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OPTIMIZATION PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

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Abstract. In this article we study rearrangement optimization problems related to boundary-value problems involving the fractional Laplacian. We establish the existence and uniqueness of a solution under suitable assumptions.

1. Introduction

A rearrangement optimization problem is referred to an optimization problem in which the admissible set consists of functions that are rearrangements of a prescribed function. The theory of rearrangement optimizations has been established by Burton [5, 6]. Since then, this topic has been widely studied by many authors in different aspects, see for example [7, 10, 11, 12, 16, 17, 18, 19]. Burton [6] proved that both the minimization and maximization problems for the boundary value problem involving the Laplacian have solutions. However, the results obtained in [6] can not be directly applied to the optimization problems for the boundary value problem involving p-Laplacian($1 < p < \infty$). So by using a new approach, Cuccu et al [10] proved that the minimization problem has a solution for $1 < p < \infty$. But their approach is not efficient for the maximization problem. Marras [17] obtained the solvability of the maximization for $1 < p < \infty$ by using another method. While Cuccu et al [12] obtained a result of uniqueness for a class of p-Laplace equations under non-standard assumptions. Recently, Qiu et al [19] considered a rearrangement optimization problem related to the quasilinear elliptic boundary value problem, where under suitable assumptions, it is shown that both the minimization and maximization problems are solvable, which extends the results in [6, 10, 17].

It is worth to note that most of the rearrangement optimization problems considered in the above papers are related to boundary value problems involving the Laplacian or p-Laplacian. In recent years, fractional and nonlocal operators of elliptic type have been attracted a lot of interests since these operators appear in concrete applications in fields such as minimal surface [9], thin obstacle problem [8], anomalous diffusion [1], phase transition [21], hemivariational inequality [23, 25], shape optimization [13], optimal transportation theory [14] and so on. Interesting

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studies involving nonlocal fractional problems by variational methods can be found in [2, 3] and the references therein.

In this article, we study several rearrangement optimization problems related to a class of boundary value problems involving the fractional Laplacian. We need to overcome the difficulties coming from both the fractional Laplacian and the rearrangement optimization problem. As our knowledge, this kind of problems has not been considered in literature.

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 2$) and let $k \in L^q(\Omega)$ with $1 \leq q \leq \infty$. We recall that a rearrangement of k is an element of the set $\mathscr{R}(k)$ of all measurable functions g on Ω satisfying

$$\operatorname{meas}(\{x \in \Omega : g(x) \geq a\}) = \operatorname{meas}(\{x \in \Omega : k(x) \geq a\}), \quad \forall a \in \mathbb{R}.$$

It is easy to prove that if $g \in \mathscr{R}(k)$, then $g \in L^q(\Omega)$ and $\|g\|_{L^q} = \|k\|_{L^q}$ (cf. [6, Lemma 2.1]).

Let $h(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $f \in L^\infty(\Omega)$. Under suitable assumptions we can show that the boundary value problem

$$\begin{gathered} -L_\theta^s u + h(x,u) = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega \end{gathered} \tag{1.1}$$

has a unique solution $u_f \in H^s(\Omega)$ (cf. Proposition 3.1), where L_θ^s is the fractional Laplace type operator defined as

$$L_\theta^s u(x) = \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \theta(y) dy, \quad x \in \mathbb{R}^N \tag{1.2}$$

with $0 < s < 1$ and $\theta : \mathbb{R}^N \to (0, +\infty)$.

In particular, if $h \equiv 0$, then the boundary value problem (1.1) becomes

$$\begin{gathered} -L_\theta^s u = f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.3}$$

and it obviously has a unique solution u_f (cf. Proposition 4.1), thus it deduces that (cf. Remark 4.2),

$$\begin{aligned} &\sup_{v \in H^s(\Omega)} \Big(\int_\Omega 2fv dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \theta(x-y)\, dx\, dy \Big) \\ &= \int_\Omega 2f u_f\, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x-y|^{N+2s}} \theta(x-y)\, dx\, dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x-y|^{N+2s}} \theta(x-y)\, dx\, dy = \int_\Omega f u_f\, dx. \end{aligned} \tag{1.4}$$

Therefore, we can define the functional Φ on $L^\infty(\Omega)$ as

$$\Phi(f) = \int_\Omega f u_f\, dx \tag{1.5}$$

and we are able to consider the following two optimization problems:

(Opt1) Find $f_1 \in \mathscr{R}(f_0)$ such that $\Phi(f_1) = \sup_{f \in \mathscr{R}(f_0)} \Phi(f)$,
(Opt2) Find $f_2 \in \mathscr{R}(f_0)$ such that $\Phi(f_2) = \inf_{f \in \mathscr{R}(f_0)} \Phi(f)$,

where $f_0 \in L^\infty(\Omega)$ is a given function and $\mathscr{R}(f_0)$ is the set of all rearrangement of f_0.

If $h \not\equiv 0$, then it is difficult to consider the above optimization problems related to the boundary value problem (1.1) since the functional Ψ on $L^\infty(\Omega)$, corresponding to the optimization problems, defined by

$$\begin{aligned}\Psi(f) = \frac{1}{2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_f(x)-u_f(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\Big)\\ + \int_\Omega H(x,u_f)\,dx - \int_\Omega f u_f\,dx\end{aligned} \tag{1.6}$$

is hard to be reduced to a simple formula like (1.5). However, in the case of $h \not\equiv 0$, we can consider the following minimization optimization problem related to (1.1):

(Opt*) Find $f_* \in \mathscr{R}(f_0)$ such that $\Psi(f_*) = \inf_{f\in\mathscr{R}(f_0)} \Psi(f)$,

where $f_0 \in L^\infty(\Omega)$ is a given function and $\mathscr{R}(f_0)$ is the set of all rearrangement of f_0.

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to discussing the minimization problem (Opt*) in detail. After establishing the uniqueness result of solutions for the problem (1.1), we show that the minimization problem (Opt*) is solvable. In Section 4, we prove that both the maximization and minimization optimization problems (Opt1) and (Opt2) are solvable. To our best of knowledge, the results of this paper are new and nontrivial.

2. Preliminaries

Given $0 < s < 1$, we define the fractional Sobolev space

$$H^s(\Omega) = \big\{u \in L^2(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N\backslash\Omega, \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy < \infty\big\}$$

with the inner product

$$\langle u,v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy \quad \forall u,v \in H^s(\Omega).$$

Then the norm of u is

$$\|u\| = \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}.$$

Throughout this article, we denote by $\|u\|$ and $\|u\|_p$ the usual norm in spaces $H^s(\Omega)$ and $L^p(\Omega)$ $(1 \le p \le \infty)$, respectively. As usual, "$\to$" and "$\rightharpoonup$" denote the strong and weak convergence.

We now list some lemmas which are useful in the proof of our main results.

Lemma 2.1 ([20, Lemma 8]). *$H^s(\Omega) \hookrightarrow L^r(\mathbb{R}^N)$, for $1 \le r \le \frac{2N}{N-2s}$, and the embedding is compact if $1 \le r < \frac{2N}{N-2s}$.*

Lemma 2.2 ([6, Lemma 2.2]). *Assume that $1 \le r < \infty$ and for $f \in L^r(\Omega)$ denote by $\overline{\mathscr{R}(f)}$ the weak closure of $\mathscr{R}(f)$ in $L^r(\Omega)$. Then $\overline{\mathscr{R}(f)}$ is convex and weakly compact in $L^r(\Omega)$.*

Lemma 2.3 ([6, Lemma 2.9] or [11, Lemma 2.1])). *Let $f,g : \Omega \to \mathbb{R}$ be measurable functions and suppose that for each $t \in \mathbb{R}$, the level set of g at t, i.e., $\{x \in \Omega : g(x) = t\}$, has zero measure. Then there exists an increasing (decreasing) function φ such that $\varphi \circ g$ is a rearrangement of f where $\varphi \circ g$ denotes a composite function defined by*

$$(\varphi \circ g)(x) = \varphi(g(x)), \forall x \in \Omega.$$

Lemma 2.4 ([6, Lemma 2.4] or [11, Lemma 2.2])**.** *For any $1 \le r < \infty$ define $r' = \frac{r}{r-1}$ if $r > 1$ and $r' = \infty$ if $r = 1$. Let $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$. Suppose that there exists an increasing (decreasing) function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ g \in \mathscr{R}(f)$. Then $\varphi \circ g$ is the unique maximizer (minimizer) of the linear functional $\int_\Omega hg\,dx$, relative to $h \in \overline{\mathscr{R}(f)}$.*

Lemma 2.5 ([15, Lemma 2.3])**.** *Suppose that $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$. Then there exists $\hat{f} \in \mathscr{R}(f)$ which maximizes (minimizes) the linear functional $\int_\Omega hgdx$, relative to $h \in \overline{\mathscr{R}(f)}$.*

As in the proof of [22, Lemma 2.1] we have the following result.

Lemma 2.6. *Given $u \in H^{1/2}(\Omega)$, there exists a unique extension $v \in H^1(\mathbb{R}^{N+1}_+)$ of u such that*

$$\begin{gathered} -\Delta v(x,y) = 0, \quad \text{for } x \in \mathbb{R}^N, y > 0, \\ v(x,0) = u(x), \quad \text{for } x \in \mathbb{R}^N. \end{gathered}$$

Moreover,

$$-\partial_y v(x,0) = (-\Delta)^{1/2} u(x)$$

in the sense that

$$-\int_{\mathbb{R}^N \times \{0\}} \frac{\partial v}{\partial y} \varphi dx = \int_{\mathbb{R}^N} \varphi (-\Delta)^{1/2} u dx$$

for every $\varphi \in H^{1/2}(\mathbb{R}^N)$.

3. Minimization related to (1.1)

Recall that the energy functional $I : H^s(\Omega) \to \mathbb{R}$ corresponding to (1.1) is

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \theta(x-y)\,dx\,dy + \int_\Omega H(x,u)dx - \int_\Omega fudx, \tag{3.1}$$

where $H(x,u) = \int_0^u h(x,t)dt$.

We use the following hypotheses on the functions $h : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R}^N \to (0,+\infty)$:

- (H1) $h(x,t)$ is Carathéodory and is non-decreasing with respect to the second variable for almost all $x \in \Omega$.
- (H2) There exist $a(x), b(x) \in L^\infty(\Omega)$, $0 < l < 1$, such that $|h(x,t)| \le a(x) + b(x)|t|^l$, for all $t \in \mathbb{R}$, a.e. $x \in \Omega$.
- (H3) $\theta(x) = \theta(-x)$ for any $x \in \mathbb{R}^N \backslash \{0\}$;
- (H4) $\theta \in L^\infty(\mathbb{R}^N)$ and there exists $\theta_0 \in \mathbb{R}_+$ such that $\theta(x) \ge \theta_0$, for any $x \in \mathbb{R}^N$.

Proposition 3.1. *Suppose that* (H1)–(H4) *hold. Then* (1.1) *has a unique solution $u_f \in H^s(\Omega)$ and $I(u_f) = \inf_{v \in H^s(\Omega)} I(v)$.*

Proof. First, we show that the problem (1.1) has a solution. Let C be a positive constant. From (H2), (H4), and the Sobolev embedding inequality it follows that

$$\Big| \int_\Omega H(x,u)dx \Big| \le \int_\Omega \Big| \int_0^u h(x,v)\,dv \Big|\,dx \le C\|u\| + C\|u\|^{l+1}, \tag{3.2}$$

$$\Big| \int_\Omega fu\,dx \Big| \le \|f\|_\infty \|u\|_1 \le C\|u\|, \tag{3.3}$$

$$\theta_0\|u\|^2 \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy, \quad \forall u \in H^s(\Omega). \tag{3.4}$$

Hence we deduce from (3.1), (3.2), (3.3) and (3.4) that

$$I(u) \ge \frac{\theta_0}{2}\|u\|^2 - C\|u\|^{l+1} - C\|u\| \to \infty$$

as $\|u\| \to \infty$, which shows that the functional I is coercive. We will prove that the functional I is weakly lower semi-continuous. To do this, let $v_n \rightharpoonup v$ in $H^s(\Omega)$ as $n \to \infty$, noting both the embeddings $H^s(\Omega) \hookrightarrow L^1(\Omega)$ and $H^s(\Omega) \hookrightarrow L^{l+1}(\Omega)$ are compact, we see that $v_n \to v$ in $L^{l+1}(\Omega)$ and $L^1(\Omega)$ as $n \to \infty$. Therefore, $H(x, v_n) \to H(x, v)$ in $L^1(\Omega)$ as $n \to \infty$ by the continuity of the operator $u \mapsto H(x, u)$ from $L^{l+1}(\Omega)$ to $L^1(\Omega)$, which implies that

$$\int_\Omega H(x, v_n)dx \to \int_\Omega H(x, v)\,dx \tag{3.5}$$

as $n \to \infty$. Then we have

$$\begin{aligned}
&\liminf_{n\to\infty} I(v_n)\\
&= \liminf_{n\to\infty} \int_\Omega \Big(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy + H(x, v_n) - fv_n\Big)\,dx\\
&\ge \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy + \int_\Omega H(x, v)dx - \limsup_{n\to\infty}\int_\Omega fv_n dx\\
&\ge I(v) - \limsup_{n\to\infty}\|f\|_\infty\|v_n - v\|_1\\
&= I(v).
\end{aligned}$$

Thus the functional I is weakly lower semi-continuous (which we will denote by w.l.s.c for short). So that the functional I has a minimizer $u_f \in H^s(\Omega)$ with $I(u_f) = \inf_{v\in H^s(\Omega)} I(v)$. By assumptions (H1), (H2), and using a standard argument [24, Lemma 2.16], we can easily show that $I \in C^1(H^s(\Omega), \mathbb{R})$, therefore u_f is a critical point of I, i.e.,

$$\begin{aligned}
I'(u_f)v = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_f(x)-u_f(y))(v(x)-v(y))}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\\
&+ \int_\Omega (h(x, u_f)v - fv)dx = 0, \quad \forall v \in H^s(\Omega).
\end{aligned} \tag{3.6}$$

By the definition of the fractional Laplace type operator L_θ^s (see (1.2)), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} -L_\theta^s u(x)v(x)dx\\
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\theta(y)v(x)\,dy\,dx\\
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{N+2s}}\theta(y)v(x)\,dy\,dx\\
&\quad - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x-y)-u(x)}{|y|^{N+2s}}\theta(y)v(x)\,dy\,dx.
\end{aligned} \tag{3.7}$$

Let $z = x + y$, $t = x - y$. By using (H3) we obtain

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|y|^{N+2s}}\theta(y)v(x)\,dy\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(z)}{|x-z|^{N+2s}}\theta(x-z)v(x)\,dz\,dx, \\
&-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x-y)-u(x)}{|y|^{N+2s}}\theta(y)v(x)\,dy\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(t)}{|x-t|^{N+2s}}\theta(x-t)v(x)\,dt\,dx.
\end{aligned} \tag{3.8}
$$

Obviously,

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(z)}{|x-z|^{N+2s}}\theta(x-z)v(x)\,dz\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(t)}{|x-t|^{N+2s}}\theta(x-t)v(x)\,dt\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\theta(x-y)v(x)\,dy\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(y)-u(x)}{|x-y|^{N+2s}}\theta(x-y)v(y)\,dy\,dx.
\end{aligned} \tag{3.9}
$$

It follows from (3.7), (3.8) and (3.9) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} -L_\theta^s u(x)v(x)dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\theta(x-y)v(x)\,dy\,dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(y)-u(x)}{|x-y|^{N+2s}}\theta(x-y)v(y)\,dy\,dx \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\theta(x-y)\,dy\,dx.
\end{aligned}
$$

This, combined with (3.6) yields

$$
\int_{\mathbb{R}^N} -L_\theta^s u_f(x)v(x)dx + \int_\Omega (h(x,u_f(x)) - f(x))v(x)dx = 0, \quad \forall v \in H^s(\Omega).
$$

Thus, u_f is a solution of problem (1.1).

Next, we show that u_f is the unique solution of (1.1). Assume that $w \in H^s(\Omega)$ is another solution of (1.1) and $u_f \neq w$. Then

$$
\|u_f - w\| > 0. \tag{3.10}
$$

Since $h(x,\cdot)$ is non-decreasing,

$$
\int_\Omega (h(x,u_f) - h(x,w))(u_f - w)dx \geq 0. \tag{3.11}
$$

From (3.6) we obtain that for every $v \in H^s(\Omega)$,

$$
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_f(x)-u_f(y))(v(x)-v(y))}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy + \int_\Omega h(x,u_f)v dx = \int_\Omega fv\,dx, \tag{3.12}
$$

$$
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w(x)-w(y))(v(x)-v(y))}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy + \int_\Omega h(x,w)v dx = \int_\Omega fv\,dx. \tag{3.13}
$$

From these two equalities, we obtain that for every $v \in H^s(\Omega)$,

$$\begin{aligned}&\int_\Omega (h(x,u_f) - h(x,w))v dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(w(x) - u_f(x) - w(y) + u_f(y))(v(x) - v(y))}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy.\end{aligned}$$

Let $v = u_f - w$ we have

$$\begin{aligned}&\int_\Omega (h(x,u_f) - h(x,w))(u_f - w)dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{-(v(x) - v(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy \\ &\le -\theta_0\|v\|^2 < 0,\end{aligned} \tag{3.14}$$

the last inequality above comes from (3.10). So, (3.14) contradicts (3.11). Therefore, we have proved that u_f is the unique solution of (1.1). $\square$

Let u_f be the unique solution of (1.1). Recall that $\Psi(f)$ is defined by (1.6). considering the optimization problem (Opt*), we have the following result.

Theorem 3.2. *Suppose that* (H1–(H4) *hold. Then for a fixed nonnegative function* $f_0 \in L^\infty(\Omega)$ *there exists* $f_* \in \mathscr{R}(f_0)$ *which solves the minimization optimization problem* (Opt*)*, i.e.,*

$$\Psi(f_*) = \inf_{f \in \mathscr{R}(f_0)} \Psi(f).$$

Proof. Let $A = \inf_{f\in\mathscr{R}(f_0)} \Psi(f)$, then A is well-defined. Indeed, for each $f \in \mathscr{R}(f_0)$, we have

$$\begin{aligned}\Psi(f) &= \frac{1}{2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\Big) \\ &\quad + \int_\Omega H(x,u_f)dx - \int_\Omega f u_f\,dx \\ &\ge \frac{\theta_0}{2}\|u_f\|^2 - C(\|f\|_\infty\|u_f\| + \|u_f\|^{l+1}).\end{aligned} \tag{3.15}$$

Noting that $\|f\|_\infty = \|f_0\|_\infty$ and $l+1 < 2$, we deduce that A must be finite.

Let $\{f_i\} \subset \mathscr{R}(f_0)$ be such that $\Psi(f_i) \to A$ as $i \to \infty$ and we denote $u_i = u_{f_i}$. It follows from (3.15) that $\{u_i\}$ is bounded in $H^s(\Omega)$, then it has a subsequence (still denoted $\{u_i\}$) which converges weakly to $u \in H^s(\Omega)$ and strongly to u in $L^2(\Omega)$. On the other hand, since $\|f_i\|_{L_\infty} \equiv \|f_0\|_{L_\infty}$, $\{f_i\}$ is bounded in $L^2(\Omega)$, it must contain a subsequence (still denoted $\{f_i\}$) converging weakly to some $\bar f \in \overline{\mathscr{R}(f_0)}$, the weak closure of $\mathscr{R}(f_0)$ in $L^2(\Omega)$. Then

$$\Big|\int_\Omega (f_i - \bar f)u dx\Big| \to 0 \quad \text{as } i \to \infty,$$

since $u \in L^2(\Omega)$. It follows from the Hölder inequality that

$$\begin{aligned}\Big|\int_\Omega (f_i u_i - \bar f u)dx\Big| &\le \Big|\int_\Omega f_i(u_i - u)dx\Big| + \Big|\int_\Omega (f_i - \bar f)u dx\Big| \\ &\le \|f_i\|_2\|u_i - u\|_2 + \Big|\int_\Omega (f_i - \bar f)u\,dx\Big| \to 0\end{aligned} \tag{3.16}$$

as $i \to \infty$. By (3.5), (3.16) and the weak lower semi-continuity of the norm in $H^s(\Omega)$, we obtain

$$\begin{aligned} A &= \lim_{i\to\infty} \Psi(f_i) \\ &\geq \frac{1}{2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\Big) \\ &\quad + \int_\Omega H(x,u)dx - \int_\Omega \bar{f}udx. \end{aligned} \tag{3.17}$$

From Lemma 2.5 we infer the existence of $\hat{f} \in \mathscr{R}(f_0)$ which maximizes the linear functional $\int_\Omega hudx$, relative to $h \in \overline{\mathscr{R}(f_0)}$. As a consequence,

$$\int_\Omega \bar{f}udx \leq \int_\Omega \hat{f}udx.$$

Combining this with (3.17), we obtain

$$A \geq \frac{1}{2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\Big) + \int_\Omega H(x,u)dx - \int_\Omega \hat{f}u\,dx. \tag{3.18}$$

By Proposition 3.1,

$$\begin{aligned} \Psi(\hat{f}) &= \inf_{v\in H^s(\Omega)}\Big(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy + \int_\Omega (H(x,v)-\hat{f}v)dx\Big) \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy + \int_\Omega (H(x,u)-\hat{f}u)dx. \end{aligned} \tag{3.19}$$

It follows from (3.18) and (3.19) that $\Psi(\hat{f}) \leq A$.

On the other hand, recall that $A = \inf_{f\in\mathscr{R}(f_0)} \Psi(f)$ and $\hat{f} \in \mathscr{R}(f_0)$, we must have $A \leq \Psi(\hat{f})$. So that $A = \Psi(\hat{f})$. We complete the proof by letting $f_* = \hat{f}$. $\square$

4. Maximization and minimization related to (1.3)

In this section, we consider two optimization problems (Opt1) and (Opt2) related to (1.3). The energy functional of (1.3) is

$$J(u) = \frac{1}{2}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\Big) - \int_\Omega f(x)u(x)dx.$$

It is easy to see $J \in C^1(H^s(\Omega),\mathbb{R})$ and

$$J'(u)v = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy - \int_\Omega f(x)v(x)dx.$$

$u \in H^s(\Omega)$ is a solution of the problem (1.3) if and only if $J'(u)v = 0, \forall v \in H^s(\Omega)$.

By proposition 3.1, we have the following result.

Proposition 4.1. *Assume that θ satisfies* (H3) *and* (H4). *Then for each f in $L^\infty(\Omega)$, (1.3) has a unique solution $u_f \in H^s(\Omega)$, and $J(u_f) = \inf_{v\in H^s(\Omega)} J(v)$.*

Remark 4.2. Since $J'(u_f)u_f = 0$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_f(x)-u_f(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy = \int_\Omega f(x)u_f(x)\,dx. \tag{4.1}$$

Recall that $\Phi(f) = \int_\Omega f u_f dx$, we have

$$
\begin{aligned}
&\Phi(f)\\
&= -2\Big(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_f(x)-u_f(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy - \int_\Omega f(x)u_f(x)dx\Big)\\
&= -2\inf_{v\in H^s(\Omega)}\Big(\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v(x)-v(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy - \int_\Omega f(x)v(x)dx\Big)\\
&= -2J(u_f).
\end{aligned}
$$

Theorem 4.3. *Assume that θ satisfies* (H3) *and* (H4)*. Then for each non-negative function $f_0 \in L^\infty(\Omega)$, there exists $f_1 \in \mathscr{R}(f_0)$ which solves the maximization optimization problem* (Opt1)*, i.e.,*

$$
\Phi(f_1) = \int_\Omega f_1 u_1\,dx = \sup_{f\in\mathscr{R}(f_0)}\int_\Omega f u_f\,dx = \sup_{f\in\mathscr{R}(f_0)}\Phi(f)
$$

where $u_1 = u_{f_1}$. Moreover, if $s = \frac{1}{2}$ then there exists an increasing function ϕ such that $f_1 = \phi(u_1)$ almost everywhere in Ω.

Proof. Let

$$
M = \sup_{f\in\mathscr{R}(f_0)}\int_\Omega f u_f\,dx.
$$

We first show that M is finite. Let $f \in \mathscr{R}(f_0)$. From (H4), (4.1) and Hölder's inequality we find

$$
\begin{aligned}
\theta_0\|u_f\|^2 &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_f(x)-u_f(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\\
&= \int_\Omega f u_f\,dx \le \|f\|_\infty\|u_f\|_1.
\end{aligned}
\tag{4.2}
$$

Since $\|f\|_\infty = \|f_0\|_\infty$, it follows from (4.2) and Lemma 2.1 that M is finite.

Let $\{f_i\}$ be a maximizing sequence and let $u_i = u_{f_i}$. From (4.2) it is clear that $\{u_i\}$ is bounded in $H^s(\Omega)$, hence it has a subsequence (still denoted $\{u_i\}$) that converges weakly to $u \in H^s(\Omega)$. We also infer that $\{u_i\}$ converges strongly to u in $L^2(\mathbb{R}^N)$. On the other hand, since $\{f_i\}$ is bounded in $L^\infty(\Omega)$, it must contain a subsequence(still denoted $\{f_i\}$) converging weakly to $\bar f \in L^2(\Omega)$. Note that $\bar f \in \overline{\mathscr{R}(f_0)}$, the weak closure of $\mathscr{R}(f_0)$ in $L^2(\Omega)$. Thus, using the weak lower semi-continuity of the $H^s(\Omega)$ norm and (3.2) we obtain

$$
\begin{aligned}
M &= \lim_{i\to\infty}\int_\Omega f_i u_i\,dx\\
&= \lim_{i\to\infty}\Big(\int_\Omega 2f_i u_i\,dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_i(x)-u_i(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy\Big)\\
&\le \int_\Omega 2\bar f u dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy.
\end{aligned}
\tag{4.3}
$$

Note that from Lemma 2.5 we infer the existence of $\hat f \in \mathscr{R}(f_0)$ that maximizes the linear functional $\int_\Omega hu\,dx$, relative to $h \in \overline{\mathscr{R}(f_0)}$. As a consequence we obtain

$$
\int_\Omega \bar f u\,dx \le \int_\Omega \hat f u\,dx. \tag{4.4}
$$

Applying (4.1), (4.3) and (4.4) we find

$$\begin{aligned} M &\le \int_\Omega 2\hat{f}udx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy \\ &\le \int_\Omega 2\hat{f}\hat{u}\,dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\hat{u}(x)-\hat{u}(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy \\ &= \int_\Omega \hat{f}\hat{u}\,dx \le M \end{aligned}$$

where $\hat{u} = u_{\hat{f}}$. Thus let $f_1 = \hat{f}$ we complete the first part of the proof.

We then show that f_1 is also a maximizer of the functional $\int_\Omega hu_1dx$, relative to $h \in \overline{\mathscr{R}(f_0)}$. In fact, we notice that for each $g \in \mathscr{R}(f_0)$,

$$\begin{aligned} \int_\Omega f_1u_1dx &\ge \int_\Omega gu_gdx \\ &\ge \int_\Omega 2gu_1dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_1(x)-u_1(y))^2}{|x-y|^{N+2s}}\theta(x-y)\,dx\,dy \\ &= \int_\Omega 2gu_1dx - \int_\Omega f_1u_1dx, \end{aligned}$$

which implies that

$$\int_\Omega f_1u_1dx \ge \int_\Omega gu_1dx, \quad \forall g \in \mathscr{R}(f_0). \tag{4.5}$$

If $g \in \overline{\mathscr{R}(f_0)}$ then we may choose a sequence $\{g_n\} \subset \mathscr{R}(f_0)$ such that $\{g_n\}$ converge weakly to g in $L^2(\Omega)$. By (4.5), we obtain

$$\int_\Omega f_1u_1\,dx \ge \int_\Omega g_nu_1dx \to \int_\Omega gu_1\,dx$$

as $n \to \infty$. So that

$$\int_\Omega f_1u_1\,dx \ge \int_\Omega gu_1\,dx, \quad \forall g \in \overline{\mathscr{R}(f_0)} \tag{4.6}$$

and our claim is valid.

Let $E = \{x \in \Omega : f_1(x) = 0\}$ and define $S = sup_{x\in E}(u_1(x))$, we claim that $u_1(x) \ge S$ on E^c almost everywhere. Arguing by contradiction suppose the claim is false. Therefore there exists a number $S_1 < S$ and a subset A of E^c with $|A| > 0$ such that $u_1(x) < S_1$ on A almost everywhere. Now let $S_1 < S_2 < S$. We can find a set D of positive measure contained in E such that $u_1(x) > S_2$ on D almost everywhere. We can assume $|A| = |D|$. Next, consider a measure-preserving map $T : A \to D$. Using T we define a particular rearrangement of f_1, denoted by $\overline{f}$, as follows:

$$\overline{f} = \begin{cases} f_1(Tx) & x \in A, \\ f_1(T^{-1}x) & x \in D, \\ f_1(x) & x \in \Omega\backslash(A \cup D). \end{cases}$$

Thus

$$\begin{aligned} &\int_\Omega \overline{f}u_1dx - \int_\Omega f_1u_1dx \\ &= \int_{A\bigcup D} \overline{f}u_1dx - \int_{A\cup D} f_1u_1dx \end{aligned}$$

$$
\begin{aligned}
&= \int_A \overline{f} u_1 dx + \int_A (u_1 \circ T) f_1 dx - \int_A f_1 u_1 dx - \int_A (u_1 \circ T) \overline{f} dx \\
&= \int_A (u_1 \circ T - u_1)(f_1 - \overline{f}) dx > (S_2 - S_1) \int_A f_1 dx > 0.
\end{aligned}
$$

Therefore $\int_\Omega \overline{f} u_1 dx > \int_\Omega f_1 u_1 dx$, which contradicts (4.6).

Since $s = \frac{1}{2}$, by Lemma 2.6 we know that u_1 has no level set of positive measure on E^c, then by Lemma 2.3 we infer the existence of an increasing function ϕ_1 such that $\phi_1(u_1)$ is a rearrangement of f_1 on E^c. Now we define an increasing function ϕ_2 as

$$
\phi_2(t) = \begin{cases} 0 & t < S, \\ \phi_1(t) & t \geq S. \end{cases}
$$

It is easy to check that $\phi_2(u_1)$ is a rearrangement of f_1 on Ω. By (4.6) and Lemma 2.4 we infer that $f_1 = \phi_2(u_1)$, so that we complete the proof. $\square$

Theorem 4.4. *Suppose that f_0 is positive, $\theta(x) \equiv m \geq \theta_0$ and $s = 1/2$. Then there exists $f_2 \in \mathscr{R}(f_0)$ which solves the minimization optimization problem* (Opt2)*, i.e.,*

$$
\Phi(f_2) = \int_\Omega f_2 u_2 \, dx = \inf_{f \in \mathscr{R}(f_0)} \int_\Omega f u_f \, dx = \inf_{f \in \mathscr{R}(f_0)} \Phi(f)
$$

where $u_2 = u_{f_2}$.

We need some lemmas before we give the proof of the above theorem.

Lemma 4.5. *The functional $\Phi|_{\overline{\mathscr{R}(f_0)}}$ is strictly convex.*

Proof. Let $g, h \in \overline{\mathscr{R}(f_0)}$ and $v \in H^s(\Omega)$, then for all $t \in (0,1)$, we have

$$
\begin{aligned}
&2 \int_\Omega (tg + (1-t)h) v dx - m\|v\|^2 \\
&= t\Big(\int_\Omega 2gv dx - m\|v\|^2 \Big) + (1-t)\Big(\int_\Omega 2hv dx - m\|v\|^2 \Big).
\end{aligned}
$$

By (1.4) and (1.5), and taking the superior relative to $v \in H^s(\Omega)$ in both sides of the above equality, we obtain

$$
\Phi(tg + (1-t)h) \leq t\Phi(g) + (1-t)\Phi(h);
$$

that is, the convexity of Φ has been proved. Now, suppose that equality holds in the above inequality for some $t \in (0,1)$. Then, denote by u_t the solution of the problem (1.3) corresponding to $tg + (1-t)h$, we have

$$
\begin{aligned}
&t\Big(\int_\Omega 2g u_t \, dx - m\|u_t\|^2 \Big) + (1-t)\Big(\int_\Omega 2h u_t \, dx - m\|u_t\|^2 \Big) \\
&= t\Big(\int_\Omega 2g u_g \, dx - m\|u_g\|^2 \Big) + (1-t)\Big(\int_\Omega 2h u_h \, dx - m\|u_g\|^2 \Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_\Omega 2g u_t \, dx - m\|u_t\|^2 &= \int_\Omega 2g u_g \, dx - m\|u_g\|^2, \\
\int_\Omega 2h u_t \, dx - m\|u_t\|^2 &= \int_\Omega 2h u_h \, dx - m\|u_h\|^2.
\end{aligned}
$$

By the uniqueness of the minimizer of the functional J, we must have $u_t = u_g = u_h$. Moreover, since

$$
\begin{aligned}
m\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_g(x)-u_g(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy &= \int_\Omega g(x)v(x)dx, \quad \forall v\in H^s(\Omega),\\
m\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy &= \int_\Omega h(x)v(x)dx, \quad \forall v\in H^s(\Omega),
\end{aligned}
$$

if $u_g = u_h$, we must have $g(x) = h(x)$ a.e. in Ω, and the strict concavity is proved. $\square$

Lemma 4.6. *The functional $\Phi|_{\overline{\mathscr{R}(f_0)}}$ is weakly continuous.*

Proof. Suppose $\{g_i\}\subset\overline{\mathscr{R}(f_0)}$ and $g_i \rightharpoonup g\in\overline{\mathscr{R}(f_0)}$. If u_i and u_g are the corresponding solutions of (1.3) we have

$$
\begin{aligned}
\Phi(g)+\int_\Omega 2(g_i-g)u_g\,dx &= \int_\Omega 2g_iu_gdx - m\|u_g\|^2\\
&\le \Phi(g_i)\\
&= \int_\Omega 2gu_idx - m\|u_i\|^2 + \int_\Omega 2(g_i-g)u_i\,dx\\
&\le \Phi(g)+\int_\Omega 2(g_i-g)u_i\,dx.
\end{aligned}
$$

We have

$$\lim_{i\to\infty}\int_\Omega (g_i-g)u_gdx = 0.$$

We only need to prove that

$$\lim_{i\to\infty}\int_\Omega (g_i-g)u_idx = 0. \tag{4.7}$$

Since $J(u_i)\le J(0)=0$ and

$$J(u_i)\ge \frac{m}{2}\|u_i\|^2 - C\|g_i\|\|u_i\|,$$

it follows that $\{u_i\}$ is bounded in $H^s(\Omega)$. Hence,

$$\Big|\int_\Omega (g_i-g)u_i\,dx\Big| \le C\|u_i\|\le C.$$

Now we can choose a subsequence $\{u_{n_j}\}$ such that

$$\lim_{j\to\infty}\Big|\int_\Omega (g_{n_j}-g)u_{n_j}\,dx\Big| = \limsup_{i\to\infty}\Big|\int_\Omega (g_i-g)u_i\,dx\Big|.$$

Noting that $\{u_{n_j}\}$ is also bounded in $H^s(\Omega)$, going if necessary to a subsequence, we may assume that $u_{n_j}\rightharpoonup u$ in $H^s(\Omega)$ and $u_{n_j}\to u$ in $L^2(\Omega)$ as $j\to\infty$. By the Hölder inequality, we obtain

$$
\begin{aligned}
\Big|\int_\Omega (g_{n_j}-g)u_{n_j}\,dx\Big| &\le \Big|\int_\Omega (g_{n_j}-g)(u_{n_j}-u)dx\Big| + \Big|\int_\Omega (g_{n_j}-g)udx\Big|\\
&\le \|g_{n_j}-g\|_{L^2}\|u_{n_j}-u\|_{L^2} + \Big|\int_\Omega (g_{n_j}-g)udx\Big| \to 0
\end{aligned}
$$

as $j \to \infty$. So that
$$0 \le \liminf_{i\to\infty} \Big| \int_\Omega (g_i - g) u_i \, dx \Big| \le \limsup_{i\to\infty} \Big| \int_\Omega (g_i - g) u_i \, dx \Big| \le 0,$$
which implies (4.7), and then $\Phi(g_i) \to \Phi(g)$. □

Proof of Theorem 4.4. Denote by $\overline{\mathscr{R}(f_0)}$ the weak closure of $\mathscr{R}(f_0)$ in $L^2(\Omega)$, then $\overline{\mathscr{R}(f_0)}$ is convex and weakly compact in $L^2(\Omega)$ by Lemma 2.2. By using Lemma 4.5 and Lemma 4.6, one can prove easily that Φ has a unique minimizer in $\overline{\mathscr{R}(f_0)}$. Let $\underline{g} \in \overline{\mathscr{R}(f_0)}$ be the minimizer of Φ. If $g \in \overline{\mathscr{R}(f_0)}$ and $0 < t < 1$ we have $g_t = t\underline{g} + (1-t)g \in \overline{\mathscr{R}(f_0)}$. By the minimality condition, we have
$$\Phi(\underline{g}) \le \Phi(g_t). \tag{4.8}$$
If u_g, $u_{\underline{g}}$ and u_t are solutions of (1.3) corresponding to g, $\underline{g}$ and g_t, respectively, we have
$$u_t = t u_{\underline{g}} + (1-t) u_g.$$
Therefore, by (1.4) (1.5) and (4.8), we have
$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_{\underline{g}}(x) - u_{\underline{g}}(y))^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Big(\frac{t(u_{\underline{g}}(x) - u_{\underline{g}}(y)) + (1-t)(u_g(x) - u_g(y))}{|x-y|^{\frac{N+2s}{2}}} \Big)^2 \, dx \, dy.
\end{aligned}$$
After simplification, we obtain
$$\begin{aligned}
&(1+t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_{\underline{g}}(x) - u_{\underline{g}}(y))^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\le 2t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_{\underline{g}}(x) - u_{\underline{g}}(y))(u_g(x) - u_g(y))}{|x-y|^{N+2s}} \, dx \, dy \\
&\quad + (1-t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))^2}{|x-y|^{N+2s}} \, dx \, dy.
\end{aligned}$$
Letting $t \to 1$, we find
$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_{\underline{g}}(x) - u_{\underline{g}}(y))^2}{|x-y|^{N+2s}} \, dx \, dy \le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_{\underline{g}}(x) - u_{\underline{g}}(y))(u_g(x) - u_g(y))}{|x-y|^{N+2s}} \, dx \, dy.$$
We can rewrite the latter inequality as
$$\int_\Omega \underline{g} u_{\underline{g}} dx \le \int_\Omega g u_{\underline{g}} dx, \quad \forall g \in \overline{\mathscr{R}(f_0)}.$$
Since f_0 is positive it is easy to deduce that $\underline{g} > 0$ a.e. $x \in \Omega$. By Lemma 2.6 we have that each level set of $u_{\underline{g}}$ has zero measure. By Lemmas 2.3 and 2.4, there exists a decreasing function ϕ such that $\phi \circ u_{\underline{g}}$ is a rearrangement of $\underline{g}$ and the unique minimizer of the linear functional $\int_\Omega g u_{\underline{g}} dx$, related to $g \in \overline{\mathscr{R}(f_0)}$. So that $\underline{g} = \phi \circ u_{\underline{g}} \in \mathscr{R}(f_0)$. We complete the proof by letting $f_2 = \underline{g}$. □

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