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INVERSE BOUNDARY PROBLEMS FOR INTERMEDIATE SPRINGS ON A ROD WITH GEOMETRICAL SYMMETRY

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Abstract. In this article we try to solve the inverse problem of determining the coefficients of stiffness of the intermediates on the springs on the rod from the two known natural frequencies. We find sufficient conditions for the existence of a unique solution to the inverse problem of determining the stiffness on the intermediates of the springs of non-terminal points of the rod from the two known natural frequencies. It was shown that for the determination of the coefficients of stiffness of the springs played an essential role in the geometric symmetry of the arrangement of the spring relative to the rod center.

1. Introduction

In various problems of mechanics and physics, many structural elements can be represented by a set of different rods. Therefore, in some areas of modern technology the problem of oscillations of rod systems, including lumped masses or intermediate springs have to be solved. As an example, one can specify the machine-building equipment, aerospace engineering [1, 3, 4, 7, 9, 10, 13, 14]. (See. Figures 1, 2).

Since the 1930's, one of the most important practical problems was finding the upper and lower bounds of eigenvalues of systems with a finite number of degrees of freedom [3]. The motion of system with a finite number of degrees of freedom is based on the methods of Rayleigh-Ritz and Weinstein, which consist of the successive approximation of an elastic medium by finite systems.

In recent years, methods for the analysis of direct and inverse problems for differential operators of lumped mass and elastic connections were actively developing [1, 5, 6, 8, 10, 15, 16]. These methods are very important, as they provide an opportunity to develop the technology and to ensure the safety of people. For the solution of inverse problems, it is very important to have a unique solution. These issues are also addressed in terms of boundary controls [12]. Relation [2] studied the spectral and oscillation properties of boundary value problems which describe the movement of partially enshrined rod the concentrated on the free end of the additional weight.

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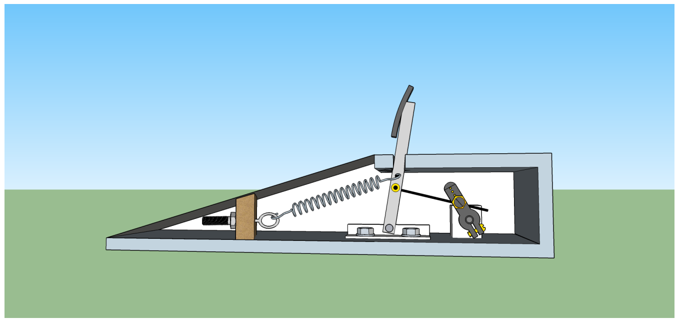

FIGURE 1. The rod with intermediate springs

FIGURE 2. Beam with three lumped mass

2. Problem of transverse vibrations of a rod with intermediate springs

This paper considers a rod with two intermediate springs with stiffness coefficients c_1 and c_2 ($\mathrm{N\,m^{-1}}$). Units of measurement and reduction of all physical parameters considered in the paper as standard. In contrast to [1, 8, 10], in this paper, we find sufficient conditions for the existence of a unique solution to the inverse problem of determining the stiffness of the springs in the intermediate non-terminal points of the rod from the known two natural frequencies. The idea of the proof of sufficient conditions an ideological close to [8]. In contrast to [8] in this paper it is proved (Theorem 3.3) that for an unambiguous determination of the stiffness coefficients of the springs plays an important role in the geometric symmetry of the arrangement of the spring relative to the rod center.

Suppose the first spring is located on the left end of the rod at a distance a, second spring, respectively, at a distance b (fig.3). As a result, the rod is divided into 3 sections: $-l/2 < x < a - l/2$, $a - l/2 < x < b - l/2$, $b - l/2 < x < l/2$.

The equation of the free transverse vibrations of a rod of length l of the following

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EJ \frac{\partial^4 w(x,t)}{\partial x^4} = 0,$$

where $w(x,t)$ is the transverse movement, (m) when $-l/2 < x < l/2$, $t > 0$; ρ is the density of material, ($\mathrm{kg\,m^{-3}}$); A is the cross-sectional area, ($\mathrm{m^2}$); E is Young's

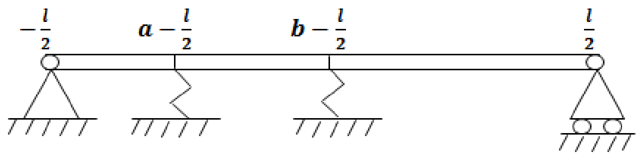

Figure 3. The rod with intermediate springs

modulus,(N m^{-2}); J is the moment of inertia of the cross-sectional area of relative to the neutral axis, (m^2).

This method is applicable to rods with different types of fixings. To be specific, we consider only the hinged rod.

The problem of transverse vibrations of a rod of length l with the replacement of $w(x,t) = y(x)\sin(\omega t)$ reduced to the following spectral problem:

$$EJy^{IV}(x) = \omega^2 \rho A y(x), \quad -l/2 < x < l/2,\ x \neq a - l/2,\ x \neq b - l/2, \tag{2.1}$$

$$[EJy'''(x)]_{x=a-l/2} = c_1 y(a - l/2), \tag{2.2}$$

$$[y(x)]_{x=a-l/2} = 0, \quad [Ey'(x)]_{x=a-l/2} = 0, \quad [EJy''(x)]_{x=a-l/2} = 0, \tag{2.3}$$

$$[EJy'''(x)]_{x=b-l/2} = c_2 y(b - l/2), \tag{2.4}$$

$$[y(x)]_{x=b-l/2} = 0, \quad [Ey'(x)]_{x=b-l/2} = 0, \quad [EJy''(x)]_{x=b-l/2} = 0, \tag{2.5}$$

$$y(x)|_{x=-l/2} = 0, \quad EJy''(x)|_{x=-l/2} = 0, \tag{2.6}$$

$$y(x)|_{x=l/2} = 0, \quad EJy''(x)|_{x=l/2} = 0, \tag{2.7}$$

where $[f(x)]_{x=c} = \lim_{\varepsilon\to+0}[f(c-\varepsilon) - f(c+\varepsilon)]$ it means the jump of the function at the point $x = c$.

Let $p^4 = \omega^2 \rho A/EJ$, where ω is the frequency parameter, Hz. Then the equation (2.1) has the form

$$y^{IV}(x) = p^4 y(x), \quad -l/2 < x < l/2, \quad x \neq a - l/2, \quad x \neq b - l/2. \tag{2.8}$$

We will need the following lemma.

Lemma 2.1. *The frequencies of problem* (2.1)–(2.7) *are defined by the equation*

$$\Delta(a,b,l,p,c_1,c_2) = \alpha(a,b,l,p)\varphi\psi + \beta(a,l,p)\varphi + \gamma(b,l,p)\psi + \Delta_0(l,p) = 0, \tag{2.9}$$

where

$$\begin{aligned}
\alpha(a,b,l,p) = {} & \cos(pl)\{ch(p(2a-l)) + ch(p(2b-l)) - 2ch(pl)\} \\
& + ch(pl)\{\cos(p(2a-l)) + \cos(p(2b-l))\} \\
& + \sin(pl)\{-sh(p(2a-l)) + sh(p(2b-l)) + sh(p(2a-2b+l))\} \\
& + sh(pl)\{\sin(p(2a-l)) - \sin(p(2a-2b+l)) - \sin(p(a+b-l))\} \\
& + 2\cos(p(a+b-l))\{ch(p(a+b-l)) - ch(p(a-b+l))\} \\
& + 2\cos(p(a-b+l))\{ch(p(a-b+l)) - ch(p(a+b-l))\} \\
& - \cos(p(2b-l))ch(p(2a-l)) - \cos(p(2a-l))ch(p(2b-l)),
\end{aligned}$$

$$\beta(a,l,p) = -4p^3\{sh(pl)(\cos(p(2a-l)) - \cos(pl)) + \sin(pl)(ch(p(2a-l)) - ch(pl))\},$$

$$\gamma(b,l,p) = -4p^3\{sh(pl)(\cos(p(2b-l)) - \cos(pl)) + \sin(pl)(ch(p(2b-l)) - ch(pl))\},$$

$$\Delta_0(l,p) = -16p^6\sin(pl)sh(pl), \quad \varphi = c_1/EJ, \quad \psi = c_2/EJ.$$

Proof. To find the characteristic determinant of (2.1)–(2.7), we must take into account that each of the intervals $-l/2 < x < a-l/2$, $a-l/2 < x < b-l/2$, $b-l/2 < x < l/2$ it has a differential equation of fourth order with constant coefficients. The clearly written out the general solution for each equation which has four free coefficients in considered three intervals from twelve independent variables. From the boundary conditions (2.6) and (2.7) are directly determined four coefficients. Conditions "conjugation" at points $a-l/2$, $b-l/2$ give a system of eight linear homogeneous equations with eight unknowns. The determinant of this system is the characteristic determinant of problem (2.1)–(2.7).

Detailing our reasoning. We are looking for the solution of equation (2.8) in the form

$$\begin{aligned} y(x) &= B_1\cos\big(p(x+\frac{l}{2})\big) + B_2\sin\big(p(x+\frac{l}{2})\big) \\ &\quad + B_3ch\big(p(x+\frac{l}{2})\big) + B_4sh\big(p(x+\frac{l}{2})\big) \end{aligned}$$

by $-l/2 < x < a-l/2$. From conditions (2.6) it follows that $B_1 = B_3 = 0$. We are looking for the solution of (2.8) in the form

$$\begin{aligned} y(x) &= D_1\cos\big(p(x-\frac{l}{2})\big) + D_2\sin\big(p(x-\frac{l}{2})\big) \\ &\quad + D_3ch\big(p(x-\frac{l}{2})\big) + D_4sh\big(p(x-\frac{l}{2})\big) \end{aligned}$$

by $b-l/2 < x < l/2$.

From conditions (2.7) it follows that $D_1 = D_3 = 0$. We are looking for the solution of (2.8) in the form

$$\begin{aligned} y(x) &= K_1\cos\big(p(x-a+\frac{l}{2})\big) + K_2\sin\big(p(x-a+\frac{l}{2})\big) \\ &\quad + K_3ch\big(p(x-a+\frac{l}{2})\big) + K_4sh\big(p(x-a+\frac{l}{2})\big) \end{aligned}$$

by $a-l/2 < x < b-l/2$.

From conditions (2.2), (2.3) we have

$$y(a-\frac{l}{2}-0) - y(a-\frac{l}{2}+0) = B_2\sin(pa) + B_4sh(pa) - K_1 - K_3 = 0, \tag{2.10}$$

$$y'(a-\frac{l}{2}-0) - y'(a-\frac{l}{2}+0) = p(B_2\cos(pa) + B_4ch(pa) - K_2 - K_4) = 0, \tag{2.11}$$

$$\begin{aligned} &y''(a-\frac{l}{2}-0) - y''(a-\frac{l}{2}+0) \\ &= p^2(-B_2\sin(pa) + B_4sh(pa) + K_1 - K_3) = 0, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &y'''(a-\frac{l}{2}-0) - y'''(a-\frac{l}{2}+0) - \frac{c_1}{EJ}y(a-\frac{l}{2}-0) \\ &= -B_2(p^3\cos(pa) + \frac{c_1}{EJ}\sin(pa)) + B_4(p^3ch(pa) - \frac{c_1}{EJ}sh(pa)) \\ &\quad + p^3(K_2 - K_4) = 0. \end{aligned} \tag{2.13}$$

From conditions (2.4), (2.5) we have

$$
\begin{aligned}
&y(b-\frac{l}{2}-0)-y(b-\frac{l}{2}+0)\\
&=K_1\cos(p(b-a))+K_2\sin(p(b-a))+K_3ch(p(b-a))\\
&\quad+K_2sh(p(b-a))-D_2\sin(p(b-l))-D_4sh(p(b-l))=0,
\end{aligned}\tag{2.14}
$$

$$
\begin{aligned}
&y'(b-\frac{l}{2}-0)-y'(b-\frac{l}{2}+0)\\
&=p\big(K_1\sin(p(b-a))+K_2\cos(p(b-a))\big)\\
&\quad+p\big(K_3sh(p(b-a))+K_2ch(p(b-a))-D_2\cos(p(b-l))\\
&\quad-D_4ch(p(b-l))\big)=0,
\end{aligned}\tag{2.15}
$$

$$
\begin{aligned}
&y''(b-\frac{l}{2}-0)-y''(b-\frac{l}{2}+0)\\
&=p^2(-K_1\cos(p(b-a))-K_2\sin(p(b-a)))\\
&\quad+p^2\big(K_3ch(p(b-a))+K_4sh(p(b-a))+D_2\sin(p(b-l))\\
&\quad-D_4sh(p(b-l))\big)=0,
\end{aligned}\tag{2.16}
$$

$$
\begin{aligned}
&y'''(b-\frac{l}{2}-0)-y'''(b-\frac{l}{2}+0)-\frac{c_2}{EJ}y(b-\frac{l}{2}-0)\\
&=K_1(p^3\sin(p(b-a))-\frac{c_2}{EJ}\cos(p(b-a)))\\
&\quad-K_2(p^3\cos(p(b-a))+\frac{c_2}{EJ}\sin(p(b-a)))\\
&\quad+K_3(p^3sh(p(b-a))-\frac{c_2}{EJ}ch(p(b-a)))\\
&\quad+K_4(p^3ch(p(b-a))-\frac{c_2}{EJ}sh(p(b-a)))\\
&\quad+p^3(D_2\cos(p(b-l))-D_4ch(p(b-l)))=0.
\end{aligned}\tag{2.17}
$$

Thus, we have obtained a system of eight linear homogeneous equations (2.10)-(2.17) with 8 unknowns B_2, B_4, K_i, D_2, D_4, $i=\overline{1,4}$. As is well known from the theory of linear operators that the system (2.10)-(2.17) has nontrivial solutions if and only if $\Delta(a,b,l,p,c_1,c_2)$, the characteristic determinant, is equal to zero [11]. Where

$$
\begin{gathered}
\Delta(a,b,l,p,c_1,c_2)=\begin{vmatrix} B & K_{\rm up} & O\\ O & K_{\rm down} & D\end{vmatrix},\\
B=\begin{pmatrix}\sin(pa) & sh(pa)\\ \cos(pa) & ch(pa)\\ -\sin(pa) & sh(pa)\\ -p^3\cos(pa)-\frac{c_1}{EJ}\sin(pa) & p^3ch(pa)-\frac{c_1}{EJ}sh(pa)\end{pmatrix},\\
O=\begin{pmatrix}0&0\\0&0\\0&0\\0&0\end{pmatrix},\quad K_{\rm up}=\begin{pmatrix}-1&0&-1&0\\0&-1&0&-1\\1&0&-1&0\\0&p^3&0&-p^3\end{pmatrix},
\end{gathered}
$$

$$D = \begin{pmatrix} -\sin(p(b-l)) & -sh(p(b-l)) \\ -\cos(p(b-l)) & -ch(p(b-l)) \\ \sin(p(b-l)) & -sh(p(b-l)) \\ p^3\cos(p(b-l)) & -p^3 ch(p(b-l)) \end{pmatrix},$$

$$K_{\rm down} = \begin{pmatrix} \cos(p(b-a)) & \sin(p(b-a)) & ch(p(b-a)) & sh(p(b-a)) \\ -\sin(p(b-a)) & \cos(p(b-a)) & sh(p(b-a)) & ch(p(b-a)) \\ -\cos(p(b-a)) & -\sin(p(b-a)) & ch(p(b-a)) & sh(p(b-a)) \\ \frac{1}{p}{k'}_{d1}(p(b-a)) & k_{d1}(p(b-a)) & \frac{1}{p}k'_{d2}(p(b-a)) & k_{d2}(p(b-a)) \end{pmatrix},$$

$$k_{d1}(p(x-a)) = -p^3\cos(p(x-a)) - \frac{c_2}{EJ}\sin(p(x-a)),$$
$$k_{d2}(p(x-a)) = p^3 ch(p(x-a)) - \frac{c_2}{EJ} sh(p(x-a)).$$

We calculate the characteristic determinant $\Delta(a,b,l,p,c_1,c_2)$ and obtain the formula (2.9). The proof is complete. $\square$

In this setting $\Delta_0(l,p)$ is the characteristic determinant in the absence of springs, $\Delta_{c_2}(b,l,p) = \gamma(b,l,p)\psi + \Delta_0(l,p)$ is the characteristic determinant in the case of only the second spring, $\Delta_{c_1}(a,l,p) = \beta(a,l,p)\varphi + \Delta_0(l,p)$ is the characteristic determinant in the case of only the first spring. Thus, to find frequency of oscillation of the relation (2.9) is first determined value and then to find $\omega = \sqrt{EJ/(\rho A p^2)}$.

3. Determination of the magnitude of the stiffness coefficient of the springs on the prescribed frequency of oscillation

Now consider the inverse problem of determining the stiffness coefficients of the springs. Knowing all the physical parameters, the location of the intermediate springs, as well as the first two natural frequencies of transverse vibrations of a rod, it is then required to determine the value of intermediate stiffness coefficients of the springs.

Let p_1, p_2 be zeros of $\Delta(p) := \Delta(a,b,l,p,c_1,c_2)$. Then the following two equations hold

$$\begin{gathered} \alpha_1\psi\varphi + \beta_1\varphi + \gamma_1\psi + \Delta_{01} = 0, \\ \alpha_2\psi\varphi + \beta_2\varphi + \gamma_2\psi + \Delta_{02} = 0. \end{gathered} \tag{3.1}$$

There $\alpha_i = \alpha(a,b,l,p_i)$, $i = 1,2$. Similar designations is valid for β, γ, Δ_0. The following Plücker relations

$$N_1N_4 - N_2N_5 + N_3N_6 = 0 \tag{3.2}$$

are true for the matrix

$$A = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \Delta_{01} \\ \alpha_2 & \beta_2 & \gamma_2 & \Delta_{02} \end{pmatrix},$$

where

$$N_1 = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}, \quad N_2 = \begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \gamma_2 \end{vmatrix}, \quad N_3 = \begin{vmatrix} \alpha_1 & \Delta_{01} \\ \alpha_2 & \Delta_{02} \end{vmatrix},$$
$$N_4 = \begin{vmatrix} \gamma_1 & \Delta_{01} \\ \gamma_2 & \Delta_{02} \end{vmatrix}, \quad N_5 = \begin{vmatrix} \beta_1 & \Delta_{01} \\ \beta_2 & \Delta_{02} \end{vmatrix}, \quad N_6 = \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix}.$$

The system of nonlinear equations (3.1) is reduced to a system of quadratic equations

$$\begin{aligned} N_1\varphi^2 + (N_3 - N_6)\varphi + N_4 = 0, \\ N_2\psi^2 + (N_3 + N_6)\psi + N_5 = 0. \end{aligned} \tag{3.3}$$

Note that between φ and ψ the linear relationship also holds

$$N_1\varphi + N_2\psi + N_3 = 0. \tag{3.4}$$

The equality $D_\varphi - D_\psi = 0$ is right for discriminants D_φ, D_ψ of system of quadratic equations because $D_\varphi = (N_3 - N_6)^2 - 4N_1N_4$, $D_\psi = (N_3 + N_6)^2 - 4N_2N_5$. Let $D_\varphi = D_\psi \geq 0$. Then the first equation in (3.3) has real roots. In order for quadratic equation has only one positive root is sufficient that

$$\operatorname{sgn}(N_1)\operatorname{sgn}(N_4) < 0. \tag{3.5}$$

ψ can be found from (3.4) selecting positive φ. In this case, there is a positive value ψ from the following inequality

$$-2N_2N_3 + N_2(N_3 - N_6) - N_2\sqrt{(N_3 - N_6)^2 - 4N_1N_4} \geq 0. \tag{3.6}$$

The result is the solution of the inverse problem of determining the pair φ and ψ two natural frequencies - only, when $D_\varphi = D_\psi \geq 0$ and the conditions (3.5) and (3.6) met.

For examples 3.1 and 3.2 experimental model consisting of a steel rod having a radius of 0.01 m and a length of 6 m which is hinged - simply supported at the ends. Then $EJ = 1649.34\,\mathrm{N\,m^2}$, $\rho = 7800\,\mathrm{kg\,m^{-3}}$, $A = 3.14 \cdot 10^{-4}\,\mathrm{m^2}$.

Table 1. Determination of the magnitude of the stiffness coefficient of the springs

ω_1	ω_2	a	b	c_1	c_2
7.17	28.48	1	2	10.000	5.000
7.17	28.48	1	2	-10528.221	32489.777

Example 3.1. Table 1 shows that magnitude of the coefficient stiffness of the first spring is equal $10\,\mathrm{Nm^{-1}}$, and the coefficient stiffness of the second spring is equal $5\,\mathrm{Nm^{-1}}$ and they are defined uniquely by the following two values $\omega_1 = 7.17\,\mathrm{Hz}$ and $\omega_2 = 28.48\,\mathrm{Hz}$ (a negative value does not satisfy the physical meaning of the problem). Notice that to calculate the natural frequencies of the problem (2.1) - (2.7) take the accuracy of $\varepsilon = 10^{-6}$. For ε, it is means that $|\Delta(a, b, l, \omega_i, c_1, c_2)| < \varepsilon$, $i = 1, 2$, for fixed values a, b, l, c_1, c_2.

In example 3.1, $N_1 = -0.00004$, $N_2 = -0.00001$, $N_3 = 0.0005$, $N_4 = 4.54142$, $N_5 = -2.27322$, $N_6 = 0.45421$. Then

$$N_1N_4 - N_2N_5 + N_3N_6 = 0, \quad \operatorname{sgn}(N_1)\operatorname{sgn}(N_4) < 0.$$

Also

$$-2N_1N_3 + N_1(N_3 + N_6) - N_1\sqrt{(N_3 + N_6)^2 - 4N_2N_5} = 0.00001 > 0.$$

Consequently, the value of the stiffness coefficients of intermediate springs are uniquely determined.

TABLE 2. Determination of the magnitude of the stiffness coefficient of the springs

ω_1	ω_2	a	b	c_1	c_2
7.29	28.49	2	4	14.997	10.002
7.29	28.49	2	4	10.002	14.997

Example 3.2. Table 2 shows that the value of the stiffness coefficients of the springs are not uniquely defined.

In example 3.2, $N_1 = -0.00079$, $N_2 = -0.00079$, $N_3 = 0.01988$, $N_4 = -0.11929$, $N_5 = -0.11929$, $N_6 = 4 \cdot 10^{-10}$. Then

$$N_1N_4 - N_2N_5 + N_3N_6 = 0, \quad \operatorname{sgn}(N_1)\operatorname{sgn}(N_4) > 0, \quad \operatorname{sgn}(N_2)\operatorname{sgn}(N_5) > 0.$$

Consequently, the value of the stiffness coefficients of the springs are not uniquely defined. But here it should be noted another very important moment, which gives us the opportunity to consider this situation from the other side. This is due to the locations of intermediate springs. In this case, we note that the intermediate springs located geometrically symmetrically about the center rod, i.e. with the condition

$$a + b = l. \tag{3.7}$$

In Example 3.2, $a = 2$, $b = 4$, $l = 6$ i.e. with the condition of geometric symmetry of fixing springs relative to the rod center (3.7). Note that analytical frequency in Tables 1, 2 calculated using the Maple. The following theorem is the main result of the paper.

Theorem 3.3. *If intermediate springs located geometrically symmetrically about the center rod then the quadratic equations of* (3.3) *have a pair of identical roots accurate to their permutations.*

Proof. In view of condition (3.7) the geometric symmetry of fixing springs relative to the rod center, we have

$$\beta(l - b, l, p) = \gamma(b, l, p).$$

Then $N_6 = 0$. It remains to show that $N_1 = N_2$ and $N_4 = N_5$. Since the first columns N_1 and N_2 is equal and $\beta(l - b, l, p_i) = \gamma(b, l, p_i)$, $i = 1, 2$, it follows that $N_1 = N_2$. Since the second columns N_4 and N_5 is equal and $\beta(l-b, l, p_i) = \gamma(b, l, p_i)$, $i = 1, 2$ then $N_4 = N_5$. The proof is complete. $\square$

In practice, intervals changes are already known for values of the stiffness of the first and second springs for a particular design. The physical meaning is that the problem of the stiffness coefficients can be only decrease. And this fact makes it possible to logically uniquely define the values of coefficients of stiffness of the first and second spring, in the case even where they are located symmetrically relative to the center rod. Such problems arise when using the methods of vibration diagnostics relating to the methods of non-destructive testing.

Conclusions. This article solved the inverse problem of determining the stiffness coefficients of the springs on the intermediate rod from the known the two natural frequencies. It is shown that in the general case this inverse problem has two solutions. We found sufficient conditions the inverse problem for the existence of a unique solution of determining the stiffness of the springs in the intermediate non-terminal points of the rod from the known the two eigen frequencies. It is proved that for the unique determination of the stiffness coefficients of springs geometric symmetry of the location of the spring relative to the center rod played an important role.

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