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UNIQUE CONTINUATION OF POSITIVE SOLUTIONS FOR DOUBLY DEGENERATE QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider quasilinear elliptic equations that are degenerate in two ways. One kind of degeneracy is due to the particular structure of the given vector fields. Another is a consequence of the weights that we impose to the quadratic form of the associated differential operator. Nonetheless we prove that positive solutions satisfy unique continuation property.

1. Introduction

This note is a further step in the study of qualitative properties of generalized solutions of degenerate elliptic PDE's. Indeed, in the last decades many papers have been devoted to investigations of qualitative and quantitative properties of solutions. Among several kind of degeneracy, we focus our attention to the case where the operator is associated to a non commuting system of vector fields of a particular kind. Moreover, we also study the possibility of regularity for the weak solutions where minimal assumptions on the lower order coefficients are assumed (see [5, 6, 16, 17, 18]). A different kind of degeneracy has been investigated by Fabes, Kenig and Serapioni and Gutierrez in [21] and [25] respectively (see also [9, 27]). There the degeneracy was given by a Muckenhoupt weight of the class A_2. However, degeneracy given by a A_2 weight is not the only possibility. Studies about a different kind of degeneracy started in [23] and [24] (see also [10, 11, 12, 14, 19, 26]) where the weight is a suitable power of a function that belongs to the class of strong A_∞ weights introduced by David and Semmes in [7].

In this note we prove that the positive solutions of a quasilinear strongly degenerate elliptic equation satisfy the unique continuation property. To make the statement precise, we let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$, let $z = (x, y) \in \mathbb{R}^N$ and denote points in $\mathbb{R}^N$ in such a way that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $n + m = N$.

Let us consider a quasilinear elliptic equation of the kind

$$\operatorname{div} A(z, u, \nabla_\lambda u) + B(z, u, \nabla_\lambda u) = 0 \tag{1.1}$$

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where A and B are measurable functions satisfying the following structure conditions

$$|A(z,u,\xi)| \le a\omega(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + b|u|^{p-1}$$
$$|B(z,u,\xi)| \le c[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + d|u|^{p-1}$$
$$\xi \cdot A(z,u,\xi) \ge \omega(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p}{2}} - d|u|^p .$$

Here $\omega = v^{1-\frac{p}{N}}$, $1 < p < N$, v is a strong A_∞ weight, $\nabla_\lambda u = (\nabla_x u(z), \lambda(x)\nabla_y u(z))$ and λ is a function satisfying (H1), (H2), (H3) (see Section 2).

Regarding equation (1.1) we will show that if u is a non negative solution of (1.1), then u cannot have zeros of infinite order unless it is identically zero in Ω.

We remark here that the equation we consider is doubly degenerate in the following sense. It is a Grushin operator which exhibits further degeneracy given by a strong A_∞ weight. Moreover, we assume the lower order coefficients to belong to suitable Stummel - Kato classes (see Section 4). This kind of assumption is necessary for regularity of solutions at least in some cases (see [8]).

We use the following two key ideas. The first is based on a method in [4] to prove that a suitable power of any positive solution satisfies a doubling inequality. The other one is the clever use of a Fefferman Poincare' type inequality

$$\int_B |V(z)||u(z)|^p\,\omega(z)\,dz \le C\int_B |\nabla_\lambda u(z)|^p \omega\,dz\,, \tag{1.2}$$

proved in the Section 4 (see also [5, 15, 13]) that is needed in order to handle the lower order terms. We then get our result because the solution has zeros of infinite order and, satisfies a doubling inequality at the same time. This is possible if and only if the solution is identically zero.

2. Preliminaries

To fix the notation we denote by $z = (x,y)$ points in $\mathbb{R}^N$, where x and y belong to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $n + m = N$. Let us denote by $\lambda = \lambda(x)$ a function such that verifies the following assumptions

- (H1) λ is a continuous nonnegative function vanishing only at a finite number of points.
- (H2) λ^n is a strong A_∞ weight (see Definition 2.5).
- (H3) λ satisfies an infinite order reverse Hölder inequality, i.e. for any $x_0 \in \mathbb{R}^n$, $r > 0$ we have

$$\fint_{|x-x_0|<r} \lambda(x)dx \sim \max_{|x-x_0|<r} \lambda(x)\,.$$

Let $u : \mathbb{R}^N \to \mathbb{R}$ be a almost everywhere differentiable function in $\mathbb{R}^N$. We denote its λ-gradient by $\nabla_\lambda u(z) = (\nabla_x u(z), \lambda(x)\nabla_y u(z))$ and $|\nabla_\lambda u|^2 = |\nabla_x u|^2 + \lambda^2(x)|\nabla_y u|^2$.

Then, we define the Carnot-Carathéodory metric (C-C metric) ρ in $\mathbb{R}^N$ with respect to ∇_λ in the following way.

Definition 2.1 (λ sub unit curve)**.** An absolutely continuous curve $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N) : [0,T] \to \mathbb{R}^N$ is said to be a λ-sub unitary curve if the following inequality holds

$$\langle \gamma'(t), z\rangle^2 \le |x|^2 + \lambda(\gamma_1(t), \dots, \gamma_n(t))|y|^2$$

for any $z = (x,y) \in \mathbb{R}^N$, and almost any $t \in [0,T]$.

Definition 2.2 (λ Carnot-Caratheodory distance)**.** If z_1,z_2 are points in $\mathbb{R}^N$, let us consider the set of all λ-sub unitary curves connecting z_1 and z_2. We set

$$\begin{aligned}\rho(z_1,z_2) = \inf\big\{&T \ge 0 : \exists \text{ a sub-unit curve } \gamma : [0,T] \to \mathbb{R}^N,\\ &\text{such that } \gamma(0) = z_1 \text{ and } \gamma(T) = z_2\big\}.\end{aligned}$$

If there are no such curves we set $\rho(z_1,z_2) = +\infty$.

We denote by $B(z_0,r)$ the λ Carnot-Caratheodory metric ball centered at z_0 of radius r. We also write B or B_r if the center of the ball has no relevance.

We recall the definition of the A_q and the strong A_∞ weights with respect to the λ metric (see [23, 24]).

Definition 2.3 (A_q weights)**.** Let $q > 1$ and let v be a nonnegative locally integrable function in $\mathbb{R}^N$. We say that v is a weight of the Muckenhoupt class A_q if

$$\sup_B \Big(\frac{1}{|B|}\int_B v(z)\,dz\Big)\Big(\frac{1}{|B|}\int_B [v(z)]^{\frac{-1}{q-1}}\,dz\Big)^{q-1} \equiv C_0 < +\infty$$

the supremum being taken over all Carnot-Carathéodory metric balls B in $\mathbb{R}^N$. The number C_0 is called the A_q constant of v.

We recall the doubling property.

Lemma 2.4 (Doubling property)**.** *Let $v \in A_q$. There exists a positive constant c such that for any C-C metric ball B_r,*

$$v(B_r) \le c\,v(B_{r/2}).$$

For a proof of the above lemma, see [3]. Now we give the definition of strong A_∞ weight with respect to the function λ.

Definition 2.5. Let v be a A_q weight for some $q > 1$. For any z_1, z_2 in $\mathbb{R}^N$ we set

$$\delta(z_1,z_2) = \inf\Big(\int_B v(z)\lambda^{\frac{m}{N-1}}(x)dz\Big)^{1/N}$$

where the infimum is taken over the C-C balls B such that $z_1, z_2 \in B$.

For any curve $\gamma : [0,T] \to \mathbb{R}^N$, we define its v-length as

$$l(\gamma) = \liminf_{|\sigma|\to 0} \sum_{i=0}^{p-1} \delta(\gamma(t_{i+1}),\gamma(t_i))$$

where $\sigma = \{t_0,\dots,t_p\}$ is a partition of $[0,T]$.

Then we define a distance $d(z_1,z_2)$ as the infimum of the v-lengths of λ-sub unitary curves connecting z_1 and z_2. If there exist positive constants c_1 and c_2 such that

$$c_1\delta(z_1,z_2) \le d(z_1,z_2) \le c_2\delta(z_1,z_2)$$

we say that v is a strong A_∞ weight for the metric ρ.

Example 2.6. A strong A_∞ weight is the function $v(z) = \rho(z,z_0)^\alpha$ with $\alpha \ge 0$ and $z_0 \in \mathbb{R}^N$ (see [24]).

Proposition 2.7. *Let $0 < \alpha < 1$ and let v be a strong A_∞ weight. Then the measure v^α satisfies the doubling property.*

Proof. Since v is a strong A_∞ there exists q such that $v \in A_q$. This implies that $v^\alpha \in A_{(q-1)\alpha+1}$ and, as a consequence, v^α satisfies the doubling property. $\square$

Lemma 2.8. *Let v be a strong A_∞ weight. Then for every compact set E there exist two positive constant r_0 and γ such that*

$$v(B(z_0,r)) \le \gamma r^N$$

for all $z_0 \in E$ and for all $0 < r < r_0$.

The above lemma is a consequence of [22, Theorem 2.3]. We can define Lebesgue and Sobolev classes with respect to the strong A_∞ weights.

Definition 2.9. Let v be a strong A_∞ weight and $\omega = v^{1-p/N}$, $\Omega \subset \mathbb{R}^N$. For any $u \in C_0^\infty(\Omega)$ we set

$$\|u\|_{L^p_v(\Omega)} = \Big(\int_\Omega |u(z)|^p\,\omega(z)\,dz\Big)^{1/p} \quad 1 \le p < \infty\,.$$

We define $L^p_v(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the above norm. In a similar way we define Sobolev classes. Let $1 < p < N$. For any $u \in C^\infty(\Omega)$ we set

$$\|u\|_{H^{1,p}_v(\Omega)} = \Big(\int_\Omega |u(z)|^p\,\omega(z)\,dz\Big)^{1/p} + \Big(\int_\Omega |\nabla_\lambda u(z)|^p\,\omega(z)\,dz\Big)^{1/p}. \tag{2.1}$$

We define $H^{1,p}_{0,v}(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the norm (2.1) and $H^{1,p}_v(\Omega)$ to be the completion of $C^\infty(\Omega)$ with respect to the same norm.

Now, to recall the Sobolev embedding theorem and the representation formula proved in [24, Theorem I and Corollary 3.2], we need another assumption on strong A_∞ weights.

A strong A_∞ weight v satisfies the local boundedness condition near the zeros of λ if the following condition holds

(H4) if $\lambda(x_1) = 0$, then $v(x,y)$ is bounded as $x \to x_1$ uniformly in y for y in any bounded set.

Theorem 2.10. *Let u_1, $u_2 \in L^1_{\rm loc}$ be positive functions such that u_1 is doubling, $1 \le p < q < +\infty$. Assume that there exist positive constants c_1 and c_2 such that, for all C-C balls $B_0 = B(z_0,r_0)$ and $B_r = B(z,r) \subset c_1B_0$, we have*

$$\frac{r}{r_0}\Big(\frac{u_1(B_r)}{u_1(B_0)}\Big)^{1/q} \le c_2\Big(\frac{u_2(B_r)}{u_2(B_0)}\Big)^{1/p},$$

where $u_1(B_0) = \int_{B_0} u_1(z)dz$.

If there exists a strong A_∞ weight w satisfying condition (H4) *such that $u_2 w^{-(N-1)/N}$ belongs to the class A_p with respect to the (doubling) measure $w^{(N-1)/N}dz$, then the following Sobolev-Poincaré inequality hold*

$$\Big(\fint_{B(z_0,r)} |g-\mu|^q u_1(z)dz\Big)^{1/q} \le Cr\Big(\fint_{B(z_0,r)} |\nabla_\lambda g|^p u_2(z)dz\Big)^{1/p}$$

for any Lipschitz continuous function g, where μ can be chosen to be the u_1-average of g over $B(z_0,r)$.

If v is a strong A_∞ weight, by taking $u_1 = u_2 = v^{1-p/N}$ in Theorem 2.10 the result follows by observing that a positive power of a strong A_∞ weight is doubling.

Theorem 2.11. *Let $1 < p < N$ and let v be a strong A_∞. If there exists a strong A_∞ weight w satisfying* (H4) *such that $v^{1-p/N}w^{-(N-1)/N}$ belongs to A_p with respect to the (doubling) measure $w^{(N-1)/N}dz$, then there exists a constant $q > p$ such that*

$$\Big(\fint_{B(z_0,r)} |g-\mu|^q v^{1-p/N}(z)dz\Big)^{1/q} \le Cr\Big(\fint_{B(z_0,r)} |\nabla_\lambda g|^p v^{1-p/N}(z)dz\Big)^{1/p} \tag{2.2}$$

for any Lipschitz continuous function g, where μ can be chosen to be the $v^{1-p/N}$-average of g over $B(z_0,r)$.

Remark 2.12. We stress that if we take the weights $v = w = \rho^\alpha(0,z)$ and the function $\lambda(x) = |x|^\sigma$, $(\alpha,\ \sigma > 0)$, the assumptions of Theorem 2.11 are satisfied (see also [23]).

3. BMO_ω, $A_2(\omega)$ and functions with zeros of infinite order

In this section we denote by v be a strong A_∞ weight, $1 < p < N$ and $\omega = v^{1-p/N}$.

Definition 3.1. *Let f be a locally integrable function in $\mathbb{R}^N$ with respect to ω. We say that f belongs to the class BMO_ω if*

$$\sup_B \frac{1}{\omega(B)}\int_B |f(\zeta)-f_B|\omega(\zeta)d\zeta < +\infty\,,$$

where the supremum is taken over the balls B.

Definition 3.2. *A locally integrable function u such that $\frac{1}{u}$ is also locally integrable u belongs to the class $A_2(\omega)$ if*

$$\int_B u(\zeta)\omega(\zeta)d\zeta\int_B \frac{1}{u(\zeta)}\omega(\zeta)d\zeta \le C(\omega(B))^2$$

for any C-C ball B.

Theorem 3.3. *Let u be a function in the class $A_2(\omega)$. Then*

$$\int_{B_{2r}} u\,\omega dz \le C\int_{B_r} u\,\omega dz$$

for any C-C ball B_r.

The proof of the above theorem is similar to that of [20, Theorem 4.5], and is omitted.

Definition 3.4. Let Ω be a bounded domain of $\mathbb{R}^N$. Let μ be a locally integrable function such that $\frac{1}{\mu}$ is locally integrable and $u \in L^1_{\rm loc}(\Omega)$. We say that u has a zero of infinite order at $z_0 \in \Omega$ if

$$\lim_{\sigma\to 0}\frac{\int_{B(z_0,\sigma)} u(z)\mu(z)dz}{(\mu(B(z_0,\sigma)))^k} = 0\,, \quad \forall k > 0\,.$$

Theorem 3.5. *Let $u \in L^1_{\rm loc}(\Omega)$, $u \ge 0$ a.e. in Ω, $u \not\equiv 0$. If there exist positive numbers C and σ_0 such that*

$$\int_{B(z,2\sigma)} u\,\omega dz \le C\int_{B(z,\sigma)} u\,\omega dz$$

for any $0 < \sigma < \sigma_0$ and any $z \in \Omega$ then u cannot have zeros of infinite order in Ω.

Proof. We argue by contradiction. Let $z_0 \in \Omega$ be a zero of infinite order for u. Using Theorem 3.3 k times, Hölder inequality and Lemma 2.8 k times we obtain

$$\begin{aligned}\int_{B_r} u\omega dz &\le C^k \int_{B_{\frac{r}{2^k}}} u\omega dz \\ &= C^k \omega(B_{r/2^k})^\beta 1/(\omega(B_{r/2^k})^\beta) \int_{B_{r/2^k}} u\omega\, dx \\ &\le C^k \gamma^{\beta(1-p/N)} \Big(\frac{r}{2^k}\Big)^{\beta(N-p)} |B_r|^{\frac{\beta p}{N}} 1/\omega(B_{r/2^k})^\beta \int_{B_{r/2^k}} u\omega\, dx.\end{aligned} \tag{3.1}$$

Now, if we choose $\beta = \log_{2^{N-p}} C$, we obtain

$$\int_{B_r} u\omega dz \le \gamma^{\beta(1-p/N)} r^{\beta(N-p)} |B_r|^{\beta p/N} 1/\omega(B_{r/2^k})^\beta \int_{B_{r/2^k}} u\omega\, dx.,$$

from which, as $k \to \infty$, we have that $u\omega \equiv 0$ in B_r and, since $\omega \not\equiv 0$, $u \equiv 0$. $\square$

4. Stummel- Kato classes and embedding theorem

In this section we give the definition of Stummel-Kato classes with respect to a strong A_∞ weight and then prove our embedding result. Our starting point is a representation formula (see [24, Corollary 3.2]) proved by Franchi, Gutierrez and Wheeden.

Lemma 4.1 (Representation formula). *Let Ω be a bounded domain in $\mathbb{R}^N$, v a strong A_∞ weight satisfying* (H4) *and u a compactly supported smooth function in a metric ball $B = B_R \subset \Omega$. Then there exists c independent of u such that*

$$|u(z)| \le c \int_B |\nabla_\lambda u(\xi)| v^{1-\frac{1}{N}}(\xi) k(z,\xi) d\xi$$

for almost all $z \in B$, where

$$k(z,\xi) = \Big(\int_{B_{\rho(z,\xi)}(z)} v(\zeta) \lambda^{\frac{m}{N-1}}(\chi) d\zeta\Big)^{\frac{1-N}{N}}.$$

Definition 4.2. Let V be a locally integrable function in Ω, $r > 0$, and $p \in]1, N[$. Let v be a strong A_∞ weight. We set $\omega(z) \equiv v^{1-\frac{p}{N}}(z)$ and

$$\phi(V;r) \equiv \sup_{z\in\Omega} \Big(\int_{B(z,r)} k(z,\xi) v(\xi) \Big(\int_{B(z,r)} |V(\zeta)| k(\zeta,\xi) \omega(\zeta) d\zeta\Big)^{\frac{1}{p-1}} d\xi\Big)^{p-1}.$$

We say that V belongs to the class $\tilde{S}_v(\Omega)$ if $\phi(V;r)$ is bounded in a right neighborhood of the origin.

Remark 4.3. If $v(z) = 1$, $\lambda(x) = 1$ and $p = 2$ the previous definitions give back the classical Stummel-Kato class (see [1]).

By using the representation formula we now prove an embedding Theorem related to the Stummel-Kato classes.

Theorem 4.4. *Let v be a strong A_∞ weight satisfying* (H4) *and $1 < p < N$. If V belongs to the class $\tilde{S}_v(\Omega)$, then there exists a constant C such that $\forall u \in C_0^\infty(\Omega)$*

$$\Big(\int_B |V(z)||u(z)|^p \omega(z)\, dz\Big)^{1/p} \le C\phi^{1/p}(V;2R) \Big(\int_B |\nabla_\lambda u(z)|^p \omega\, dz\Big)^{1/p}, \tag{4.1}$$

where $\omega(z) \equiv v^{1-\frac{p}{N}}(z)$ *and* R *is the radius of a metric ball* B*, containing the support of* u*.*

Proof. From the representation formula in Lemma 4.1 it follows that

$$
\begin{aligned}
&\int_B |V(z)||u(z)|^p \omega(z)\,dz \\
&\le C\int_B |V(z)||u(z)|^{p-1}\Big(\int_B |\nabla_\lambda u(\xi)| k(z,\xi) v^{1-\frac{1}{N}}(\xi)\,d\xi\Big)\omega(z)\,dz \\
&= C\int_B |\nabla_\lambda u(\xi)| v^{1-\frac{1}{N}}(\xi)\Big(\int_B |V(z)||u(z)|^{p-1} k(z,\xi)\,\omega(z)\,dz\Big)\,d\xi \\
&\le C\Big(\int_B |\nabla_\lambda u(\xi)|^p v^{1-\frac{p}{N}}(\xi)\,d\xi\Big)^{1/p} \\
&\quad\times\Big(\int_B\Big(\int_B |V(z)||u(z)|^{p-1}k(z,\xi)\omega(z)\,dz\Big)^{p'} v(\xi)\,d\xi\Big)^{1/p'}.
\end{aligned}
$$

By considering the last integral we can write

$$
\begin{aligned}
&\Big(\int_B |V(z)||u(z)|^{p-1}k(z,\xi)\omega(z)\,dz\Big)^{p'} \\
&\le \Big(\int_B |V(z)|k(z,\xi)\omega(z)\,dz\Big)^{p'/p}\int_B |V(z)||u(z)|^p k(z,\xi)\omega(z)\,dz
\end{aligned}
$$

and then, we obtain

$$
\begin{aligned}
&\int_B |V(z)||u(z)|^p \omega(z)\,dz \\
&\le \Big(\int_B |\nabla_\lambda u(\xi)|^p v^{1-\frac{p}{N}}(\xi)\,d\xi\Big)^{1/p}\Big(\int_B\Big(\int_B |V(\zeta)|k(\zeta,\xi)\omega(\zeta)\,d\zeta\Big)^{1/p-1} \\
&\quad\times\int_B |V(z)|k(z,\xi)|u(z)|^p\,\omega(z)\,dz v(\xi)\,d\xi\Big)^{1/p'}
\end{aligned}
$$

so

$$
\begin{aligned}
&\int_B |V(z)||u(z)|^p \omega(z)\,dz \\
&= \Big(\int_B |\nabla_\lambda u(\xi)|^p v^{1-\frac{p}{N}}(\xi)\,d\xi\Big)^{1/p}\int_B |V(z)||u(z)|^p\int_B k(z,\xi) \\
&\quad\times\Big(\int_B |V(\zeta)|k(\zeta,\xi)\omega(\zeta)\,d\zeta\Big)^{\frac{1}{p-1}} v(\xi)\,d\xi\,\omega(z)\,dz \\
&\le \Big(\int_B |\nabla_\lambda u(\xi)|^p \omega(\xi)\,d\xi\Big)^{1/p}\Big(\int_B \phi^{\frac{1}{p-1}}(V;R)|V(z)||u(z)|^p\,\omega(z)\,dz\Big)^{1/p'} \\
&= \phi^{1/p}(V;R)\Big(\int_B |\nabla_\lambda u(\xi)|^p \omega(\xi)\,d\xi\Big)^{1/p}\Big(\int_B |V(z)||u(z)|^p\,\omega\,dz\Big)^{1/p'}
\end{aligned}
$$

from which (4.1) easily follows. $\square$

5. Unique continuation property

In this section we show how to apply the results obtained in the previous section in order to get unique continuation property for positive solutions of equation (1.1). Let Ω be a bounded domain in $\mathbb{R}^N$ and λ be a function satisfying (H1), (H2), (H3). Moreover, let v be a strong A_∞ weight satisfying (H4).

We consider the quasilinear equation

$$\operatorname{div} A(z,u,\nabla_\lambda u) + B(z,u,\nabla_\lambda u) = 0 \tag{5.1}$$

where A and B are given measurable functions that satisfy the following structure conditions

$$\begin{gathered} |A(z,u,\xi)| \le a\omega(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + b(z)|u|^{p-1} \\ |B(z,u,\xi)| \le c(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + d(z)|u|^{p-1} \\ \xi \cdot A(z,u,\xi) \ge \omega(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p}{2}} - d(z)|u|^p\,, \end{gathered} \tag{5.2}$$

where $\omega = v^{1-\frac{p}{N}}$, $1<p<N$.

Definition 5.1. A function $u \in H^{1,p}_v(\Omega)$ is a local weak solution of (5.1) in Ω if

$$\int_\Omega A(z,u(z),\nabla_\lambda u(z))\, \nabla_\lambda \varphi(z)\, dz + \int_\Omega B(z,u(z),\nabla_\lambda u(z))\; \varphi(z)\, dz = 0\,, \tag{5.3}$$

for every $\varphi \in H^{1,p}_{0,v}(\Omega)$.

Theorem 5.2. *Let u be a non negative weak solution of* (5.1) *that is not identically zero in Ω. Let us assume that the structure conditions* (5.2) *hold assuming that*

$$a \in \mathbb{R}\,, \quad \Big(\frac{b}{\omega}\Big)^{p/p-1}, \Big(\frac{c}{\omega}\Big)^p, \frac{d}{\omega}, \in \tilde{S}_v(\Omega)\,. \tag{5.4}$$

Then u has no zero of infinite order in Ω.

Proof. Let $z_0 \in \Omega$ and $B(z_0,r)$ be a ball such that $B(z_0,2r) \subset \Omega$. Let $B_h \subset B(z_0,r)$ and $\eta \in W^{1,p}_0(\Omega)$ be a non negative function supported in B_{2h}. Using $\varphi = \eta^p u^{1-p}$ as test function in (5.3) we obtain (see [12])

$$\int_\Omega \eta^p |\nabla_\lambda \log u|^p \omega\, dz \le C_1 \Big\{ \int_\Omega |\nabla_\lambda \eta|^p \omega\, dz + \int_\Omega V \eta^p\, dz \Big\}\,, \tag{5.5}$$

where

$$V = \frac{b^{\frac{p}{p-1}}}{\omega^{\frac{1}{p-1}}} + \frac{c^p}{\omega^{p-1}} + d\,.$$

From (5.4) $\frac{V}{\omega} \in \tilde{S}_v(\Omega)$ and by Theorem 4.4 we obtain

$$\int_\Omega V\eta^p\, dz \le C_2 \int_\Omega |\nabla_\lambda \eta|^p \omega\, dz$$

and then, from (5.5), we have

$$\int_\Omega \eta^p |\nabla_\lambda \log u|^p \omega\, dz \le C_3(p,a,\phi_{\frac{V}{\omega}}, \operatorname{diam}\Omega) \int_\Omega |\nabla_\lambda \eta|^p \omega\, dz\,.$$

Choosing $\eta(z)$ so that $\eta(z) \equiv 1$ in B_h, $0 \le \eta \le 1$ in $B_{2r} \setminus B_h$ and $|\nabla_\lambda \eta| \le 3/h$, we obtain

$$\int_{B_h} |\nabla_\lambda \log u|^p \omega\, dz \le C_4(p,a,\phi_{\frac{V}{\omega}}, \operatorname{diam}\Omega) \frac{\omega(B_h)}{h^p}\,.$$

Therefore, by Poincaré-Sobolev inequality (2.2) and John-Nirenberg lemma (see [2]), we have $\log u \in BMO_\omega$.

Then there exists a constant $\delta > 0$ such that $u^\delta \in A_2(\omega)$. By Theorem 3.3, u^δ has the doubling property. Then from Theorem 3.5 u^δ and u have no zero of infinite order. $\square$

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