Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 78, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE SOLUTIONS FOR p-LAPLACIAN PROBLEMS INVOLVING GENERAL SUBCRITICAL GROWTH IN BOUNDED DOMAINS

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Abstract. Using variational methods, we study the existence of multiple solutions for a class of p-Laplacian problems with concave-convex nonlinearities in bounded domains. Our result improves those in [8, 9] stated only for subcritical growth.

1. Introduction

In this article, we are interested in the existence of solutions for p-Laplacian problems of the form

$$\begin{gathered} -\Delta_p u = g(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth bounded domain, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying subcritical growth condition.

Problem (1.1) has been studied extensively for many years. Since Ambrosetti and Rabinowitz proposed the mountain pass theorem in 1973 (see [1]), critical point theory has become one of the main tools for finding solutions to elliptic equations and systems of variational type. To apply this theorem, the authors introduced one of very important conditions (Ambrosetti and Rabinowitz type condition) on the nonlinear term g as follows:

(AR) For some $\theta > p$, and $R > 0$, we have

$$0 < \theta G(x,t) \leq g(x,t)t, \quad \forall |t| \geq R, \quad \text{a.e. } x \in \Omega,$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. This condition ensures that the energy functional associated to the problem satisfies the Palais-Smale condition ((PS) condition for short). Clearly, if the condition (AR) is satisfied then there exist two positive constants d_1, d_2 such that

$$G(x,t) \geq d_1 |t|^{\mu} - d_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

2010 *Mathematics Subject Classification.* 35D05, 35J60.
Key words and phrases. p-Laplacian problems; general subcritical growth; concave-convex nonlinearities; variational method.

Submitted February 26, 2016. Published March 18, 2016.

This means that g is p-superlinear at infinity in the sense that

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{|t|^p} = +\infty.$$

In recent years, there have been many authors considering problem (1.1) without the (AR) type condition, we refer to some interesting papers on this topic [3, 6, 7, 10, 11, 12, 14, 15] and the references cited there. Miyagaki et al [12], studied problem (1.1) in the semilinear case $p = 2$ by proposing the following non-global condition on the superlinear term $g(x,t)$: There exists $t_0 > 0$ such that

$$\frac{g(x,t)}{t} \text{ is increasing for } t \geq t_0 \text{ and decreasing for } t \leq -t_0, \quad \forall x \in \Omega.$$

Using the mountain pass theorem with the (PS) condition in [1], the authors obtained the existence of a non-trivial weak solution. This result was extended to the p-Laplace operator $-\Delta_p u$ by Li et al [10]. It should be noticed that in [10, 12], the authors need the following subcritical growth condition

- (A0') $|g(x,t)| \leq C(1+|t|^{r-1})$ for all $t \in \mathbb{R}$, a.e. $x \in \Omega$, $r \in [1, p^*)$, where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = +\infty$ if $p \geq N$.

Recently Lan et al [8, 9] studied problem (1.1) by introducing a general type of subcritical growth condition, where $r = p^*$. Using mountain pass theorem [1], they obtained the existence of at least one nontrivial weak solution of (1.1) without (AR) condition. In this article, we consider (1.1) when $g(x,u) = \lambda|u|^{q-2}u + f(x,u)$, i.e.

$$\begin{gathered} -\Delta_p u = \lambda|u|^{q-2}u + f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $1 < q < p$, λ is a positive parameter, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following general subcritical growth condition

- (A0) $\lim_{|t|\to+\infty} f(x,t)/|t|^{p^*-1} = 0$ uniformly a.e. $x \in \Omega$.

In particular, as in [8, 9], we do not use the (AR) condition for the nonlinear term f, see condition (A4) as well as some examples and comments in the papers [8, 9]. Using the mountain pass theorem [1] combined with Ekeland variational principle [5], we will obtain the existence of at least two nontrivial weak solutions for problem (1.1). Our result introduced here is a natural extension from the previous ones for elliptic problems with concave-convex nonlinearities [2, 13]. Regarding this interesting topic, we refer the readers to the paper [4], in which the authors studied elliptic problems with local superlinearity and sublinearity.

To state the main result of this paper, let us introduce the following conditions on the function f:

- (A1) There exists a positive constant $\bar{t} > 0$ such that $F(x,t) \geq 0$ a.e. $x \in \Omega$ and all $t \in [0, \bar{t}]$, where $F(x,t) := \int_0^t f(x,s)\,ds$.
- (A2) $\limsup_{|t|\to 0} \frac{F(x,t)}{|t|^p} < \lambda_1$ uniformly a.e. $x \in \Omega$, where λ_1 is the first eigenvalue of $-\Delta_p$.
- (A3) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^p} = +\infty$ uniformly a.e. $x \in \Omega$.
- (A4) There exist constants $\theta \geq 1$, $C_* > 0$ such that

$$\theta H(x,t) + C_* \geq H(x,st)$$

for all $t \in \mathbb{R}$, $x \in \Omega$, $s \in [0,1]$, where $H(x,t) = f(x,t)t - pF(x,t)$.

It should be noticed that the function $f(x,t) = |t|^{p-2}t\log(1+|t|)$ satisfies (A1)–(A4). We refer the readers to [8, 9] for more details. In this article, we look for weak solutions to (1.2) in the usual Sobolev space $W_0^{1,p}(\Omega)$ which is equipped with the norm $\|u\| = \big(\int_\Omega |\nabla u|^p\,dx\big)^{1/p}$.

Definition 1.1. We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.2) if
$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx - \lambda\int_\Omega |u|^{q-2}uv\,dx - \int_\Omega f(x,u)v\,dx = 0$$
for all $v \in W_0^{1,p}(\Omega)$.

Our main result is given by the following theorem.

Theorem 1.2. *Suppose that* (A0)–(A4) *are satisfied. Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0,\lambda^*)$, problem* (1.2) *has two nontrivial weak solutions.*

2. Multiple solutions

In this section, we prove our main result. Let us denote by c_i general positive constants. As we will see, in order to obtain the existence of at least two weak solutions for problem (1.2) we use variational methods (mountain pass theorem and Ekeland variational principle).

We look for the weak solutions of (1.2) which are the same as the critical points of the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega F(x,u)\,dx.$$
We can see that $J \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ and
$$J'(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx - \lambda\int_\Omega |u|^{q-2}uv\,dx - \int_\Omega f(x,u)v\,dx$$
for all $u, v \in W_0^{1,p}(\Omega)$.

Lemma 2.1. *There exists $\lambda^* > 0$ such that for any $\lambda \in (0,\lambda^*)$, we can choose $\alpha, \rho > 0$ so that $J(u) \geq \alpha$ for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho$.*

Proof. From (A0) and (A2), for any $\epsilon > 0$, there exists $c(\epsilon) > 0$ depending on ϵ, such that
$$F(x,t) \leq \frac{1}{p}(\lambda_1 - \epsilon)|t|^p + c(\epsilon)|t|^{p^*} \tag{2.1}$$
for all $t \in \mathbb{R}$ and a.e. $x \in \Omega$. Hence, using Sobolev's embedding, we have
$$\begin{aligned} J(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p}\|u\|^p - \frac{\lambda}{q}c_1\|u\|^q - \frac{1}{p}(\lambda_1-\epsilon)\int_\Omega |u|^p\,dx - c(\epsilon)\int_\Omega |u|^{p^*}\,dx \\ &\geq \frac{1}{p}\Big(1 - \frac{\lambda_1-\epsilon}{\lambda_1}\Big)\|u\|^p - \frac{\lambda}{q}c_1\|u\|^q - \overline{c}(\epsilon)\|u\|^{p^*} \\ &= \Big(\frac{\epsilon}{p\lambda_1} - \frac{\lambda}{q}c_1\|u\|^{q-p} - \overline{c}(\epsilon)\|u\|^{p^*-p}\Big)\|u\|^p, \end{aligned} \tag{2.2}$$
where $\overline{c}(\epsilon)$ and c_1 are positive constants.

For each $\lambda > 0$, we consider the function $\gamma_\lambda : (0, +\infty) \to \mathbb{R}$ defined by

$$\gamma_\lambda(t) = \frac{\lambda}{q} c_1 t^{q-p} - \overline{c}(\epsilon) t^{p^*-p}. \tag{2.3}$$

It is clear that $\gamma_\lambda(t)$ is a continuous function on $(0, +\infty)$. Since $p^* > p > q > 1$, it follows that

$$\lim_{t\to 0^+} \gamma_\lambda(t) = \lim_{t\to +\infty} \gamma_\lambda(t) = +\infty. \tag{2.4}$$

Hence, we can find $t_* > 0$ such that $0 < \gamma_\lambda(t_*) = \min_{t\in(0,+\infty)} \gamma_\lambda(t)$, in which t_* is defined by the equation

$$0 = \gamma_\lambda'(t_*) = \frac{\lambda c_1}{q}(q-p)t_*^{q-p-1} + \overline{c}(\epsilon)(p^*-p)t_*^{p^*-p-1}$$

or

$$t_* = \Big(\frac{\lambda c_1(p-q)}{q\overline{c}(\epsilon)(p^*-p)}\Big)^{\frac{1}{p^*-q}}.$$

Some simple computations show that

$$\gamma_\lambda(t_*) = c_2.\lambda^{\frac{p^*-p}{p^*-q}} \to 0 \quad \text{as } \lambda \to 0^+. \tag{2.5}$$

From relations (2.3), (2.4) and (2.5), there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, we can choose $\alpha > 0$ and $\rho > 0$ so that $J(u) \geq \alpha > 0$ for all $u \in W_0^{1,p}(\Omega)$ with $\|u\| = \rho$. $\square$

Lemma 2.2. *There exists $\phi \in W_0^{1,p}(\Omega)$, $\phi > 0$ such that $J(t\phi) \to -\infty$ as $t \to +\infty$.*

Proof. (ii) From (A3), it follows that for any $M > 0$ there exists a constant $c_M = c(M) > 0$ depending on M, such that

$$F(x,t) \geq M|t|^{p^+} - c_M, \quad \text{for a.e. } x \in \Omega,\ \forall t \in \mathbb{R}. \tag{2.6}$$

Take $\phi \in C_0^\infty(\Omega)$ with $\phi > 0$, from (2.6) and the definition of J, we obtain

$$\begin{aligned} J(t\phi) &= \frac{1}{p}\int_\Omega |\nabla t\phi|^p\,dx - \lambda\int_\Omega \frac{1}{q}|t\phi|^q\,dx - \int_\Omega F(x,t\phi)\,dx \\ &\leq \frac{1}{p}\|t\phi\|^p - M\int_\Omega |t\phi|^p\,dx - \frac{\lambda}{q}\int_\Omega |t\phi|^q\,dx + c_M|\Omega| \\ &\leq t^p\Big(\frac{1}{p}\|\phi\|^p - M\int_\Omega |\phi|^p\,dx\Big) - \frac{\lambda t^q}{q}\int_\Omega |\phi|^q\,dx + c_M|\Omega|, \end{aligned} \tag{2.7}$$

where $t > 0$ and $|\Omega|$ denotes the Lebesgue measure of Ω.

From (2.7) and the fact that $1 < q < p$, if M is large enough such that

$$\frac{1}{p}\|\phi\|^p - M\int_\Omega |\phi|^p\,dx < 0,$$

then we have $\lim_{t\to+\infty} J(t\phi) = -\infty$. $\square$

Lemma 2.3. *There exists $\psi \in W_0^{1,p}(\Omega)$, $\psi > 0$ such that $J(t\psi) < 0$ for all $t > 0$ small enough.*

Proof. Take $\psi \in C_0^\infty(\Omega)$ with $\psi > 0$, from the definition of J and condition (A1), for all $t \in \left(0, \frac{\bar{t}}{\|\psi\|_{L^\infty(\Omega)}}\right)$ small enough, we obtain

$$\begin{aligned} J(t\psi) &= \frac{1}{p}\int_\Omega |\nabla t\psi|^p\,dx - \frac{\lambda}{q}\int_\Omega |t\psi|^q\,dx - \int_\Omega F(x,t\psi)\,dx \\ &\le \frac{t^p}{p}\|\psi\|^p - \frac{\lambda t^q}{q}\int_\Omega |\psi|^q\,dx. \end{aligned} \tag{2.8}$$

From this inequality, taking

$$0 < \delta < \frac{\lambda p \int_\Omega |\psi|^q\,dx}{q\|\psi\|^p}$$

we conclude that $J(t\psi) < 0$ for all $0 < t < \min\{\delta^{\frac{1}{p-q}}, \frac{\bar{t}}{\|\psi\|_{L^\infty(\Omega)}}\}$. The proof of Lemma 2.3 is complete. □

Lemma 2.4. *The functional J satisfies the (Ce) condition.*

Proof. Let $\{u_m\} \subset W_0^{1,p}(\Omega)$ be a (C_c) sequence of the functional J, that is,

$$J(u_m) \to \bar{c}, \quad \|J'(u_m)\|_*(1 + \|u_m\|) \to 0 \quad \text{as } m \to \infty,$$

which shows that

$$\bar{c} = J(u_m) + o(1), \quad J'(u_m)(u_m) = o(1), \tag{2.9}$$

where $o(1) \to 0$ as $m \to \infty$.

We prove that $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. Indeed, by contradiction, we assume that $\|u_m\| \to +\infty$ as $m \to \infty$. Let $w_m = \frac{u_m}{\|u_m\|}$ we obtain $w_m \in W_0^{1,p}(\Omega)$ with $\|w_m\| = 1$. Then there exists $w \in W_0^{1,p}(\Omega)$ such that $\{w_m\}$ converges weakly to w in $W_0^{1,p}(\Omega)$ and

$$w_m(x) \to w(x), \quad \text{a.e. in } \Omega,\ m \to \infty, \tag{2.10}$$

$$w_m \to w \quad \text{strongly in } L^r(\Omega),\ m \to \infty,\ 1 \le r < p^*, \tag{2.11}$$

$$|w_m|_{p^*}^{p^*} \le c_3. \tag{2.12}$$

Let $\Omega_{\neq} := \{x \in \Omega : \ w(x) \neq 0\}$. If $x \in \Omega_{\neq}$ then it follows from (2.10) that $\lim_{m\to\infty} w_m(x) = \lim_{m\to\infty} \frac{u_m(x)}{\|u_m\|} = w(x)$ and thus $|u_m(x)| = |w_m(x)|\|u_m\| \to +\infty$ as $m \to \infty$ for a.e. $x \in \Omega_{\neq}$.

Using (A3) we have

$$\lim_{m\to\infty} \frac{F(x,u_m(x))}{|u_m(x)|^p} = +\infty, \quad \text{a.e. } x \in \Omega_{\neq}. \tag{2.13}$$

This implies

$$\lim_{m\to\infty} \frac{F(x,u_m(x))}{|u_m(x)|^p}|w_m(x)|^p = +\infty, \quad \text{a.e. } x \in \Omega_{\neq}. \tag{2.14}$$

Using condition (A3) again, there exists $t_0 > 0$ such that

$$\frac{F(x,t)}{|t|^p} > 1 \tag{2.15}$$

for all $x \in \Omega$ and $|t| > t_0 > 0$. Since $F(x,t)$ is continuous on $\overline{\Omega} \times [-t_0, t_0]$, there exists a positive constant c_4 such that

$$|F(x,t)| \le c_4 \tag{2.16}$$

for all $(x,t) \in \overline{\Omega} \times [-t_0, t_0]$. From (2.15) and (2.16) there exists $c_5 \in \mathbb{R}$ such that

$$F(x,t) \geq c_5 \tag{2.17}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$. From (2.17), for all $x \in \Omega$ and m, we have

$$\frac{F(x, u_m(x)) - c_5}{\|u_m\|^p} \geq 0$$

or

$$\frac{F(x, u_m(x))}{|u_m(x)|^p} |w_m(x)|^p - \frac{c_5}{\|u_m\|^p} \geq 0, \quad \forall x \in \Omega, \ \forall m. \tag{2.18}$$

Using (2.9) and the Sobolev embedding, there exists $c_6 > 0$ such that

$$\begin{aligned}
\overline{c} &= J(u_m) + o(1) \\
&= \frac{1}{p} \int_\Omega |\nabla u_m|^p \, dx - \frac{\lambda}{q} \int_\Omega |u_m|^q \, dx - \int_\Omega F(x, u_m) \, dx + o(1) \\
&\geq \frac{1}{p} \|u_m\|^p - \frac{\lambda c_6}{q} \|u_m\|^q - \int_\Omega F(x, u_m) \, dx + o(1);
\end{aligned}$$

since $1 < q < p$, this implies

$$\int_\Omega F(x, u_m) \, dx \geq \frac{1}{p} \|u_m\|^p - \frac{\lambda c_6}{q} \|u_m\|^q - \overline{c} + o(1) \to +\infty \quad \text{as } m \to \infty. \tag{2.19}$$

Also we have

$$\begin{aligned}
\|u_m\|^p &= p \int_\Omega F(x, u_m) \, dx + \frac{\lambda p}{q} \int_\Omega |u_m|^q \, dx + p\overline{c} - o(1) \\
&\geq p \int_\Omega F(x, u_m) \, dx + p\overline{c} - o(1) > 0 \quad \text{for } m \text{ large enough.}
\end{aligned} \tag{2.20}$$

Next, we claim that $|\Omega_{\neq}| = 0$. In fact, if $|\Omega_{\neq}| \neq 0$, then by relations (2.18), (2.19), (2.20) and the Fatou lemma, we have

$$\begin{aligned}
+\infty &= (+\infty)|\Omega_{\neq}| \\
&= \int_{\Omega_{\neq}} \liminf_{m \to \infty} \frac{F(x, u_m(x))}{|u_m(x)|^p} |w_m(x)|^p \, dx - \int_{\Omega_{\neq}} \limsup_{m \to \infty} \frac{c_5}{\|u_m\|^p} \, dx \\
&= \int_{\Omega_{\neq}} \liminf_{m \to \infty} \Big(\frac{F(x, u_m(x))}{|u_m(x)|^p} |w_m(x)|^p - \frac{c_5}{\|u_m\|^p} \Big) \, dx \\
&\leq \liminf_{m \to \infty} \int_{\Omega_{\neq}} \Big(\frac{F(x, u_m(x))}{|u_m(x)|^p} |w_m(x)|^p - \frac{c_5}{\|u_m\|^p} \Big) \, dx \\
&\leq \liminf_{m \to \infty} \int_\Omega \Big(\frac{F(x, u_m(x))}{|u_m(x)|^p} |w_m(x)|^p - \frac{c_5}{\|u_m\|^p} \Big) \, dx \\
&= \liminf_{m \to \infty} \int_\Omega \frac{F(x, u_m(x))}{\|u_m\|^p} \, dx - \limsup_{m \to \infty} \int_\Omega \frac{c_5}{\|u_m\|^p} \, dx \\
&= \liminf_{m \to \infty} \int_\Omega \frac{F(x, u_m(x))}{\|u_m\|^p} \, dx \\
&\leq \liminf_{m \to \infty} \frac{\int_\Omega F(x, u_m(x)) \, dx}{p \int_\Omega F(x, u_m) \, dx + p\overline{c} - o(1)}.
\end{aligned} \tag{2.21}$$

From (2.19) and (2.21), we obtain

$$+\infty \le \frac{1}{p},$$

which is a contradiction. This shows that $|\Omega_{\neq}| = 0$ and thus $w(x) = 0$ a.e. in Ω.

Since the function $t \mapsto J(tu_m)$ is continuous in $t \in [0,1]$, for each m there exists $t_m \in [0,1]$ such that

$$J(t_m u_m) := \max_{t\in[0,1]} J(tu_m), \quad m = 1, 2, \dots. \tag{2.22}$$

It is clear that $t_m > 0$ and $J(t_m u_m) \ge \bar{c} > 0 = J(0) = J(0.u_m)$. If $t_m < 1$ then $\frac{d}{dt} J(tu_m)|_{t=t_m} = 0$ which gives $J'(t_m u_m)(t_m u_m) = 0$. If $t_m = 1$, then $J'(u_m)(u_m) = o(1)$. So we always have

$$J'(t_m u_m)(t_m u_m) = o(1). \tag{2.23}$$

Now, we fix a big integer $k \ge 1$ and define the sequence $\{v_m\}$ by

$$v_m = (2p\|u_k\|^p)^{1/p}\, w_m, \quad m = 1, 2, \dots. \tag{2.24}$$

From the dominated convergence theorem and since $w = 0$ we have

$$\lim_{m\to\infty} \int_\Omega |v_m|^q \,dx = 0. \tag{2.25}$$

Furthermore, by (A0), for every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that

$$|F(x,t)| \le \frac{1}{c_3}\epsilon |t|^{p^*} + c(\epsilon), \quad \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Let $\delta = \frac{\epsilon}{2c(\epsilon)} > 0$, $E \subseteq \Omega$, $|E| < \delta$ we have

$$\begin{aligned} \Big|\int_E F(x, v_m)\,dx\Big| &\le \int_E |F(x, v_m)|\,dx \\ &\le \int_E c(\epsilon)\,dx + \frac{1}{2c_3}\epsilon \int_E |v_m|^{p^*}\,dx \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2}, \end{aligned}$$

hence $\{\int_\Omega F(x, v_m)\,dx : m \in \mathbb{N}\}$ is equi-absolutely-continuous. It follows easily from Vitali convergence theorem that

$$\int_\Omega F(x, v_m)\,dx \to \int_\Omega F(x, 0)\,dx = 0 \quad \text{as } m \to \infty.$$

Since $\|u_m\| \to +\infty$ as $m \to \infty$, we can find $m_k \ge k$ such that

$$0 < \frac{(2p\|u_k\|^p)^{1/p}}{\|u_m\|} < 1, \quad \forall m > m_k. \tag{2.26}$$

Hence, using relations (2.22), (2.24)-(2.26), it follows that

$$\begin{aligned}
&J(t_m u_m)\\
&\geq J\Big(\frac{(2p\|u_k\|^p)^{1/p}}{\|u_m\|}u_m\Big)\\
&= J(v_m)\\
&= \frac{1}{p}\int_\Omega |\nabla v_m|^p\,dx - \frac{\lambda}{q}\int_\Omega |v_m|^q\,dx - \int_\Omega F(x,v_m)\,dx\\
&\geq \frac{1}{p}\int_\Omega \Big(\|u_k\|^p.(2p)^{\frac{p}{p}}.|\nabla w_m|^p\Big)\,dx - \frac{\lambda}{q}\int_\Omega |v_m|^q\,dx - \int_\Omega F(x,v_m)\,dx\\
&\geq 2\|u_k\|^p - \frac{\lambda}{q}\int_\Omega |v_m|^q\,dx - \int_\Omega F(x,v_m)\,dx\\
&\geq \|u_k\|^p
\end{aligned} \tag{2.27}$$

for any $m > m_k \geq k$ large enough.

On the other hand, using condition (A4) and relation (2.23), for all $m > m_k > k$ large enough, we have

$$\begin{aligned}
&J(t_m u_m)\\
&= J(t_m u_m) - \frac{1}{p}J'(t_m u_m)(t_m u_m) + o(1)\\
&= \frac{1}{p}\int_\Omega |\nabla t_m u_m|^p\,dx - \frac{\lambda}{q}\int_\Omega |t_m u_m|^q\,dx - \int_\Omega F(x,t_m u_m)\,dx\\
&\quad - \frac{1}{p}\int_\Omega |\nabla t_m u_m|^p\,dx + \frac{\lambda}{p}\int_\Omega |t_m u_m|^q\,dx\\
&\quad + \frac{1}{p}\int_\Omega f(x,t_m u_m)t_m u_m\,dx + o(1)\\
&= \lambda\big(\frac{1}{p} - \frac{1}{q}\big)\int_\Omega |t_m u_m|^q\,dx + \frac{1}{p}\int_\Omega H(x,t_m u_m)\,dx\\
&\leq \frac{1}{p}\int_\Omega \big(\theta H(x,u_m) + C_*\big)\,dx + o(1)\\
&= \theta\Big(\frac{1}{p}\int_\Omega |\nabla u_m|^p\,dx - \frac{\lambda}{q}\int_\Omega |u_m|^q\,dx - \int_\Omega F(x,u_m)\,dx\Big)\\
&\quad - \frac{\theta}{p}\Big(\int_\Omega |\nabla u_m|^p\,dx - \lambda\int_\Omega |u_m|^q\,dx - \int_\Omega f(x,u_m)u_m\,dx\Big)\\
&\quad + \lambda\theta\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega |u_m|^q\,dx + \frac{\theta C_*|\Omega|}{p} + o(1)\\
&= \theta J(u_m) - \frac{\theta}{p}J'(u_m)(u_m) + \lambda\theta\big(\frac{1}{q} - \frac{1}{p}\big)\int_\Omega |u_m|^q\,dx + \frac{\theta C_*|\Omega|}{p} + o(1)\\
&\leq \theta J(u_m) - \frac{\theta}{p}J'(u_m)(u_m) + \lambda\theta c_7\big(\frac{1}{q} - \frac{1}{p}\big)\|u_m\|^q + \frac{\theta C_*|\Omega|}{p} + o(1).
\end{aligned} \tag{2.28}$$

From (2.27) and (2.28), we deduce that for all $m > m_k > k$ large enough,

$$\|u_k\|^p \leq \theta J(u_m) - \frac{\theta}{p}J'(u_m)(u_m) + \lambda\theta c_7\big(\frac{1}{q} - \frac{1}{p}\big)\|u_m\|^q + \frac{\theta C_*|\Omega|}{p} + o(1)$$

or

$$\|u_k\|^p - \lambda\theta c_7\big(\frac{1}{q}-\frac{1}{p}\big)\|u_m\|^q \le \theta J(u_m) - \frac{\theta}{p}J'(u_m)(u_m) + \frac{\theta C_*|\Omega|}{p} + o(1) \quad (2.29)$$

Recall that $k \ge 1$ is an arbitrarily big integer and $m > m_k > k$. In (2.29), let $k \to \infty$ we have $m \to \infty$ and the left hand side of (2.29) tends to $+\infty$ since $q < p$. In the right hand side of (2.29), $J(u_m) \to c$ and $\frac{\theta}{p}J'(u_m)(u_m) \to 0$ as $m \to \infty$. Thus, we have a contradiction. This proves that the sequence $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$.

Now, since the Banach space $W_0^{1,p}(\Omega)$ is reflexive, there exists $u \in W_0^{1,p}(\Omega)$ such that passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in $W_0^{1,p}(\Omega)$ and converges strongly to u in $L^r(\Omega)$, $1 \le r < p^*$. Moreover, $\{u_m\}$ converges weakly to u in $L^{p^*}(\Omega)$ and we have $|u_m|_{p^*}^{p^*} \le c_8$. From (A0), for every $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that

$$|f(x,t)t| \le \frac{1}{2c_8}\epsilon|t|^{p^*} + c(\epsilon), \quad \forall t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

Let $\delta = \frac{\epsilon}{2c(\epsilon)} > 0$, $E \subseteq \Omega$, $|E| < \delta$ we have

$$\begin{aligned}
\big|\int_E f(x,u_m)u_m\,dx\big| &\le \int_E |f(x,u_m)u_m|\,dx \\
&\le \int_E c(\epsilon)\,dx + \frac{1}{2c_8}\epsilon\int_E |u_m|^{p^*}\,dx \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2},
\end{aligned}$$

hence $\{\int_\Omega f(x,u_m)u_m\,dx : m \in \mathbb{N}\}$ is equi-absolutely-continuous. It follows easily from Vitali convergence theorem that

$$\int_\Omega f(x,u_m)u_m\,dx \to \int_\Omega f(x,u)u\,dx \quad \text{as } m \to \infty. \quad (2.30)$$

Using (A0) again, for any $\epsilon > 0$ there exists $c(\epsilon) > 0$ such that

$$|f(x,t)| \le \frac{1}{2c_9c_{10}}\epsilon|t|^{p^*-1} + c(\epsilon), \quad \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

where

$$c_9 \ge \Big(\int_\Omega |u_m|^{p^*}\,dx\Big)^{\frac{p^*-1}{p^*}}, \quad \forall m; \quad c_{10} := \Big(\int_\Omega |u|^{p^*}\,dx\Big)^{1/p^*}.$$

From the Hölder inequality, for every $E \subseteq \Omega$, we have

$$\begin{gathered}
\int_E c(\epsilon)|u|\,dx \le c(\epsilon)|E|^{\frac{p^*-1}{p^*}}\Big(\int_E |u|^{p^*}\,dx\Big)^{1/p^*} = c(\epsilon)|E|^{\frac{p^*-1}{p^*}}c_{10}, \\
\int_E |u_m|^{p^*-1}|u|\,dx \le \Big(\int_E |u_m|^{p^*}\,dx\Big)^{\frac{p^*-1}{p^*}}\Big(\int_E |u|^{p^*}\,dx\Big)^{1/p^*} \le c_9c_{10}.
\end{gathered}$$

Let $\delta = (\frac{\epsilon}{2c_{10}c(\epsilon)})^{\frac{p^*}{p^*-1}} > 0$, $E \subseteq \Omega$, $|E| < \delta$ we have

$$\begin{aligned}
\big|\int_E f(x,u_m)u\,dx\big| &\le \int_E |f(x,u_m)u|\,dx \\
&\le \int_E c(\epsilon)|u|\,dx + \frac{1}{2c_9c_{10}}\epsilon\int_E |u_m|^{p^*-1}|u|\,dx
\end{aligned}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

hence $\{\int_\Omega f(x,u_m)u_m\,dx : m \in \mathbb{N}\}$ is equi-absolutely-continuous. It follows easily from Vitali convergence theorem that

$$\int_\Omega f(x,u_m)u\,dx \to \int_\Omega f(x,u)u\,dx \quad \text{as } m \to \infty. \tag{2.31}$$

From (2.30) and (2.31) we have

$$\int_\Omega f(x,u_m)(u_m - u)\,dx \to 0 \quad \text{as } m \to \infty. \tag{2.32}$$

We also have

$$\begin{aligned} \int_\Omega |u_m|^{q-2}u_m(u_m-u)\,dx &\leq \int_\Omega |u_m|^{q-1}|u_m-u|\,dx \\ &\leq \Big(\int_\Omega |u_m|^q\,dx\Big)^{\frac{q-1}{q}}\Big(\int_\Omega |u_m-u|^q\,dx\Big)^{1/q} \to 0 \end{aligned} \tag{2.33}$$

as $m \to \infty$. Since $J'(u_m)(u_m-u) \to 0$ as $m \to \infty$, we deduce from (2.32) and (2.33) that

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m(\nabla u_m - \nabla u)\,dx \to 0 \quad \text{as } m \to \infty,$$

which gives us that $\{u_m\}$ converges strongly to u in $W_0^{1,p}(\Omega)$ and the functional J satisfies the (Ce) condition. □

Proof Theorem 1.2. By Lemmas 2.1, 2.2 and 2.4, there exists $\lambda^* > 0$ such that for any $\lambda \in (0,\lambda^*)$, the functional J satisfies all the assumptions of the mountain pass theorem. Then we deduce u_1 as a non-trivial critical point of the functional J with $J(u_1) = \overline{c} > 0$ and thus a non-trivial weak solution of problem (1.2).

We now prove that there exists a second weak solution $u_2 \in W_0^{1,p}(\Omega)$ such that $u_2 \neq u_1$. Indeed, by (2.2), the functional J is bounded from below on the ball $\overline{B}_\rho(0)$.

Applying the Ekeland variational principle in [5] to the functional $J : \overline{B}_\rho(0) \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B}_\rho(0)$ such that

$$J(u_\epsilon) < \inf_{u \in \overline{B}_\rho(0)} J(u) + \epsilon,$$
$$J(u_\epsilon) < J(u) + \epsilon\|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

By Lemmas 2.1 and 2.2, we have

$$\inf_{u \in \partial B_\rho(0)} J(u) \geq R > 0 \quad \text{and} \quad \inf_{u \in \overline{B}_\rho(0)} J(u) < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{u \in \partial B_\rho(0)} J(u) - \inf_{u \in \overline{B}_\rho(0)} J(u).$$

Then, $J(u_\epsilon) < \inf_{u \in \partial B_\rho(0)} J(u)$ and thus, $u_\epsilon \in B_\rho(0)$.

Now, we define the functional $I : \overline{B}_\rho(0) \to \mathbb{R}$ by $I(u) = J(u) + \epsilon\|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + tv) - I(u_\epsilon)}{t} \geq 0$$

for all $t > 0$ small enough and all $v \in B_\rho(0)$. The above information shows that

$$\frac{J(u_\epsilon + tv) - J(u_\epsilon)}{t} + \epsilon\|v\| \geq 0.$$

Letting $t \to 0^+$, we deduce that $\langle J'(u_\epsilon), v\rangle \geq -\epsilon\|v\|$. It should be noticed that $-v$ also belongs to $B_\rho(0)$, so replacing v by $-v$, we obtain

$$\langle J'(u_\epsilon), -v\rangle \geq -\epsilon\| - v\|, \langle J'(u_\epsilon), v\rangle \leq \epsilon\|v\|,$$

which helps us to deduce that $\|J'(u_\epsilon)\|_* \leq \epsilon$.

Then, there exists a sequence $\{u_m\} \subset B_\rho(0)$ such that

$$J(u_m) \to \underline{c} = \inf_{u \in \overline{B}_\rho(0)} J(u) < 0, \quad J'(u_m) \to 0 \quad \text{in } W^{-1,p}(\Omega) \text{ as } m \to \infty. \tag{2.34}$$

From Lemma 2.4, the sequence $\{u_m\}$ converges strongly to some $u_2 \in W_0^{1,p}(\Omega)$ as $m \to \infty$. Moreover, since $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, by (2.9) it follows that $J(u_2) = \underline{c}$ and $J'(u_2) = 0$. Thus, u_2 is a non-trivial weak solution of (1.2).

Finally, we point out that $u_1 \neq u_2$ since $J(u_1) = \bar{c} > 0 > \underline{c} = J(u_2)$. The proof of Theorem 1.2 is complete. □

Acknowledgments. The authors would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript. This work is supported by Quang Binh University (Grant N. CS.05.2016).

References

[1] A. Ambrosetti, P.H. Rabinowitz; Dual variational methods in critical points theory and applications, *J. Funct. Anal.*, **04** (1973), 349-381.

[2] A. Ambrosetti, H. Brezis, G. Cerami; Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, **122** (1994), 519-543.

[3] D. G. Costa, C. A. Magalhães; Variational elliptic problems which are nonquadratic at infinity, *Nonlinear Anal. (TMA)*, **23** (1994), 1401-1412.

[4] D. G. De Figueiredo, J. P. Gossez, P. Ubilla; Local superlinearity and sublinearity for indefinite semilinear elliptic problems, *J. Funct. Anal.*, **199** (2003), 452-467.

[5] I. Ekeland; On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324-353.

[6] L. Iturriaga, S. Lorca, P. Ubilla; A quasilinear problem without the Ambrosetti-Rabinowitz-type condition, *Proc. Roy. Soc. Edinburgh Sect. A*, **140** (2010), 391-398.

[7] A. Kristály, H. Lisei, C. Varga; Multiple solutions for p-Laplacian type equations, *Nonlinear Anal. (TMA)*, **68** (2008), 1375-1381.

[8] Y. Y. Lan; Existence of solutions to p-Laplacian equations involving general subcritical growth, *Electronic J. of Diff. Equ.*, **2014** (151) (2014), 1-9.

[9] Y. Y. Lan, C. L. Tang; Existence of solutions to a class of semilinear elliptic equations involving general subcritical growth, *Proc. Roy. Soc. Edinburgh Sect. A*, **144** (2014), 809-818.

[10] G. Li, C. Yang; The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition, *Nonlinear Anal. (TMA)*, **72** (2010), 4602-4613.

[11] S. Liu; On superlinear problems without the Ambrosetti and Rabinowitz condition, *Nonlinear Anal. (TMA)*, **73** (2010), 788-795.

[12] O. H. Miyagaki, M. A. S. Souto; Superlinear problems without Ambrosetti and Rabinowitz growth condition, *J. Differential Equations*, **245** (2008), 3628-3638.

[13] N. S. Papageorgiou, E. M. Rocha; Pairs of positive solutions for p-Laplacian equations with sublinear and superlinear nonlinearities which do not satisfy the AR-condition, *Nonlinear Anal. (TMA)*, **70** (2009), 3854-3863.

[14] M. Z. Sun; Multiple solutions of a superlinear p-Laplacian equation without AR-condition, *Appl. Anal.*, **89** (2010), 325-336.

[15] J. Wang, C. L. Tang; Existence and multiplicity of solutions for a class of superlinear p-Laplacian equations, *Boundary Value Problems* , **2006** (2006),1-12, Art ID 47275.

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