Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 163, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

VIBRATIONS MODELED BY THE STANDARD LINEAR MODEL OF VISCOELASTICITY WITH BOUNDARY DISSIPATION

PEDRO GAMBOA, HUY HOANG NGUYEN, OCTAVIO VERA, VERONICA POBLETE

Communicated by Dung Le

Abstract. We consider vibrations modeled by the standard linear solid model of viscoelasticity with boundary dissipation. We establish the well-posedness and the exponential stability.

1. Introduction

We consider in this paper vibrations modeled by the standard linear solid model of viscoelasticity

$$\alpha u_{ttt} + u_{tt} - a^2\Delta u - a^2\alpha\Delta u_t = 0, \quad (t,x) \in (0,\infty)\times\Omega \tag{1.1}$$

with boundary dissipation. Here, Ω is a bounded open connected domain with smooth boundary $\Gamma = \partial\Omega$ in $\mathbb{R}^n$ ($n \geq 1$) and α is a positive constant. The function $u = u(x,t)$ represents the vibrations of flexible structures. In the above equation, $a > 0$ is the constant wave velocity. The initial conditions are given by

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u_{tt}(0,x) = u_2(x), \quad x \in \Omega \tag{1.2}$$

and the dissipative boundary condition is given by

$$\frac{\partial u}{\partial \nu} + \alpha\frac{\partial u_t}{\partial \nu} = -u_t - \alpha u_{tt} \quad \text{on } \Gamma. \tag{1.3}$$

We denote by ν, the unit normal of Γ pointing towards the exterior of Ω.

Equation (1.1) is given by the standard linear model of viscoelasticity. We refer to [1] for mathematical formulation of problem (1.1)-(1.3). Let us consider

$$v = \alpha u_t + u, \quad (t,x) \in [0,\infty)\times\overline{\Omega}. \tag{1.4}$$

Then, (1.1) can be rewritten as

$$v_{tt} - a^2\Delta v = 0 \quad \text{in } (0,\infty)\times\Omega, \tag{1.5}$$

$$\frac{\partial v}{\partial \nu} = -v_t \quad \text{on } \Gamma. \tag{1.6}$$

2010 *Mathematics Subject Classification.* 35A01.
Key words and phrases. Strongly continuous; infinitesimal generator; exponential stability; boundary dissipation.

Submitted November 16, 2016. Published July 4, 2017.

The initial conditions of problem (1.5)-(1.6) are given by

$$\begin{aligned} v(0,x) &= \alpha u_t(0,x) + u(0,x) = v_0(x), \\ v_t(0,x) &= \alpha u_{tt}(0,x) + u_t(0,x) = v_1(x), \end{aligned} \tag{1.7}$$

for $x \in \Omega$.

For vibrations modeled for coupled system of thermoviscoelastic equations type, there are relatively few mathematical results, see for instance [13] and references therein. The asymptotic behaviour of solutions to the equations of linear viscoelasticity as t tends to infinity has been studied by many authors (see the book of Liu and Zheng [12] for a general survey on these topics). Research in the stabilization of mathematical models of vibrating, flexible structure has been considerably stimulated by an increasing number of questions of practical concern among others. Gorain [8] considered the stabilization for the vibrations modeled by the standard linear model of viscoelastic defined in Ω subject to the undamped Dirichlet and Neumann boundary where the boundary Γ consists of two parts Γ_0 and Γ_1 such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$. He proved that the amplitude of the vibrations remains bounded in the sense of a suitable norm in an appropriate space. In Alves et. al. [1] the authors generalized the work given by Gorain [8] for coupled system with a thermal effect and they proved the similar result. Other results can be founded in [1, 7, 9] and references therein.

We have a result related to the total energy of the system (1.5)-(1.7) as follows.

Lemma 1.1. *For every solution of the system* (1.5)-(1.7) *the total energy* $\mathcal{E} : [0,\infty) \to [0,\infty)$ *is given in time* t *by*

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega \left(|v_t|^2 + a^2\,|\nabla v|^2\right) dx$$

and satisfies

$$\frac{d}{dt}\mathcal{E}(t) = -a^2 \int_\Gamma |v_t|^2\, d\Gamma \le 0. \tag{1.8}$$

Proof. Multiplying (1.5) by v_t and using the Green formula together with the boundary dissipation condition (1.6) the lemma follows. □

In this work we extend the results given in [1] and [8], where in this paper we consider the case $\alpha = \beta$ and we put the boundary dissipation.

This paper is organized as follows: Section 2 briefly outlines the notations and the well-posedness of the system is established. In Section 3, we show the exponential stability of the system (1.9)-(1.11), using suitable multiplier techniques.

Remark 1.2. The negativity of the integral on the right hand side of (1.8) shows that some amount of energy of the system is dissipating throughout the domain due to consideration of small internal damping of the structure.

Remark 1.3. To make the problem more realistic, the internal material damping of the structure is incorporated (see [4, 7, 8]). The term $2\xi v_t$ is the damping one appearing in governing differential equation (1.5). Since v_t is order v (displacements) from (1.4), and is not equal to orders v_t (velocities), then, to obtain uniform stability of the system, it is necessary to incorporate a separate damping mechanism order of v_t either in the governing equation or in the boundary. So, damping coefficients $\xi > 0$ are crucially important in this discussion. Taking into account the internal material damping of the structure in the governing differential equation (1.5), we have the following problem

$$v_{tt} + 2\xi v_t - a^2 \Delta v = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.9}$$

$$\frac{\partial v}{\partial \nu} = -v_t \quad \text{on } \Gamma. \tag{1.10}$$

The initial conditions of problem (1.9)-(1.10) are given by

$$\begin{aligned} v(0, x) &= \alpha u_t(0, x) + u(0, x) = v_0(x), \\ v_t(0, x) &= \alpha u_{tt}(0, x) + u_t(0, x) = v_1(x), \end{aligned} \tag{1.11}$$

for $x \in \Omega$. We remark that the existence and the regularity of the solution v of (1.9)-(1.11) have similar properties of (1.5)-(1.7).

Finally, throughout this paper, C is a generic constant, not necessarily the same at each occasion. It could be changed from line to line and depends on an increasing way on the indicated quantities.

2. Setting of the semigroup

In this section we study the setting of the semigroup and establish the well-posedness of the system (1.1)-(1.7). We will use the following standard $L^2(\Omega)$ space, where the scalar product and the norm are denoted by

$$\langle \varphi, \psi \rangle_{L^2(\Omega)} = \int_\Omega \varphi \overline{\psi} \, d\mathbf{x}, \quad \|\psi\|^2_{L^2(\Omega)} = \int_\Omega |\psi|^2 \, d\mathbf{x}.$$

Remark 2.1. Before considering the equation

$$v_{tt} - a^2 \Delta v = 0 \tag{2.1}$$

with the boundary condition

$$\frac{\partial v}{\partial \nu} = -v_t \quad \text{on } \Gamma, \tag{2.2}$$

we go back to the classical non-homogeneous Neumann's problem as follows: Find u such that

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \text{ on } \Gamma \tag{2.3}$$

where $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\Gamma)$ satisfy the compatibility condition

$$\int_\Omega f dx + \langle g, 1 \rangle_\Gamma = 0.$$

Note that we never have the uniqueness of the solution u because problem (2.3) only involves the derivative of u. It leads us to seek a solution u in the quotient space $H^1(\Omega)/\mathbb{R}$ with the quotient norm, denoted by $[u]_{H^1(\Omega)}$ as follows

$$[u]_{H^1(\Omega)} = \|u\|_{H^1(\Omega)/\mathbb{R}} = \inf_{Q \in \mathbb{R}} \|u + Q\|_{H^1(\Omega)}.$$

The space $H^1(\Omega)/\mathbb{R}$ is a Hilbert space and the semi-norm $|\cdot|_{H^1(\Omega)}$ defines on $H^1(\Omega)/\mathbb{R}$ a norm which is equivalent to the quotient norm. This property is called the Poincaré-type inequality (see [11, 2, 3, 14]). Returning to (2.3), the problem is equivalent to a variational formulation and by using Lax-Milgram's theorem, we can show there exists a unique solution $u \in H^1(\Omega)/\mathbb{R}$. In addition, u belongs to

$H^2(\Omega)/\mathbb{R}$ and then, in particular, $g \in H^{1/2}(\Gamma)$. Moreover, we have the following estimate

$$|u|_{H^1(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\Gamma)}).$$

The proof can be found in [6].

We now consider the phase space $\mathcal{H} = H^1(\Omega)/\mathbb{R} \times L^2(\Omega)$ with the inner product

$$\langle U, V\rangle_{\mathcal{H}} = a^2 \int_\Omega \nabla v \cdot \nabla\overline{\phi}\, d\mathbf{x} + \int_\Omega w\overline{\psi}\, d\mathbf{x},$$

where $U = (v, w)^T$ and $V = (\phi, \psi)^T \in \mathcal{H}$. The norm in this space is

$$\|U\|_{\mathcal{H}}^2 = a^2 \int_\Omega |\nabla v|^2\, d\mathbf{x} + \int_\Omega |w|^2\, d\mathbf{x}.$$

We define the operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} w \\ a^2\Delta v \end{pmatrix}. \tag{2.4}$$

We want to find $U = (v, w)^T \in \Omega$ such that for $t \ge 0$,

$$\begin{gathered} \frac{d}{dt}U(t) = \mathcal{A}U(t), \\ U(0) = (v_0, v_1)^T, \end{gathered} \tag{2.5}$$

where

$$\begin{aligned} \mathcal{D}(\mathcal{A}) &= \big\{U = (v, w)^T \in \mathcal{H}; w \in H^1(\Omega),\ v \in H^2(\Omega),\ \frac{\partial v}{\partial \nu} = -v_t = w \text{ on } \Gamma\big\} \\ &= \mathcal{V} \times H^1(\Omega). \end{aligned}$$

Here $\mathcal{V} \times H^1(\Omega)$ is defined as follows

$$\mathcal{V} \times H^1(\Omega) := \big\{\varphi \in H^2(\Omega)/\mathbb{R}; \frac{\partial \varphi}{\partial \nu} = -\varphi_t = \theta \text{ on } \Gamma\big\} \times \{\theta \in H^1(\Omega)\}.$$

Before showing the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on the space $\mathcal{H}$, we will consider the two following lemmas.

Lemma 2.2. *The operator $\mathcal{A}$ is dissipative.*

Proof. We observe that if $U = (v, w)^T \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= a^2 \int_\Omega \nabla w \cdot \overline{\nabla v}\, dx + a^2 \int_\Omega \Delta v \cdot \overline{w}\, dx \\ &= a^2 \int_\Omega \nabla w \cdot \overline{\nabla v}\, dx - a^2 \int_\Omega \nabla v \cdot \overline{\nabla \omega}\, dx + a^2 \int_\Gamma \frac{\partial v}{\partial \nu}\overline{\omega}\, d\Gamma \\ &= 2\, a^2\, i \int_\Omega \mathrm{Im}(\nabla\omega \cdot \nabla\overline{v}) - a^2 \int_\Gamma |\omega|^2\, d\Gamma. \end{aligned}$$

Taking the real part of the above relation, we obtain

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -a^2 \int_\Gamma |\omega|^2\, d\Gamma \le 0, \quad U \in \mathcal{D}(\mathcal{A}). \tag{2.6}$$

$\square$

Let H be a Hilbert space and A be an operator in H. We define the resolvent set of A as follows

$$\varrho(A) = \{\lambda \in \mathbb{C} : w \mapsto (\lambda I - A)^{-1} w \in \mathcal{L}(X)\}$$

where $\mathcal{L}(X)$ is the linear continuous mapping from X into X.

Lemma 2.3. *We have $0 \in \varrho(\mathcal{A})$.*

Proof. We prove that with a given $F = (f, g)^T \in \mathcal{H}$ satisfying the compatibility condition

$$\int_\Omega g\,dx + a^2 \langle f, 1\rangle_\Gamma = 0, \tag{2.7}$$

there exists a unique $U = (v, w)^T \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = F$. Then

$$w = f \quad \text{in } H^1(\Omega), \tag{2.8}$$

$$a^2 \Delta v = g \quad \text{in } L^2(\Omega). \tag{2.9}$$

By associating the boundary condition and by the definition of the domain $\mathcal{D}(\mathcal{A})$, we obtain

$$\begin{gathered} -\Delta v = h := \frac{g}{a^2} \in L^2(\Omega) \\ \frac{\partial v}{\partial \nu} = f|_\Gamma \in H^{-1/2}(\Gamma), \end{gathered} \tag{2.10}$$

with the compatibility condition (2.7). Thanks to Remark 2.1 and some calculation, we obtain

$$\|\mathcal{A}^{-1} F\|_{\mathcal{H}} \ \le C \,\|F\|_{\mathcal{H}}.$$

The result follows. $\square$

Proposition 2.4. *The operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}(t)$ of contractions on the space $\mathcal{H}$, $(t \in [0, \infty))$.*

Proof. Note first that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. We have showed that $\mathcal{A}$ is a dissipative operator (see Lemma 2.2) and 0 belongs to $\varrho(\mathcal{A})$. Our conclusion will follow by using Lemma 2.2, Lemma 2.3 and the well known Lumer-Phillips Theorem [15]. $\square$

The first result of this paper follows from Proposition 2.4, [10, Theorem 4.3.2] and [15] which can be stated in the following theorem.

Theorem 2.5. *If $U_0 \in \mathcal{D}(\mathcal{A})$ then $U(t) = \mathcal{S}(t)U_0$ is the unique solution of (2.5) satisfying*

$$\mathcal{S}(t)U_0 \in C([0, \infty);\ \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty);\ \mathcal{H}). \tag{2.11}$$

Remark 2.6. Note that if $U_0 \in \mathcal{D}(\mathcal{A})$, then $U(t) = \mathcal{S}(t)U_0$ is the unique solution of (1.5)-(1.7) satisfying

$$\mathcal{S}(t)U_0 \in C([0, \infty); \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H}).$$

Moreover, $\mathcal{D}(\mathcal{A}) \subseteq H^2(\Omega) \times H^1(\Omega)$. Then

$$(v, w) \in \mathcal{D}(\mathcal{A}) \Rightarrow v \in H^2(\Omega) \wedge v_t \in H^1(\Omega).$$

Hence

$$v \in C([0, \infty); H^2(\Omega)), \quad v \in C^1([0, \infty); H^1(\Omega)).$$

In addition, from (2.11) we have $v_t \in H^1(\Omega)\ \wedge\ v_{tt} \in L^2(\Omega)$. Then

$$v \in C^1([0, \infty); H^1(\Omega)), \quad v \in C^2([0, \infty); L^2(\Omega)).$$

Thus

$$v \in C([0,\infty); H^2(\Omega)) \cap C^1([0,\infty); H^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)). \tag{2.12}$$

Then from Remark 1.3, we have the following regularity result.

Corollary 2.7. *If $(v_0, v_1) \in \mathcal{V} \times H^1(\Omega)$, then there exists a unique solution v of problem* (1.5)-(1.7) *(or problem* (1.9)-(1.11)*, respectively) satisfying*

$$v \in C([0,\infty); H^2(\Omega)) \cap C^1([0,\infty); H^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)).$$

From (1.4), we get an ordinary differential equation. In this case, taking $\varphi(t) = \exp\{t/\alpha\}$ as the integrating factor, we conclude that

$$u(t) = u_0 \exp\{-t/\alpha\} + \frac{1}{\alpha}\int_0^t \exp\{-(t-s)/\alpha\} v(s)\, ds. \tag{2.13}$$

Then we deduce from (2.13) that $u(0) = u_0$ and $u_t(0) = u_1$. Also observe that from (1.6), we have

$$\begin{aligned} \frac{\partial u}{\partial \nu} &= \frac{\partial u_0}{\partial \nu} \exp\{-t/\alpha\} - \frac{1}{\alpha}\int_0^t \exp\{-(t-s)/\alpha\} v_s(s)\, ds, \\ \frac{\partial u_t}{\partial \nu} &= -\frac{1}{\alpha}\frac{\partial u_0}{\partial \nu} \exp\{-t/\alpha\} + \frac{1}{\alpha^2}\int_0^t \exp\{-(t-s)/\alpha\} v_s(s)\, ds - \frac{v_t}{\alpha}. \end{aligned}$$

Adding the results above, we obtain

$$\frac{\partial u}{\partial \nu} + \alpha \frac{\partial u_t}{\partial \nu} = -v_t = -(u + \alpha u_t)_t$$

Now we check the regularity of u.

$$u_t(t) = -\frac{u_0}{\alpha}\exp\{-t/\alpha\} + \frac{1}{\alpha}v(t) - \frac{1}{\alpha^2}\exp\{-t/\alpha\}\int_0^t \exp\{s/\alpha\} v(s)\, ds \tag{2.14}$$

$$\begin{aligned} u_{tt}(t) = \frac{u_0}{\alpha^2}\exp\{-t/\alpha\} + \frac{1}{\alpha^3}\exp\{-t/\alpha\}\int_0^t \exp\{s/\alpha\} v(s)\, ds \\ + \frac{1}{\alpha^2}v(t) + \frac{1}{\alpha}v_t(t) \end{aligned} \tag{2.15}$$

$$\begin{aligned} u_{ttt}(t) = \frac{u_0}{\alpha^3}\exp\{-t/\alpha\} - \frac{1}{\alpha^4}\exp\{-t/\alpha\}\int_0^t \exp\{s/\alpha\} v(s)\, ds \\ + \frac{1}{\alpha^3}v(t) + \frac{1}{\alpha^2}v_t(t) + \frac{1}{\alpha}v_{tt}. \end{aligned} \tag{2.16}$$

As $v \in C([0,\infty); H^2(\Omega))$, from (2.13), we conclude that $u \in C^1([0,\infty); H^2(\Omega))$. Similarly, we conclude that

$$u \in C^2([0,\infty); H^1(\Omega)) \cap C^3([0,\infty); L^2(\Omega)).$$

Theorem 2.8. *If $(u_0, u_1, u_2) \in H^2(\Omega) \times H^1(\Omega) \times L^2(\Omega)$, then there exists a unique solution u of* (1.1)-(1.3) *satisfying*

$$u \in C^1([0,\infty); H^2(\Omega)) \cap C^2([0,\infty); H^1(\Omega)) \cap C^3([0,\infty); L^2(\Omega)).$$

3. Asymptotic behaviour

In this section we state and prove the exponential stability result for system (1.9)-(1.11). The main tool is to use the multipliers technique. Now we will state the main result of this section.

Theorem 3.1. *Let v be the solution of system* (1.9)-(1.11) *given by Corollary 2.7. Then there exist positive constants C and $\mathcal{K}$ such that*

$$\mathcal{E}(t) \leq C\mathcal{E}(0)e^{-\mathcal{K}t}, \quad \forall t \geq 0,$$

where $\mathcal{E}(t)$ is the total energy of system (1.9)-(1.11).

Before proving Theorem 3.1, we give the following lemma.

Lemma 3.2. *Let v be the solution of* (1.9)-(1.11) *given by Corollary 2.7. Then the functional*

$$\mathcal{F}_1(t) = \int_\Omega v v_t \, dx + \frac{1}{2}a^2 \int_\Gamma v^2 \, d\Gamma \tag{3.1}$$

satisfies

$$\frac{d}{dt}\Big[\mathcal{F}_1(t) + \xi \int_\Omega v^2 dx\Big] = \int_\Omega |v_t|^2 \, dx - a^2 \int_\Omega |\nabla v|^2 \, dx. \tag{3.2}$$

Proof. Differentiating (3.1) in the t-variable, using Green's Theorem and (1.5)-(1.6) we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t) &= \int_\Omega |v_t|^2 \, dx + \int_\Omega v v_{tt} \, dx + \frac{a^2}{2}\frac{d}{dt}\int_\Gamma v^2 \, d\Gamma \\
&= \int_\Omega |v_t|^2 \, dx + a^2 \int_\Omega v \Delta v \, dx - \xi \frac{d}{dt}\int_\Omega v^2 \, dx + \frac{a^2}{2}\frac{d}{dt}\int_\Gamma v^2 \, d\Gamma \\
&= \int_\Omega |v_t|^2 \, dx - a^2 \int_\Omega |\nabla v|^2 \, dx + a^2 \int_\Gamma \frac{\partial v}{\partial \nu} v \, d\Gamma \\
&\quad - \xi \frac{d}{dt}\int_\Omega v^2 \, dx + \frac{a^2}{2}\frac{d}{dt}\int_\Gamma v^2 \, d\Gamma \\
&= \int_\Omega |v_t|^2 \, dx - a^2 \int_\Omega |\nabla v|^2 \, dx - a^2 \int_\Gamma v_t v \, d\Gamma \\
&\quad - \xi \frac{d}{dt}\int_\Omega v^2 \, dx + \frac{a^2}{2}\frac{d}{dt}\int_\Gamma v^2 \, d\Gamma \\
&= \int_\Omega |v_t|^2 \, dx - a^2 \int_\Omega |\nabla v|^2 \, dx - \frac{a^2}{2}\frac{d}{dt}\int_\Gamma |v|^2 \, d\Gamma \\
&\quad - \xi \frac{d}{dt}\int_\Omega v^2 \, dx + \frac{a^2}{2}\frac{d}{dt}\int_\Gamma v^2 \, d\Gamma.
\end{aligned} \tag{3.3}$$

Hence

$$\frac{d}{dt}\Big[\mathcal{F}_1(t) + \xi \int_\Omega v^2 \, dx\Big] = \int_\Omega |v_t|^2 \, dx - a^2 \int_\Omega |\nabla v|^2 \, dx. \tag{3.4}$$

□

Proof of Theorem 3.1. We define

$$\mathcal{F}(t) = \mathcal{F}_1(t) + \xi \int_\Omega v^2 \, dx, \tag{3.5}$$

$$\mathcal{G}(t) = \mathcal{E}(t) + \varepsilon\mathcal{F}(t). \tag{3.6}$$

Similarly as in Lemma 1.1, we deduce

$$
\begin{aligned}
\frac{d}{dt}\mathcal{G}(t) &= \frac{d}{dt}\mathcal{E}(t) + \frac{d}{dt}\mathcal{F}(t) \\
&= -2\xi \int_\Omega |v_t|^2\,dx - a^2 \int_\Gamma |v_t|^2\,d\Gamma + \varepsilon \int_\Omega |v_t|^2\,dx - a^2\varepsilon \int_\Omega |\nabla v|^2\,dx \\
&\le -(2\xi - \varepsilon) \int_\Omega |v_t|^2\,dx - a^2 \int_\Gamma |v_t|^2\,d\Gamma - a^2\varepsilon \int_\Omega |\nabla v|^2\,dx \\
&\le -(2\xi - \varepsilon) \int_\Omega |v_t|^2\,dx - a^2\varepsilon \int_\Omega |\nabla v|^2\,dx,
\end{aligned} \tag{3.7}
$$

where $2\xi - \varepsilon > 0$ if and only if $2\xi > \varepsilon > 0$. We consider $\kappa_1 = \min\{2\xi - \varepsilon, \varepsilon\} > 0$.
Then

$$
\frac{d}{dt}\mathcal{G}(t) \le -\kappa_1 \mathcal{E}(t). \tag{3.8}
$$

On the other hand,

$$
\begin{aligned}
&|\mathcal{G}(t) - \mathcal{E}(t)| \\
&= \varepsilon\,|\mathcal{F}(t)| = \varepsilon\,|\mathcal{F}_1(t)| + \varepsilon\xi \int_\Omega |v|^2\,dx \\
&\le \varepsilon \int_\Omega |v|\,|v_t|\,dx + \frac{a^2\varepsilon}{2} \int_\Gamma |v|^2\,d\Gamma + \varepsilon\xi \int_\Omega |v|^2\,dx \\
&\le \varepsilon\Big(\frac{1}{2} \int_\Omega |v|^2\,dx + \frac{1}{2} \int_\Omega |v_t|^2\,dx\Big) + \frac{a^2\varepsilon}{2} \int_\Gamma |v|^2\,d\Gamma + \varepsilon\xi \int_\Omega |v|^2\,dx \\
&= \frac{\varepsilon}{2}\|v\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\,\|v_t\|^2_{L^2(\Omega)} + \frac{a^2\varepsilon}{2}\,\|v\|^2_{L^2(\Gamma)} + \varepsilon\xi\|v\|^2_{L^2(\Omega)} \\
&\le \frac{\varepsilon}{2}c_P\|\nabla v\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\,\|v_t\|^2_{L^2(\Omega)} + \frac{a^2\varepsilon}{2}\,\|v\|^2_{L^2(\Gamma)} + \varepsilon\xi\,\|v\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.9}
$$

We have

$$
\|v\|_{H^{1/2}(\Gamma)} \le c\,\|v\|_{H^1(\Omega)}.
$$

Moreover, $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$. Then

$$
\|v\|_{L^2(\Gamma)} \le c_0\,\|v\|_{H^{1/2}(\Gamma)} \Longrightarrow \|v\|_{L^2(\Gamma)} \le c_1\,\|v\|_{H^1(\Omega)}.
$$

Hence

$$
\|v\|_{L^2(\Gamma)} \le c_2\left(\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right) \le c_3\,\|\nabla v\|_{L^2(\Omega)}.
$$

Replacing into (3.9) we obtain

$$
\begin{aligned}
&|\mathcal{G}(t) - \mathcal{E}(t)| \\
&\le \frac{\varepsilon}{2}c_P\|\nabla v\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\,\|v_t\|^2_{L^2(\Omega)} + \frac{a^2\varepsilon c_3^2}{2}\,\|\nabla v\|^2_{L^2(\Omega)} + \varepsilon\xi c_P\,\|v\|^2_{L^2(\Omega)} \\
&\le \varepsilon c\mathcal{E}(t).
\end{aligned} \tag{3.10}
$$

Then

$$
-\varepsilon c\mathcal{E}(t) \le \mathcal{G}(t) - \mathcal{E}(t) \le \varepsilon c\mathcal{E}(t) \Longleftrightarrow (1 - \varepsilon c)\mathcal{E}(t) \le \mathcal{G}(t) - \mathcal{E}(t) \le (1 + \varepsilon c)\mathcal{E}(t).
$$

Observe that we can choose ε such that $\varepsilon < 1/c$. Therefore,

$$
\kappa_2\mathcal{E}(t) \le \mathcal{G}(t) \le \kappa_3\mathcal{E}(t), \quad \kappa_2, \kappa_3 > 0. \tag{3.11}
$$

From (3.8) and (3.10) we conclude that

$$\mathcal{E}(t) \le C\mathcal{E}(0)e^{-\mathcal{K}t}, \quad \forall t \ge 0.$$

Then the statement of the theorem follows. □

Acknowledgements. Octavio Vera was supported by Fondecyt projects 1121120. He also wants to thank for the hospitality and financial support from the National Laboratory for Scientific Computation (LNCC), Petrópolis, Brazil.

Huy Hoang Nguyen wishes to thank the warm hospitality and the financial support of the Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour and CNRS, France. He is also supported in part by ALV'2012, Universidade Federal do Rio de Janeiro.

Verónica Poblete was supported by ENL016/15 and PAIFAC.

The authors want to thank Professor Alois Kufner for the gift [11] and for the nice discussions.

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Pedro Gamboa

Instituto de Matemática, Universidade Federal de Rio de Janeiro, Av. Athos da Silveira Ramos, P.O. Box 68530, CEP:21945-970, RJ, Brazil

E-mail address: pgamboa@im.ufrj.br

Huy Hoang Nguyen
Instituto de Matemática and Campus de Xerém, Universidade Federal de Rio de Janeiro, Av. Athos da Silveira Ramos, P.O. Box 68530, CEP:21945-970, RJ. Brazil.
Laboratoire de Mathématiques et de leurs Applications (LMAP/UMR CNRS 5142), Bat. IPRA, Avenue de l'Université, F-64013 Pau, France
E-mail address: nguyen@im.ufrj.br

Octavio Vera
Departamento de Matemática, Universidad del Bío Bío, Collao 1202, Casilla 5-C, Concepción, Chile
E-mail address: overa@ubiobio.cl

Verónica Poblete
Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Las Palmeras 3425, Ñuñoa, Santiago, Chile
E-mail address: vpoblete@uchile.cl