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MULTIPLICITY OF GROUND STATE SOLUTIONS FOR DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS WITH UNBOUNDED POTENTIALS

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Abstract. The discrete nonlinear Schrödinger equation is a nonlinear lattice system that appears in many areas of physics such as nonlinear optics, biomolecular chains and Bose-Einstein condensates. In this article, we consider a class of discrete nonlinear Schrödinger equations with unbounded potentials. We obtain some new sufficient conditions on the multiplicity results of ground state solutions for the equations by using the symmetric mountain pass lemma. Recent results in the literature are greatly improved.

1. Introduction

The discrete nonlinear Schrödinger (DNLS) equation is one of the most important inherently discrete models. DNLS equations play a crucial role in the modeling of a great variety of phenomena, ranging from solid state and condensed matter physics to biology [6, 7, 8]. For example, they have been successfully applied to the modeling of localized pulse propagation optical fibers and wave guides, to the study of energy relaxation in solids, to the behavior of amorphous material, to the modeling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand [15].

This article considers the DNLS equation

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - \gamma_n f(\psi_n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is discrete Laplacian operator, v_n and γ_n are real valued for each $n \in \mathbb{Z}$, $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and the nonlinearity $f(u)$ is gauge invariant; that is,

$$f(e^{i\theta}u) = e^{i\theta}f(u), \theta \in \mathbb{R}. \tag{1.2}$$

Since solitons are spatially localized time-periodic solutions and decay to zero at infinity, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t},$$
$$\lim_{|n|\to\infty} \psi_n = 0,$$

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where ψ_n is real valued for each $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.1) becomes

$$-\Delta u_n + v_n u_n - \omega u_n = \gamma_n f(u_n), n \in \mathbb{Z}, \tag{1.3}$$

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.4}$$

holds, where $|n|$ is the length of index n. Naturally, if we look for solitons of (1.1), we just need to get the solutions of (1.3) satisfying (1.4).

In the past decade, the existence of solitons of the DNLS equations has drawn a great deal of interest [13, 14, 16, 17, 18, 19, 24, 25, 26, 27, 28]. The existence for the periodic DNLS equations with superlinear nonlinearity [16, 17] and with saturable nonlinearity [27, 28] has been studied. And the existence results of solitons of the DNLS equations without periodicity assumptions were established in [13, 14, 26]. As for the existence of the homoclinic orbits of nonlinear Schrödinger equations, we refer to [4, 21, 22, 23]. By using the generalized Nehari manifold approach, Mai and Zhou [16] in 2013 proved the existence of a kind of special solitons of (1.3), which called ground state solutions [14], that is, nontrivial solutions with least possible energy in l^2. In this paper, we employ the Symmetric Mountain Pass Lemma instead of the generalized Nehari manifold approach to obtain the existence of ground state solutions of (1.3).

Let $F(u) = \int_0^u f(t)dt \geq 0, t \in \mathbb{R}$. Our main results are as follows.

Theorem 1.1. *Suppose that $f(u)$ is odd in u and the following hypotheses are satisfied:*

- (A1) *for any $n \in \mathbb{Z}$, we have $\underline{v} = \inf_{n\in\mathbb{Z}} v_n > \omega > 0$, and $\lim_{|n|\to\infty} v_n = +\infty$;*
- (A2) *there exist two positive constants $\underline{\gamma}$ and $\bar{\gamma}$ such that for any $n \in \mathbb{Z}$, $\underline{\gamma} \leq \gamma_n \leq \bar{\gamma}$;*
- (A3) *$f(u)$ is continuous in u and $f(u) = o(u)$ as $u \to 0$;*
- (A4) *for any $c > 0$, there exist $p = p_c > 0$, $q = q_c > 0$ and $\mu < 2$ such that*
$$\Big(2 + \frac{1}{p + q|u_n|^{\mu/2}}\Big)F(u_n) \leq f(u_n)u_n, \quad \forall n \in \mathbb{Z}, |u_n| \geq c;$$
- (A5) $\lim_{s\to+\infty}[\frac{\underline{\gamma}\min_{|u|=1} F(su)}{s^2}] = +\infty$.

Then (1.3) *has an unbounded sequence solutions satisfying* (1.4).

Theorem 1.2. *The unbounded sequence solutions $u^{(k)}$ ($k \in \mathbb{N}$ of* (1.3) *obtained in Theorem 1.1 decay exponentially at infinity:*

$$|u_n^{(k)}| \leq C^{(k)} e^{-\beta^{(k)}|n|}, \ n \in \mathbb{Z},$$

with some constants $C^{(k)} > 0$ and $\beta^{(k)} > 0$, $k \in \mathbb{N}$.

Remark 1.3. Zhang et al. [21, 22] studied (1.3) under the assumption that

$$0 < (q_1 - 1)f(u)u \leq f'(u)u^2, \ \forall u \neq 0$$

holds for some constant $q_1 \in (2, +\infty)$. This is a stronger condition than the classical Ambrosetti- Rabinowitz superlinear condition; i.e., there exist constants $q_1 > 2$ and $r_1 > 0$ such that

$$0 < q_1 \int_0^u f(s)ds \leq uf(u), \ \forall |u| \geq r_1.$$

Thus, our results improve the corresponding results in [24, 25].

As it is well known, critical point theory is a powerful tool to deal with the homoclinic solutions of differential equations [9, 10, 11, 12] and is used to study homoclinic solutions of discrete systems in recent years [1, 2, 3, 27]. Our aim in this paper is to obtain the multiplicity results of ground state solutions for the discrete nonlinear Schrödinger equations by using critical point theory. The main idea is to transfer the problem of solutions in E (defined in Section 2) of (1.3) into that of critical points of the corresponding functional. The motivation for the present work stems from the recent papers [2, 4, 12].

2. Preliminaries

To apply the critical point theory, we establish the variational framework corresponding to (1.3) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notation.

Let S be the vector space of all real sequences of the form

$$u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty},$$

namely $S = \{\{u_n\} : u_n \in \mathbb{R},\ n \in \mathbb{Z}\}$. Define

$$E = \big\{u \in S : \sum_{n=-\infty}^{+\infty} (-\Delta u_n \cdot u_n + v_n u_n^2) < +\infty\big\}.$$

The space is a Hilbert space with the inner product

$$\langle u, \nu\rangle = \sum_{n=-\infty}^{+\infty} (-\Delta u_n \cdot \nu_n + v_n u_n \nu_n), \forall u, \nu \in E, \tag{2.1}$$

and the corresponding norm

$$\|u\| = \Big[\sum_{n=-\infty}^{+\infty} (-\Delta u_n \cdot u_n + v_n u_n^2)\Big]^{1/2}, \quad \forall u \in E. \tag{2.2}$$

In what follows, l^2 denotes the space of functions whose second powers are summable on $\mathbb{Z}$ equipped with

$$\|u\|_{l^2}^2 = \sum_{n\in\mathbb{Z}} u_n^2.$$

Let

$$l^\infty(\mathbb{Z}, \mathbb{R}) = \{u \in S : \sup_{n\in\mathbb{Z}} |u_n| < +\infty\}.$$

For any $n_1, n_2 \in \mathbb{Z}$ with $n_1 < n_2$, we let $\mathbb{Z}(n_1, n_2) = [n_1, n_2] \cap \mathbb{Z}$ and for function $f : \mathbb{Z} \to \mathbb{R}$ and $a \in \mathbb{R}$, we set

$$\mathbb{Z}(f_n \ge a) = \{n \in \mathbb{Z} : f_n \ge a\}, \mathbb{Z}(f_n \le a) = \{n \in \mathbb{Z} : f_n \le a\}.$$

For all $u \in E$, define the functional J on E as follows:

$$\begin{aligned} J(u) :=& \frac{1}{2} \sum_{n=-\infty}^{+\infty} (-\Delta u_n \cdot u_n + v_n u_n^2) - \frac{\omega}{2} \sum_{n=-\infty}^{+\infty} u_n^2 - \sum_{n=-\infty}^{+\infty} \gamma_n F(u_n) \\ =& \frac{1}{2}\|u\|^2 - \frac{\omega}{2}\|u\|_{l^2}^2 - \sum_{n=-\infty}^{+\infty} \gamma_n F(u_n). \end{aligned} \tag{2.3}$$

Standard arguments show that the functional J is a well-defined C^1 functional on E and (1.3) is easily recognized as the corresponding Euler-Lagrange equation for J. Thus, to find nontrivial solutions to (1.3) satisfying (1.4), we need only to look for nonzero critical points of J in E.

For the derivative of J we have the formula

$$\langle J'(u),\nu\rangle = \sum_{n=-\infty}^{+\infty} (-\Delta u_n\cdot\nu_n + v_n u_n\nu_n - \omega u_n\nu_n - \gamma_n f(u_n)\nu_n),\ \forall u,\nu\in E. \tag{2.4}$$

Let E be a real Banach space, $J\in C^1(E,\mathbb{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u_n\}\subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$ $(n\to\infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Symmetric Mountain Pass Lemma [20])**.** *Let E be a real Banach space and $J\in C^1(E,\mathbb{R})$ with J even. Suppose that J satisfies the P.S. condition, $J(0)=0$,*

- (A6) *there exist constants $\rho,\alpha>0$ such that $J|_{\partial B_\rho}\geq\alpha$, and*
- (A7) *for each finite dimensional subspace $\tilde{E}\subset E$, there is $r=r(\tilde{E})>0$ such that $J(u)\leq 0$ for $u\in\tilde{E}\backslash B_r$, where B_r is an open ball in E of radius r centered at 0.*

Then J possesses an unbounded sequence of critical values.

Lemma 2.2. *For $u\in E$,*

$$\underline{v}\|u\|_\infty^2 \leq \underline{v}\|u\|_{l^2}^2 \leq \|u\|^2, \tag{2.5}$$

where $\|u\|_\infty=\sup_{n\in\mathbb{Z}}|u_n|$.

Proof. Since $u\in E$, it follows that $\lim_{|n|\to\infty}|u_n|=0$. Hence, there exists $n^*\in\mathbb{Z}$ such that

$$\|u\|_\infty = |u_{n^*}| = \max_{n\in\mathbb{Z}}|u_n|.$$

By (A1) and (2.2), we have

$$\|u\|^2 \geq \sum_{n\in\mathbb{Z}} v_n u_n^2 \geq \underline{v}\sum_{n\in\mathbb{Z}} u_n^2 \geq \underline{v}\|u\|_\infty^2.$$

The proof is complete. □

Lemma 2.3. *Suppose that* (A1)–(A5) *are satisfied. Then J satisfies condition* (C) [2, 20].

Proof. Let $\{u^{(k)}\}_{k\in\mathbb{N}}\subset E$ be such that $\{J(u^{(k)})\}_{k\in\mathbb{N}}$ is bounded and $(1+\|u^{(k)}\|)\|J'(u^{(k)})\|\to 0$ as $k\to\infty$. Then there is a positive constant K such that $|J(u^{(k)})|\leq K$. By (2.3) and (2.4), we have

$$\begin{aligned} 2K &\geq 2J(u^{(k)}) - \langle J'(u^{(k)}),u^{(k)}\rangle \\ &= \sum_{n=-\infty}^{+\infty}\gamma_n\big[f(u_n^{(k)})u_n^{(k)} - 2F(u_n^{(k)})\big]. \end{aligned} \tag{2.6}$$

By (A3), there exists $\eta\in(0,1)$ such that

$$|F(u_n)| \leq \frac{\underline{v}-\omega}{4\bar{\gamma}}u_n^2, \quad \forall n\in\mathbb{Z},\ |u_n|\leq\eta. \tag{2.7}$$

Then it follows from (A4) that

$$f(u_n^{(k)})u_n^{(k)} > 2F(u_n^{(k)}) \geq 0, \quad \forall n \in \mathbb{Z}, \tag{2.8}$$

$$F(u_n^{(k)}) \leq \Big[p + q|u_n^{(k)}|^{\mu/2}\Big]\Big[f(u_n^{(k)})u_n^{(k)} - 2F(u_n^{(k)})\Big], \quad \forall n \in \mathbb{Z},\ |u_n^{(k)}| \geq \eta. \tag{2.9}$$

By Lemma 2.2, (2.3), (2.7), (2.8), (2.10)) and (2.11), we have

$$\begin{aligned}
&\frac{1}{2}\|u^{(k)}\|^2 \\
&= J(u^{(k)}) + \frac{\omega}{2}\|u^{(k)}\|_{l^2}^2 + \sum_{n\in\mathbb{Z}(|u_n^{(k)}|\leq\eta)} \gamma_n F(u_n^{(k)}) + \sum_{n\in\mathbb{Z}(|u_n^{(k)}|\geq\eta)} \gamma_n F(u_n^{(k)}) \\
&\leq J(u^{(k)}) + \frac{\omega}{2\underline{v}}\|u^{(k)}\|^2 + \frac{\underline{v}-\omega}{4} \sum_{n\in\mathbb{Z}(|u_n^{(k)}|\leq\eta)} (u_n^{(k)})^2 \\
&\quad + \bar{\gamma} \sum_{n\in\mathbb{Z}(|u_n^{(k)}|\geq\eta)} \big[p + q|u_n^{(k)}|^{\mu/2}\big]\big[f(u_n^{(k)})u_n^{(k)} - 2F(u_n^{(k)})\big] \\
&\leq K + \frac{\omega}{2\underline{v}}\|u^{(k)}\|^2 + \frac{\underline{v}-\omega}{4\underline{v}}\|u^{(k)}\|^2 + 2K\bar{\gamma}\Big(p + q\underline{v}^{\mu/2}\|u^{(k)}\|^{\mu}\Big).
\end{aligned}$$

That is,

$$\frac{\underline{v}-\omega}{4\underline{v}}\|u^{(k)}\|^2 \leq K + 2K\bar{\gamma}\Big(p + q\underline{v}^{\mu/2}\|u^{(k)}\|^{\mu}\Big).$$

Since $\underline{v} > \omega$ and $\mu < 2$, it is not difficult to know that $\{u^{(k)}\}_{k\in\mathbb{N}}$ is a bounded sequence in E, i.e., there exists a constant $K_1 > 0$ such that

$$\|u^{(k)}\| \leq K_1, \quad k \in \mathbb{N}. \tag{2.10}$$

So passing to a subsequence if necessary, it can be assumed that $u^{(k)} \rightharpoonup u^{(0)}$ in E. For any given number $\varepsilon > 0$, by (A3), we can choose $\zeta > 0$ such that

$$|f(u)| \leq \varepsilon|u|, \quad \forall\ u \in \mathbb{R}, \tag{2.11}$$

where $|u| \leq \zeta$.

By (A1), we can also choose a positive integer $D \in \mathbb{R}$ such that

$$v_n \geq \frac{K_1^2}{\zeta^2}, \quad |n| \geq D. \tag{2.12}$$

By (2.10) and (2.12), we obtain

$$(u_n^{(k)})^2 = \frac{1}{v_n}v_n(u_n^{(k)})^2 \leq \frac{\zeta^2}{K_1^2}\|u^{(k)}\|^2 \leq \zeta^2, \quad |n| \geq D. \tag{2.13}$$

Since $u^{(k)} \rightharpoonup u^{(0)}$ in E, it is easy to verify that $u_n^{(k)}$ converges to $u_n^{(0)}$ pointwise for all $n \in \mathbb{Z}$, that is

$$\lim_{k\to\infty} u_n^{(k)} = u_n^{(0)}, \quad \forall n \in \mathbb{Z}. \tag{2.14}$$

Combining this with (2.13), we have

$$(u_n^{(0)})^2 \leq \zeta^2, \quad |n| \geq D. \tag{2.15}$$

It follows from (2.14) and the continuity of $f(u)$ on u that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-D}^{D} \gamma_n |f(u_n^{(k)}) - f(u_n^{(0)})| < \varepsilon, \quad k \geq k_0. \tag{2.16}$$

On the other hand, from (A3), (2.5), (2.10), (2.11), (2.13), (2.15) and Hölder inequality it follows that

$$
\begin{aligned}
&\sum_{|n|\geq D} \gamma_n |f(u_n^{(k)}) - f(u_n^{(0)})||u_n^{(k)} - u_n^{(0)}| \\
&\leq \sum_{|n|\geq D} \bar{\gamma}\Big(|f(u_n^{(k)})| + |f(u_n^{(0)})|\Big)(|u_n^{(k)}| + |u_n^{(0)}|) \\
&\leq \bar{\gamma}\varepsilon \sum_{|n|\geq D} \big[|u_n^{(k)}| + |u_n^{(0)}|\big]\big(|u_n^{(k)}| + |u_n^{(0)}|\big) \\
&\leq 2\bar{\gamma}\varepsilon \sum_{n=-\infty}^{+\infty} \big(|u_n^{(k)}|^2 + |u_n^{(0)}|^2\big) \\
&\leq \frac{2\bar{\gamma}\varepsilon}{\underline{v}}\big(K_1^2 + \|u^{(0)}\|^2\big).
\end{aligned}
\tag{2.17}
$$

Since ε is arbitrary, we obtain

$$
\sum_{n=-\infty}^{+\infty} \gamma_n \big|f(u_n^{(k)}) - f(u_n^{(0)})\big| \to 0, \quad k \to \infty. \tag{2.18}
$$

It follows from (2.2), (2.4) and (2.5) that

$$
\begin{aligned}
&\langle J'(u^{(k)}) - J'(u^{(0)}), u^{(k)} - u^{(0)}\rangle \\
&= \|u^{(k)} - u^{(0)}\|^2 - \omega\|u^{(k)} - u^{(0)}\|_{l^2}^2 - \sum_{n=-\infty}^{+\infty} \gamma_n(f(u_n^{(k)}) - f(u_n^{(0)}))(u^{(k)} - u^{(0)}) \\
&\geq \frac{\underline{v} - \omega}{\underline{v}}\|u^{(k)} - u^{(0)}\|^2 - \sum_{n=-\infty}^{+\infty} \gamma_n\Big(f(u_n^{(k)}) - f(u_n^{(0)})\Big)(u^{(k)} - u^{(0)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\underline{v} - \omega}{\underline{v}}\|u^{(k)} - u^{(0)}\|^2 &\leq \langle J'(u^{(k)}) - J'(u^{(0)}), u^{(k)} - u^{(0)}\rangle \\
&\quad + \sum_{n=-\infty}^{+\infty} \gamma_n\Big(f(u_n^{(k)}) - f(u_n^{(0)})\Big)(u^{(k)} - u^{(0)}).
\end{aligned}
$$

Since $\underline{v} > \omega > 0$ and $\langle J'(u^{(k)}) - J'(u^{(0)}), u^{(k)} - u^{(0)}\rangle \to 0, k \to \infty$. Thus, $u^{(k)} \to u^{(0)}$ in E and the proof is complete. □

3. Proofs of theorems

In this section, we shall prove our main results by using the critical point method.

Proof of Theorem 1.1. It is clear that J is even and $J(0) = 0$. We have already known that $J \in C^1(E, \mathbb{R})$ and J satisfies condition (C). Hence, it suffices to prove that J satisfies the conditions (J_1) and (A7) of Lemma 2.1.

If $\|u\| = \sqrt{\underline{v}}\eta := \rho$, then by Lemma 2.2, $|u_n| \le \eta$ for $n \in \mathbb{Z}$. Set $\alpha = \frac{\underline{v}-\omega}{4\underline{v}}\eta^2$. Hence, from (2.3), (2.5), (2.7), (A1) and (A3), we have

$$\begin{aligned}
J(u) &\ge \frac{1}{2}\|u\|^2 - \frac{\omega}{2\underline{v}}\|u\|^2 - \sum_{n=-\infty}^{+\infty} \gamma_n F(u_n) \\
&\ge \frac{\underline{v}-\omega}{2\underline{v}}\|u\|^2 - \frac{\underline{v}-\omega}{4\underline{v}} \sum_{n=-\infty}^{+\infty} v_n u_n^2 \\
&\ge \frac{\underline{v}-\omega}{2\underline{v}}\|u\|^2 - \frac{\underline{v}-\omega}{4\underline{v}}\|u\|^2 \\
&= \frac{\underline{v}-\omega}{4\underline{v}}\|u\|^2 = \alpha.
\end{aligned} \tag{3.1}$$

Equation (3.3) shows that $\|u\| = \rho$ implies that $J(u) \ge \alpha$, i.e., J satisfies assumption (A6).

In the following, we shall verify condition (A7). Let $\tilde{E} \subset E$ be a finite dimensional subspace. Consider $u \in \tilde{E}$ with $u \ne 0$. Since all norms of a finite dimensional normed space are equivalent, there is a constant $c_1 > 0$ such that

$$\|u\|^2 \le c_1 \|u\|_\infty^2, \quad \forall u \in \tilde{E}. \tag{3.2}$$

Assume that $\dim \tilde{E} = m$ and $u_1, u_2, \dots, u_m$ are the basis of $\tilde{E}$ such that for $i, j = 1, 2, \dots, m$, we have

$$\langle u_i, u_j \rangle = \begin{cases} c_1^2, & i = j, \\ 0, & i \ne j. \end{cases} \tag{3.3}$$

Since $u_i \in E$, we can choose a positive integer $D_1 > 0$ such that

$$|u_n^{(i)}| < \frac{1}{m}, \quad |n| > D_1,\ i = 1, 2, \dots, m. \tag{3.4}$$

Set $\Theta = \{u \in \tilde{E} : \|u\| = c_1\}$. Then for $u \in \Theta$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \dots, m$ such that

$$u_n = \sum_{i=1}^{m} \lambda_i u_n^{(i)}, \quad \forall n \in \mathbb{Z}. \tag{3.5}$$

It follows that

$$c_1^2 = \|u\|^2 = \langle u, u \rangle = \sum_{i=1}^{m} \lambda_i^2 \langle u^{(i)}, u^{(i)} \rangle = c_1^2 \sum_{i=1}^{m} \lambda_i^2,$$

which implies that $|\lambda_i| \le 1$ for $i = 1, 2, \dots, m$. Hence, for $u \in \Theta$, let $|u_{n_0}| = \|u\|_\infty$, then by (3.2) and (3.4) we have

$$1 \le \|u\|_\infty = |u_{n_0}| \le \sum_{i=1}^{m} |\lambda_i||u_{n_0}^{(i)}| \le \sum_{i=1}^{m} |u_{n_0}^{(i)}|, \quad \theta \in \Theta. \tag{3.6}$$

By (A5), there exists $\sigma_0 = \sigma_0(c_1, D_1) > 1$ such that

$$\frac{\underline{\gamma} \min_{|u|=1} F(n, su_n)}{s^2} \ge c_1^2, \quad \forall s \ge \sigma_0,\ n \in \mathbb{Z}(D_1, D_1). \tag{3.7}$$

By (3.4) and (3.6), there exists $n_0 = n_0(u) \in \mathbb{Z}(D_1, D_1)$ such that

$$1 \le |u_{n_0}| = \|u\|_\infty, \quad \forall u \in \Theta. \tag{3.8}$$

By (2.3), (3.7) and (3.8), we have

$$
\begin{aligned}
J(\sigma u) &= \frac{1}{2}\|\sigma u\|^2 - \frac{\omega}{2}\|\sigma u\|_{l^2}^2 - \sum_{n=-\infty}^{+\infty} \gamma_n F(\sigma u_n) \\
&\le \frac{\sigma^2}{2}\|u\|^2 - \frac{\omega}{2}\|\sigma u\|_{l^2}^2 - \underline{\gamma} F(\sigma u_{n_0}) \\
&\le \frac{\sigma^2}{2}\|u\|^2 - \frac{\omega}{2}\|\sigma u\|_{l^2}^2 - \underline{\gamma} \min_{|x|=1} F(\sigma |u_{n_0}| x) \\
&\le \frac{(c_1\sigma)^2}{2} - (c_1\sigma)^2 |u_{n_0}|^2 \\
&\le \frac{(c_1\sigma)^2}{2} - (c_1\sigma)^2 \\
&= -\frac{(c_1\sigma)^2}{2}, \quad u \in \Theta,\ \sigma \ge \sigma_0.
\end{aligned}
\tag{3.9}
$$

This implies $J(u) < 0$ for $u \in \tilde{E}$ and $\|u\| \ge c_1\sigma_0$. The condition (A7) holds. By Lemma 2.1, J possesses an unbounded sequence $\{d^{(k)}\}_{k\in\mathbb{N}}$ of critical values with $d^{(k)} = J(u^{(k)})$, where $u^{(k)}$ is such that $J'(u^{(k)}) = 0$ for $k = 1, 2, \dots$. By (2.3), we have

$$
\frac{1}{2}\|u^{(k)}\|^2 = d^{(k)} + \frac{\omega}{2}\|u^{(k)}\|_{l^2}^2 + \sum_{n=-\infty}^{+\infty} \gamma_n F(u_n^{(k)}) \ge d^{(k)}, \quad k \in \mathbb{N}.
$$

Since $\{d^{(k)}\}_{k\in\mathbb{N}}$ is unbounded, it follows that $\{\|u^{(k)}\|\}_{k\in\mathbb{N}}$ is unbounded. □

Remark 3.1. Similar to [18], we can prove that the homoclinic solutions $u^{(k)}$ decay exponentially fast at infinity. For simplicity, we omit its proof.

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