $\lambda > 0$, we have

$$\begin{aligned} -\Big(1 + b\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx\Big)\Delta u_\lambda + u_\lambda &= k(x)f(u_\lambda) + \lambda h(x)u_\lambda \\ &\geq k(x)f(u_\lambda) \geq -C(M_0)u_\lambda. \end{aligned}$$

This yields

$$-\Delta u_\lambda + Lu_\lambda \geq 0,$$

where $L = (1 + C(M_0))/(1 + b\int_{\mathbb{R}^3} |\nabla u_\lambda|^2 dx)$. Then by the maximum principle that $u_\lambda > 0$ in $\mathbb{R}^3$, then it is a positive solution of problem (1.1). $\square$

Lemma 3.3. *Suppose that* (H1)–(H6) *hold. Then* (1.1) *has at least one positive solution ω_λ with $I_\lambda(\omega_\lambda) < 0$ for $\lambda_1(h) < \lambda < \lambda_1(h) + \widetilde{\delta}$.*

Proof. By (b) of Lemma 3.1 and its proof, there exist $\widetilde{\delta}$, $\rho > 0$ with $\rho \to 0$, $\alpha > 0$ if $\lambda \in (\lambda_1(h), \lambda_1(h) + \widetilde{\delta})$ and

$$I_\lambda(u) \geq \alpha > 0 \text{ if } \|u\| = \rho.$$

Let $m_\lambda := \inf_{B_\rho} I_\lambda(u)$, where $B_\rho := \{u \in H^1\mathbb{R}^3 : \|u\| \leq \rho\}$ with ρ as in Lemma 3.1. It is clear $m_\lambda > -\infty$. Next, we prove that $m_\lambda < 0$. In fact, given $R > 0$, define $\kappa_R \in C_0^\infty(\mathbb{R}^3)$ with $0 \leq \kappa_R \leq 1$ and $|\nabla\kappa_R| \leq \frac{2}{R}$ for any $x \in \mathbb{R}^3$ and

$$\kappa_R(x) = \begin{cases} 1, & |x| \leq R, \\ 0, & |x| \geq 2R. \end{cases}$$

Then $\kappa_R e_1 \in H^1(\mathbb{R}^3)$ and we have

$$
\begin{aligned}
&I_\lambda(t\kappa_R e_1)\\
&= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla(\kappa_R e_1)|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} |\kappa_R e_1|^2 dx + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla(\kappa_R e_1)|^2 dx\Big)^2\\
&\quad - \int_{\mathbb{R}^3} k(x)F(t\kappa_R e_1)dx - \frac{\lambda t^2}{2}\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx\\
&= \frac{t^2}{2}\int_{\mathbb{R}^3} (\kappa_{\mathbb{R}}^2|\nabla e_1|^2 + |\nabla\kappa_R|^2 e_1^2 + 2\kappa_R e_1\nabla e_1\nabla\kappa_R)dx + \frac{t^2}{2}\int_{\mathbb{R}^3} |\kappa_R e_1|^2 dx\\
&\quad + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla(\kappa_R e_1)|^2 dx\Big)^2 - \int_{\mathbb{R}^3} k(x)F(t\kappa_R e_1)dx - \frac{\lambda t^2}{2}\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx.
\end{aligned}
\tag{3.8}
$$

Multiplying both sides of the equation $-\Delta e_1 + e_1 = \lambda_1(h)h(x)e_1$ by $\kappa_{\mathbb{R}}^2 e_1$ and integrate by parts, we obtain

$$
\begin{aligned}
&2\int_{\mathbb{R}^3} \kappa_R e_1\nabla e_1\nabla\kappa_R dx + \int_{\mathbb{R}^3} \kappa_{\mathbb{R}}^2 e_1^2 dx + \int_{\mathbb{R}^3} \kappa_{\mathbb{R}}^2|\nabla e_1|^2 dx\\
&= \lambda_1(h)\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx.
\end{aligned}
\tag{3.9}
$$

Inserting (3.9) to (3.8), we obtain

$$
\begin{aligned}
I_\lambda(t\kappa_R e_1) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla\kappa_R|^2 e_1^2 dx + \frac{t^2(\lambda_1(h)-\lambda)}{2}\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx\\
&\quad + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3} |\nabla(\kappa_R e_1)|^2 dx\Big)^2 - \int_{\mathbb{R}^3} k(x)F(t\kappa_R e_1)dx.
\end{aligned}
\tag{3.10}
$$

Moreover, by the definition of κ_R, the Hölder and Sobolev inequalities, we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^3} |\nabla\kappa_R|^2 e_1^2 dx &= \int_{R\le|x|\le 2R} |\nabla\kappa_R|^2 e_1^2 dx\\
&\le \Big(\int_{R\le|x|\le 2R} e_1^6 dx\Big)^{1/3}\Big(\int_{R\le|x|\le 2R} |\nabla\kappa_R|^3 dx\Big)^{2/3}\\
&\le \Big(\int_{R\le|x|\le 2R} e_1^6 dx\Big)^{1/3}\Big((\frac{2}{R})^3\int_{R\le|x|\le 2R} dx\Big)^{2/3}\\
&\le C_{39}\Big(\int_{R\le|x|\le 2R} e_1^6 dx\Big)^{1/3} \to 0
\end{aligned}
\tag{3.11}
$$

as $R\to\infty$ because $\|e_1\| = 1$. Then, multiplying both sides of the equation $-\Delta e_1 + e_1 = \lambda_1(h)h(x)e_1$ by e_1 and integrate by parts, we obtain the identity

$$
\int_{\mathbb{R}^3} h(x)e_1^2 dx = \frac{1}{\lambda_1(h)}\|e_1\|^2 = \frac{1}{\lambda_1(h)}. \tag{3.12}
$$

Furthermore, by choosing R sufficiently large, the definition of κ_R and (3.12), we obtain that

$$
\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx \ge \int_{|x|\le R} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx = \int_{|x|\le R} h(x)e_1^2 dx \ge \frac{1}{2\lambda_1(h)}. \tag{3.13}
$$

Therefore, by choosing $R_1\ge 1$ sufficiently large, by (3.11) and (3.13), we obtain

$$
\int_{\mathbb{R}^3} |\nabla\kappa_R|^2 e_1^2 dx \le \frac{\lambda-\lambda_1(h)}{2}\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx \tag{3.14}
$$

for all $R \geq R_1$ and $\lambda > \lambda_1(h)$. Inserting (3.14) to (3.10), by (2.21) and (H2), we have

$$\begin{aligned} I_\lambda(t\kappa_R e_1) &\leq \frac{t^2(\lambda_1(h)-\lambda)}{4}\int_{\mathbb{R}^3} h(x)\kappa_{\mathbb{R}}^2 e_1^2 dx + \frac{bt^4}{4}\Big(\int_{\mathbb{R}^3}|\nabla(\kappa_R e_1)|^2dx\Big)^2 \\ &\quad - \int_{\mathbb{R}^3} k(x)F(t\kappa_R e_1)dx \\ &\leq -C_{40}t^2 + C_{41}t^4 + C_{42}t^p + C_{43}t^q, \end{aligned}$$

for all $R \geq R_1$. This yields that $I_\lambda(t\kappa_R e_1) < 0$ for all $t > 0$ small enough because that $p,\ q > 4$. Then there exists t_0 such that $\|t_0\kappa_R e_1\| \leq \rho \to 0$ such that $I_\lambda(t_0\kappa_R e_1) < 0$. Thus, $m_\lambda = \inf_{B_\rho} I_\lambda(u) < 0$.

Since $m_\lambda = \inf_{B_\rho} I_\lambda(u) < 0$ and $I_\lambda(u) = I_\lambda(|u|)$, there exists a minimizing sequence $\{w_n^*\}$ for m_λ with $w_n^* \geq 0$ and $\|w_n^*\| \leq \rho$ such that

$$m_\lambda \leq I_\lambda(w_n*) < m_\lambda + \frac{1}{n}.$$

Then Ekeland's variational principle in [25] yields another sequence $\{w_n\}$ with $\|w_n\| \leq \rho$ such that

$$\begin{gathered} m_\lambda \leq I_\lambda(w_n) \leq I_\lambda(w_n^*) < m_\lambda + \frac{1}{n}, \\ \|w_n - w_n^*\| \leq \frac{1}{\sqrt{n}}, quad\|I_\lambda'(w_n)\| \leq \frac{1}{n}. \end{gathered} \tag{3.15}$$

Thus, the sequence $\{w_n\}$ satisfies

$$I_\lambda(w_n) \to m_\lambda \text{ and } I_\lambda'(w_n) \to 0$$

as $n \to \infty$. By lemma 2.4, there exists a minimizer $w_\lambda \in B_\rho$ such that $w_n \to w_\lambda$ in $H^1(\mathbb{R}^3)$ and $I_\lambda(w_\lambda) = m_\lambda$. By (3.15), $w_n^* \to w_\lambda$. Together with $w_n^* \geq 0$, we obtain that $w_\lambda \geq 0$ a.e. in $\mathbb{R}^3$ with $I_\lambda(w_\lambda) < 0$ and w_λ is a solution of problem (1.1). Since $w_\lambda \in B_\rho$ and $\rho \to 0^+$, by (H4), we have

$$\begin{aligned} -(1+b\rho)(\Delta w_\lambda + w_\lambda) &\geq -\Big(1 + b\int_{\mathbb{R}^3}|\nabla w_\lambda|^2dx\Big)\Delta w_\lambda + w_\lambda \\ &= k(x)f(w_\lambda) + \lambda h(x)w_\lambda \geq 0. \end{aligned}$$

Then we deduce that $w_\lambda > 0$ in $\mathbb{R}^3$ by strong maximum principle. The proof is complete. □

Proof of Theorem 1.1. Clearly, the first conclusion of Theorem 1.1 is deduced directly from Lemma 3.2. The second conclusion is obtained from Lemmas 3.2 and 3.3. In fact, from Lemmas 3.2 and 3.3, there is a positive solution u_λ of (1.1) with $I_\lambda(u_\lambda) > 0$ and a positive solution w_λ of (1.1) with $I_\lambda(w_\lambda) < 0$. Clearly, $u_\lambda \neq w_\lambda$. Hence, the second statement of Theorem 1.1 is proved. □

Acknowledgment. The authors are very grateful to the anonymous referees for the useful comments and remarks. The first author expresses their thanks to Professor S. M. Sun for his valuable help and suggestion during her visit to Virginia Tech. This research is supported by the Key Project in Science and Technology Research Plan of the Education Department of Hubei Province (No. D20142602). This article is dedicated to Professor Bolin Guo on his 80-th birthday.

References

[1] S. Alama, G. Tarantello; *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var., Partial Differential Equations, 1(4): 439-475,1993.

[2] S. Alama, G. Tarantello; *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. Anal. 141 (1): 159-215, 1996.

[3] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma; *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. 49(1): 85-93, 2005.

[4] C. O. Alves, G. M. Figueiredo; *Nonlinear perturbations of a periodic Kirchhoff equation in* $\mathbb{R}^N$. Nonlinear Anal., 75(5): 2750-2759, 2012.

[5] P. D'Ancona, S. Spagnolo; *Global solvability for the degenerate Kirchhoff equation with real analytic data*. Invent. Math., 108 (2): 247-262, 1992.

[6] A. Arosio, S. Panizzi; *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc., 348 (1): 305-330, 1996.

[7] A. Azzollini, P. d' Avenia, A. Pomponio; *Multiple critical points for a class of nonlinear functionals*, Ann. Mat. Pura Appl., 190(3): 507-523, 2011.

[8] H. Berestycki, P. L. Lions; *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal., 82 (4): 347-375, 1983.

[9] S. Bernstein; *Sur une classe d'équations fonctionnelles aux dérivées*, Bull. Acad. Sci. URSS, Sr 4 (1): 17-26, 1940 (Izvestia Akad. Nauk SSSR).

[10] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano; *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differential Equations, 6(6): 701-730, 2001.

[11] J. Q. Chen; *Multiple positive solutions to a class of Kirchhoff equation on* $\mathbb{R}^3$ *with indefinite nonlinearity*, Nonlinear Analysis, 96: 134-145, 2014.

[12] B. T. Cheng; *New existence and multiplicity of nontrivial solutions for nonlocal elliptic Kirchhoff type problems*, J. Math. Anal. Appl., 394(2): 488-495, 2012.

[13] D. G. Costa, H. Tehrani; *Existence of positive solutions for a class of indefinite elliptic problems in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, 13(2): 159-189, 2001.

[14] X. M. He, W. M. Zou; *Existence and concentration behavior of positive solutions for a Kirchhoff equation in* $\mathbb{R}^3$, J. Differential Equations, 252(2): 1813-1834, 2012.

[15] X. M. He, W. Zou; *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Anal., 70 (3): 1407-1414, 2009.

[16] J. H. Jin, X. Wu; *Infinitely many radial solutions for Kirchhoff-type problems in* $\mathbb{R}^N$, J. Math. Anal. Appl., 369(2): 564-574, 2010.

[17] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, 1883.

[18] J. L. Lions; *On some questions in boundary value problems of mathematical physics*, North-Holland Math. Stud., 30: 284-346, 1978.

[19] P. L. Lions; *The Concentration-Compactness Principle in the Calculus of Variations. The Locally Compact Case, Part I*, Ann. Inst. H. Poincare, Anal. Non Linaeaire, 1(2): 109-145, 1984.

[20] W. Liu, X. M. He; *Multiplicity of high energy solutions for superlinear Kirchhoff equations*, J. Appl. Math. Comput., 39(1-2): 473-487, 2012.

[21] K. Perera, Z. T. Zhang; *Nontrivial solutions of Kirchhoff-type problems via the Yang index*, J. Differential Equations, 221(1): 246-255, 2006.

[22] S. I. Pohožaev; *On a class of quasilinear hyperbolic equations*, Mat. Sb. (N.S.) 96(138) (1): 152-166, 1975. (in Russian).

[23] J. J. Sun, C. L. Tang; *Existence and multiplicity of solutions for Kirchhoff type equations*, Nonlinear Anal., 74(4): 1212-1222, 2011.

[24] X. Wu; *Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in* $\mathbb{R}^N$, Nonlinear Anal. Real World Appl., 12(2): 1278-1287, 2011.

[25] M. Willem; *Minimax Theorems*, Birkhäuser, Boston, 1996.

[26] Y. Yang, J. H. Zhang; *Nontrivial solutions of a class of nonlocal problems via local linking theory*, Appl. Math. Lett., 23(1): 377-380, 2010.

[27] Z. T. Zhang, K. Perera; *Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow*, J. Math. Anal. Appl., 317(2): 456-463, 2006.

Ling Ding
School of Mathematics and Computer Science, Hubei University of Arts and Science, Hubei 441053, China
E-mail address: dingling1975@qq.com, dingling19750118@163.com

Yi-Jie Meng
School of Mathematics and Computer Science, Hubei University of Arts and Science, Hubei 441053, China
E-mail address: 245330581@qq.com

Shi-Wu Xiao
School of Mathematics and Computer Science, Hubei University of Arts and Science, Hubei 441053, China
E-mail address: xshiwu@sina.com

Jin-Ling Zhang
School of Mathematics and Computer Science, Hubei University of Arts and Science, Hubei 441053, China
E-mail address: jinling48@163.com