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STABILITY OF BOUNDARY-VALUE PROBLEMS FOR THIRD-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We consider a boundary-value problem for the third-order partial differential equation

$$\begin{gathered}\frac{d^3u(t)}{dt^3}+Au(t)=f(t),\quad 0<t<1,\\ u(0)=\varphi,\quad u(1)=\psi,\quad u'(1)=\xi\end{gathered}$$

in a Hilbert space H with a self-adjoint positive definite operator A. Using the operator approach, we establish stability estimates for the solution of the boundary value problem. We study three types of boundary value problems and obtain stability estimates for the solution of these problems.

1. Introduction

Boundary value problems for third order partial differential equation have been considered in fields of sciences and engineering, such as modern physics, chemical diffusion and mechanic fluids. The well-posedness of various boundary-value problems for partial differential and difference equations has been studied extensively by many researchers [1, 2, 3, 16, 18, 19, 20, 22, 23, 25] and the references therein.

The following boundary-value problem for a third order partial differential equation with three points boundary condition is studied in [23],

$$\begin{gathered}\frac{\partial^3u(x,t)}{\partial t^3}+\frac{\partial}{\partial x}\Big(a(x,t),\frac{\partial u(x,t)}{\partial x}\Big)=f(x,t),\\ \int_c^1 u(x,t)dx=0,\quad t\in[0,T],\ 0\le c<1,\ T>0,\\ u(x,0)=0,\quad \frac{\partial u}{\partial t}(x,0)=0,\quad \frac{\partial^2u}{\partial t^2}(x,T)=0,\quad x\in[0,1],\end{gathered}$$

where $a(x,t)$ and its derivatives satisfy the condition $0<a_0<a(x,t)<a_1$, $|(a(x,t))_x|\le b$, and $f(x,t)$ is given smooth function in $[0,1]\times[0,T]$, It was obtained the approximate solution of the considered problem, the authors established a bounded linear operator and an orthogonal basis to use the reproducing kernel space method, numerical results are also given.

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There are several methods for solving partial differential equations. For instance, the method of operator as a tool for investigation of the stability of partial differential equation in Hilbert and Banach space has been systematically devoted by several authors (see for example [4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 17, 21, 24] and the references therein).

In this article we consider the boundary-value problem for third-order partial differential equation

$$\begin{gathered} \frac{d^3u(t)}{dt^3} + Au(t) = f(t), \quad 0 < t < 1, \\ u(0) = \varphi, \quad u(1) = \psi, \quad u'(1) = \xi \end{gathered} \tag{1.1}$$

in a Hilbert space H with a self-adjoint positive definite operator $A \geq \delta I$, where $\delta > 0$. We are interested in the stability of the solution of problem (1.1).

A function $u(t)$ is a solution of problem (1.1) if the following conditions are satisfied:

- (i) $u(t)$ is thrice continuously differentiable on the interval $(0, 1)$ and continuously differentiable on the segment $[0, 1]$. The derivatives at the end points of the segment are understood as the appropriate unilateral derivatives.
- (ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, 1]$, and function $Au(t)$ is continuous on the segment $[0, 1]$.
- (iii) $u(t)$ satisfies the equation and boundary conditions (1.1).

The outline of this article is as follows. In Section 2 the main theorem on stability of problem (1.1) is established. Section 3 proves the stability estimates for the solution of three problems for partial differential equations of third order in t. Conclusion presents in Section 4.

2. Main theorem on stability

Let us prove some lemmas needed in the sequel.

Lemma 2.1 ([15]). *For $t \geq 0$ the following estimate holds*

$$\| \exp\{\pm itA^{1/3}\}\|_{H \to H} \leq 1. \tag{2.1}$$

Lemma 2.2 ([6]). *The operator Δ defined by the formula*

$$\Delta = \frac{1}{3}\{I - (ae^{-(1+a)B} + \bar{a}e^{-(1+\bar{a})B})\}$$

has a bounded inverse $T = \Delta^{-1}$ and

$$\|T\|_{H \to H} \leq \frac{3}{1 - 2e^{-(3/2)\delta^{1/3}}}. \tag{2.2}$$

Here $a = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\bar{a} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$, $B = A^{1/3}$.

Lemma 2.3. *Suppose that $\varphi \in D(A)$, $\psi \in D(A)$, $\xi \in D(A)$ and $f(t)$ is continuously differentiable on $[0, 1]$. Then there is unique solution of problem (1.1)* *and*

$$
\begin{aligned}
u(t) = {} & e^{-Bt}u(0) + \frac{1}{1+a}B^{-1}(e^{-(1-t)B} - e^{-(a+t)B})(u'(1) + Bu(1)) \\
& + \frac{1}{a-\bar{a}}B^{-2}\{\frac{1}{1+a}(e^{-(1-t)aB} - e^{-(a+t)B}) \\
& - \frac{1}{1+\bar{a}}(e^{-(1-t)\bar{a}B} - e^{-(\bar{a}+t)B})\}(u''(1) + \bar{a}Bu'(1) - aB^2u(1)) \\
& - \frac{1}{a-\bar{a}}B^{-2}\int_0^t \Big[\frac{1}{1+a}(e^{-(t-s)B} - e^{-(t+sa)B}) \\
& - \frac{1}{1+\bar{a}}(e^{-(t-s)B} - e^{-(t+s\bar{a})B})\Big] f(s)ds,
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
u''(1) = {} & T\Big\{B^2e^{-B}u(0) + \frac{1}{1+a}B(I - e^{-(a+1)B})(u'(1) + Bu(1)) \\
& + \frac{1}{a-\bar{a}}\{\frac{1}{1+a}(a^2I - e^{-(a+1)B}) - \frac{1}{1+\bar{a}}(\bar{a}^2I - e^{-(\bar{a}+1)B})\} \\
& \times (\bar{a}Bu'(1) - aB^2u(1)) - \frac{1}{a-\bar{a}}B^{-1}\Big[e^{-(1+a)B} - e^{-(1+\bar{a})B} \\
& - \frac{1}{1+a}(e^{-(a+1)B} - I) + \frac{1}{1+\bar{a}}(e^{-(\bar{a}+1)B} - I)\Big] f(1) \\
& - \frac{1}{a-\bar{a}}B^{-2}\Big[\frac{1}{1+a}(I - e^{-(a+1)B}) - \frac{1}{1+\bar{a}}(I - e^{-(\bar{a}+1)aB})\Big] f'(1) \\
& - \frac{1}{a-\bar{a}}\int_0^1 \Big[\frac{1}{1+a}(e^{-(1-s)B} - e^{-(sa+1)B}) \\
& - \frac{1}{1+\bar{a}}(e^{-(1-s)B} - e^{-(s\bar{a}+1)B})\Big] f(s)ds\Big\}.
\end{aligned} \tag{2.4}
$$

Proof. Obviously, it can be written as the equivalent boundary-value problem for the system of first order differential equations

$$
\begin{gathered}
\frac{du(t)}{dt} + Bu(t) = v(t), u(0) = \varphi, u(1) = \psi, \\
\frac{dv(t)}{dt} - aBv(t) = w(t), u'(1) = \xi, \\
\frac{dw(t)}{dt} - \bar{a}Bw(t) = f(t), \quad 0 < t < 1.
\end{gathered} \tag{2.5}
$$

Integrating these equations, we can write

$$
\begin{gathered}
w(t) = e^{-(1-t)\bar{a}B}w(1) - \int_t^1 e^{-(s-t)\bar{a}B}f(s)ds, \\
v(t) = e^{-(1-t)aB}v(1) - \int_t^1 e^{-(p-t)aB}w(p)dp, \\
u(t) = e^{-Bt}u(0) + \int_0^t e^{-(t-p)B}v(p)dp.
\end{gathered} \tag{2.6}
$$

Applying system of equations (2.5), we obtain

$$
\begin{gathered}
v(1) = u'(1) + Bu(1), \\
w(1) = v'(1) - aBv(1) = u''(1) + \bar{a}Bu'(1) - aB^2u(1).
\end{gathered}
$$

Then, we have

$$w(t) = e^{-(1-t)\bar{a}B}[u''(1) + \bar{a}Bu'(1) - aB^2u(1)] - \int_t^1 e^{-(s-t)\bar{a}B} f(s)ds. \tag{2.7}$$

Using formulas (2.6), (2.7), we obtain

$$\begin{aligned} v(t) &= \frac{1}{a-\bar{a}} B^{-1}\Big(e^{-(1-t)aB} - e^{-(1-t)\bar{a}B}\Big)(u''(1) + \bar{a}Bu'(1) - aB^2u(1)) \\ &\quad + e^{-(1-t)aB}(u'(1) + Bu(1)) \\ &\quad - \frac{1}{a-\bar{a}} B^{-1} \int_t^1 (e^{-(s-t)aB} - e^{-(s-t)\bar{a}B}) f(s)ds. \end{aligned} \tag{2.8}$$

Using formulas (2.6), (2.8), we obtain formula (2.3). Taking the second order derivative and putting $t = 1$, we obtain the following operator equation with respect to $u''(1)$.

$$\begin{aligned} u''(1) &= B^2e^{-B}u(0) + \frac{1}{1+a}B(I - e^{-(a+1)B})(u'(1) + Bu(1)) \\ &\quad + \frac{1}{a-\bar{a}}\Big\{\frac{1}{1+a}(a^2I - e^{-(a+1)B}) - \frac{1}{1+\bar{a}}(\bar{a}^2I - e^{-(\bar{a}+1)B})\Big\} \\ &\quad \times (u''(1) + \bar{a}Bu'(1) - aB^2u(1)) \\ &\quad - \frac{1}{a-\bar{a}}B^{-1}\Big[e^{-(1+a)B} - e^{-(1+\bar{a})B} - \frac{1}{1+a}(e^{-(a+1)B} - I) \\ &\quad + \frac{1}{1+\bar{a}}(e^{-(\bar{a}+1)B} - I)\Big] f(1) \\ &\quad - \frac{1}{a-\bar{a}}B^{-2}\Big[\frac{1}{1+a}(I - e^{-(a+1)B}) - \frac{1}{1+\bar{a}}(I - e^{-(\bar{a}+1)aB})\Big] f'(1) \\ &\quad - \frac{1}{a-\bar{a}}\int_0^1 \Big[\frac{1}{1+a}(e^{-(1-s)B} - e^{-(sa+1)B}) \\ &\quad - \frac{1}{1+\bar{a}}(e^{-(1-s)B} - e^{-(s\bar{a}+1)B})\Big] f(s)ds. \end{aligned} \tag{2.9}$$

Since

$$\begin{aligned} \Delta &= I\frac{1}{a-\bar{a}}\Big\{\frac{1}{1+a}(a^2I - e^{-(a+1)B}) - \frac{1}{1+\bar{a}}(\bar{a}^2I - e^{-(\bar{a}+1)B})\Big\} \\ &= -\frac{1}{3}\{I - (ae^{-(1+a)B} + \bar{a}e^{-(1+\bar{a})B})\} \end{aligned}$$

has a bounded inverse $T = \Delta^{-1}$, using lemma 2.2, we obtain formula (2.4). The proof is complete. □

Now, we formulate the main theorem.

Theorem 2.4. *$\varphi \in D(A)$, $\psi \in D(A)$, $\xi \in D(A^{2/3})$ and $f(t)$ is continuously differentiable on $[0,1]$. Then there is a unique solution of problem (1.1) and the following inequalities hold*

$$\begin{aligned} \max_{0\le t\le 1} \|u(t)\|_H &\le M\Big\{\|\varphi\|_H + \|B^{-1}\xi\|_H + \|B^{-4}f'(1)\|_H + \|\psi\|_H \\ &\quad + \max_{0\le t\le 1}\|B^{-2}f(t)\|_H\Big\}, \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&\max_{0\le t\le 1}\|\frac{d^3u(t)}{dt^3}\|_H+\max_{0\le t\le 1}\|Au\|_H\\
&\le M\big\{\|A\varphi\|_H+\|A\psi\|_H+\|A^{2/3}\xi\|_H+\|f(0)\|_H+\max_{0\le t\le 1}\|f'(t)\|_H\big\},
\end{aligned}
\tag{2.11}
$$

where M does not depend on $f(t)$, φ, ψ, ξ.

Proof. First, we estimate $\|u(t)\|_H$ for $t\in[0,1]$. Applying (2.3), (2.1) and triangle inequality, we obtain

$$
\begin{aligned}
&\|u(t)\|_H\\
&\le \|e^{-Bt}\|_{H\to H}\|\varphi\|_H+\frac{1}{|1+a|}\|e^{-(1-t)B}-e^{-(a+t)B}\|_{H\to H}\|B^{-1}\xi+\psi\|_H\\
&\quad+\frac{1}{|a-\bar a|}\frac{1}{|1+a|}\Big\{\|e^{-(1-t)aB}-e^{-(a+t)B}\|_{H\to H}+\|e^{-(1-t)\bar aB}\\
&\quad-e^{-(\bar a+t)B}\|_{H\to H}\Big\}(\|B^{-2}u''(1)\|_H+|\bar a|\|B^{-1}\xi\|_H+|a|\|\varphi\|_H)\\
&\quad+\frac{1}{|a-\bar a|}\frac{1}{|1+a|}\int_0^t\big[\|e^{-(t-s)B}-e^{-(sa+t)B}\|_{H\to H}+\|e^{-(t-s)B}\\
&\quad-e^{-(s\bar a+t)B}\|_{H\to H}\big]\|B^{-2}f(s)\|_Hds\\
&\le M\{\|\varphi\|_H+\|B^{-1}\xi\|_H+\|\psi\|_H\\
&\quad+\max_{0\le t\le 1}\|B^{-2}f(t)\|_H+\|B^{-2}u''(1)\|_H\}
\end{aligned}
\tag{2.12}
$$

for any $t\in[0,1]$. Applying formula (2.4), estimate (2.2), and the triangle inequality, we obtain

$$
\begin{aligned}
&\|B^{-2}u''(1)\|_H\\
&\le\|T\|_{H\to H}\Big\{\|e^{-B}\|_{H\to H}\|\varphi\|_H+\frac{1}{|1+a|}\|I-e^{-(a+1)B}\|_{H\to H}\|B^{-1}\xi+\psi\|_H\\
&\quad+\frac{1}{|a-\bar a|}\frac{1}{|1+a|}\{\|a^2I-e^{-(a+1)B}\|_{H\to H}+\|\bar a^2I-e^{-(\bar a+1)B}\|_{H\to H}\}\\
&\quad\times(|\bar a|\|B^{-1}\xi\|_H+|a|\|\varphi\|_H)+\frac{1}{|a-\bar a|}\frac{1}{|1+a|}\int_0^1\big[\|e^{-(1-s)B}\\
&\quad-e^{-(sa+1)B}\|_{H\to H}+\|e^{-(1-s)B}-e^{-(s\bar a+1)B}\|_{H\to H}\big]\|B^{-2}f(s)\|_Hds\\
&\quad+\frac{1}{|a-\bar a|}\Big[\|e^{-(a+1)B}-e^{-(1+\bar a)B}\|_{H\to H}+\frac{1}{|1+a|}\big[\|I-e^{-(a+1)B}\|_{H\to H}\\
&\quad+\|I-e^{-(\bar a+1)B}\|_{H\to H}\big]\Big]\|B^{-3}f(1)\|_H+\frac{1}{|a-\bar a|}\frac{1}{|1+a|}\big\{\|I-e^{-(a+1)B}\|_{H\to H}\\
&\quad+\|I-e^{-(\bar a+1)B}\|_{H\to H}\big\}\|B^{-4}f'(1)\|_H\Big\}\\
&\le M\{\|\varphi\|_H+\|B^{-1}\xi\|_H+\|\psi\|_H+\|B^{-4}f'(1)\|_H+\max_{0\le t\le 1}\|B^{-2}f(t)\|_H\}.
\end{aligned}
\tag{2.13}
$$

From estimates (2.12) and (2.13) it follows estimate (2.10). Second, we estimate $\|Au(t)\|_H$ for $t\in[0,1]$. Since

$$
\int_0^t\big[\frac{1}{1+a}\big(e^{-(t-s)B}-e^{-(t+sa)B}\big)-\frac{1}{1+\bar a}\big(e^{-(t-s)B}-e^{-(t+s\bar a)B}\big)\big]f(s)ds,
$$

from formula (2.9) it follows that

$$
\begin{aligned}
u(t) &= e^{-Bt}u(0) + \frac{1}{1+a}B^{-1}(e^{-(1-t)B} - e^{-(a+t)B})(u'(1) + Bu(1)) \\
&\quad + \frac{1}{a-\bar{a}}B^{-2}\Big\{\frac{1}{1+a}(e^{-(1-t)aB} - e^{-(a+t)B}) - \frac{1}{1+\bar{a}}(e^{-(1-t)\bar{a}B} \\
&\quad - e^{-(\bar{a}+t)B})\Big\}(u''(1) + \bar{a}Bu'(1) - aB^2u(1)) \\
&\quad - \frac{1}{a-\bar{a}}B^{-3}\Big[[\frac{1}{1+a}(I + \bar{a}e^{-(1+a)tB}) - \frac{1}{1+\bar{a}}(I + ae^{-(1+\bar{a})tB})]f(t) \\
&\quad - [\frac{1+\bar{a}}{1+a} - \frac{1+a}{1+\bar{a}}]e^{-tB}f(0) - \int_0^t \Big[\frac{1}{1+a}(e^{-(t-s)B} + \bar{a}e^{-(t+sa)B}) \\
&\quad - \frac{1}{1+\bar{a}}(e^{-(t-s)B} + ae^{-(t+s\bar{a})B})\Big]f'(s)ds\Big].
\end{aligned} \tag{2.14}
$$

In the similarly manner, applying (2.14), (2.1) and the triangle inequality, we obtain

$$
\begin{aligned}
\|Au(t)\|_H &\le M\{\|A\varphi\|_H + \|B^2\xi\|_H + \|A\psi\|_H \\
&\quad + \|f(0)\|_H + \max_{0\le t\le 1}\|f'(t)\|_H + \|Bu''(1)\|_H\}
\end{aligned} \tag{2.15}
$$

for any $t \in [0,1]$. Applying formula (2.9), estimate (2.2), and the triangle inequality, we obtain

$$
\|Bu''(1)\|_H \le M\{\|A\varphi\|_H + \|B^2\xi\|_H + \|A\psi\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\|f'(t)\|_H\}. \tag{2.16}
$$

Applying estimates (2.15) and (2.16), we obtain

$$
\max_{0\le t\le 1}\|Au(t)\|_H \le M\big\{\|A\varphi\|_H + \|B^2\xi\|_H + \|A\psi\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\|f'(t)\|_H\}.
$$

From this, (1.1) and the triangle inequality it follows that

$$
\begin{aligned}
\max_{0\le t\le 1}\|\frac{d^3u(t)}{dt^3}\|_H &\le \max_{0\le t\le 1}\|Au(t)\|_H + \max_{0\le t\le 1}\|f(t)\|_H \\
&\le M_1\{\|A\varphi\|_H + \|B^2\xi\|_H + \|A\psi\|_H + \|f(0)\|_H + \max_{0\le t\le 1}\|f'(t)\|_H\}.
\end{aligned}
$$

The proof is complete. □

3. Applications

In this section we consider three applications of Theorem 2.4. First application. We consider the nonlocal boundary-value problem for a third-order partial dierential equation,

$$
\begin{gathered}
\frac{\partial^3 u(t,x)}{\partial t^3} - (a(x)u_x(t,x))_x + \delta u(t,x) = f(t,x), \quad 0 < t, x < 1, \\
u(0,x) = \varphi(x), \quad u(1,x) = \psi(x), \quad u_t(1,x) = \xi(x), \quad 0 \le x \le 1, \\
u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad 0 \le t \le 1
\end{gathered} \tag{3.1}
$$

This problem has a unique smooth solution $u(t,x)$ for smooth $a(x) \ge a > 0$, $x \in (0,1)$, $\delta > 0$, $a(1) = a(0)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$ $(x \in [0,1])$ and $f(t,x)$ $(t \in (0,1), x \in (0,1))$ functions. This allows us to reduce problem (1.1) in a Hilbert space $H = L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by (3.1). Let us give a number of results from the abstract Theorem 2.4.

Theorem 3.1. *For the solution of* (3.1)*, the following two stability inequalities hold:*

$$\max_{0\le t\le 1}\|u(t,\cdot)\|_{L_2[0,1]}\le M\Big[\max_{0\le t\le 1}\|f(t,\cdot)\|_{L_2[0,1]}+\|f_t(1,\cdot)\|_{L_2[0,1]}\\ +\|\varphi\|_{L_2[0,1]}+\|\psi\|_{L_2[0,1]}+\|\xi\|_{L_2[0,1]}\Big], \tag{3.2}$$

$$\max_{0\le t\le 1}\|u(t,\cdot)\|_{W_2^2[0,1]}+\max_{0\le t\le 1}\|\frac{\partial^3 u}{\partial t^3}(t,\cdot)\|_{L_2[0,1]}\\ \le M\Big[\max_{0\le t\le 1}\|f_t(t,\cdot)\|_{L_2[0,1]}+\|f(0,\cdot)\|_{L_2[0,1]}+\|\varphi\|_{W_2^2[0,1]}\\ +\|\psi\|_{W_2^2[0,1]}+\|\xi\|_{W_2^2[0,1]}\Big], \tag{3.3}$$

where M does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (3.1) can be written in the abstract form

$$\frac{d^3u(t)}{dt^3}+Au(t)=f(t),\quad 0\le t\le 1,\\ u(0)=\varphi,\quad u(1)=\psi,\quad u'(1)=\xi \tag{3.4}$$

in the Hilbert space $L_2[0,1]$, for all square integrable functions defined on $[0,1]$. Here the self-adjoint positive definite operator $A=A^x$ defined by

$$A^xu(x)=-(a(x)u_x)_x+\delta u(x) \tag{3.5}$$

with domain

$$D(A^x)=\{u(x):u,u_x,(a(x)u_x)_x\in L_2[0,1],u(0)=u(1),u'(0)=u'(1)\}.$$

where $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are respectively known and unknown abstract functions defined on $[0,1]$ with $H=L_2[0,1]$. Therefore, estimates (3.2)-(3.3) follow from estimates (2.10)-(2.11). The proof is complete. ☐

Second application. Let $\Omega\subset\mathbb{R}^n$ be a bounded open domain with smooth boundary S, $\bar\Omega=\Omega\cup S$. In $[0,1]\times\Omega$, we consider the boundary-value problem for a third-order partial differential equation

$$\frac{\partial^3 u(t,x)}{\partial t^3}-\sum_{r=1}^n(a_r(x)u_{x_r}(t,x))_{x_r}=f(t,x),\\ x=(x_1,\dots,x_n)\in\Omega,\ 0<t<1,\\ u(0,x)=\varphi(x),\quad u(1,x)=\psi(x),\quad u_t(1,x)=\xi(x),\quad x\in\bar\Omega,\\ u(t,x)=0,\quad x\in S,\quad 0\le t\le 1, \tag{3.6}$$

where $a_r(x)$, $x\in\Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x\in\bar\Omega$ and $f(t,x)$ $(x\in[0,1])$, $x\in\Omega$ are given smooth functions and $a_r(x)>0$. We introduce the Hilbert space $L_2(\bar\Omega)$, the space of integrable functions defined on $\bar\Omega$ equipped with norm

$$\|f\|_{L_2(\bar\Omega)}=\Big(\int\cdots\int_{x\in\bar\Omega}|f(x)|^2dx_1\dots dx_n\Big)^{1/2}.$$

Theorem 3.2. *For the solution of (3.6) the following two stability inequalities hold:*

$$\begin{aligned} \max_{0\le t\le 1}\|u(t,\cdot)\|_{L_2(\bar\Omega)} \le M_2\Big[\max_{0\le t\le 1}\|f(t,\cdot)\|_{L_2(\bar\Omega)}+\|f_t(1,\cdot)\|_{L_2(\bar\Omega)} \\ +\|\varphi\|_{L_2(\bar\Omega)}+\|\psi\|_{L_2(\bar\Omega)}+\|\xi\|_{L_2(\bar\Omega)}\Big], \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\max_{0\le t\le 1}\|u(t,\cdot)\|_{W_2^2[0,1]}+\max_{0\le t\le 1}\|\frac{\partial^3 u}{\partial t^3}(t,\cdot)\|_{L_2(\bar\Omega)} \\ &\le M_2\Big[\max_{0\le t\le 1}\|f_t(t,\cdot)\|_{L_2(\bar\Omega)}+\|f(0,\cdot)\|_{L_2(\bar\Omega)}+\|\varphi\|_{W_2^2(\bar\Omega)} \\ &\quad +\|\psi\|_{W_2^2(\bar\Omega)}+\|\xi\|_{W_2^2(\bar\Omega)}\Big], \end{aligned} \tag{3.8}$$

where M_2 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (3.6) can be written in abstract form (3.4) in Hilbert space $L_2(\bar\Omega)$ with self-adjoint positive definite operator $A=A^x$ defined by the formula

$$A^x u(x)=-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} \tag{3.9}$$

with domain

$$\begin{aligned} D(A^x)=\big\{u(x): u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2(\bar\Omega),\ 1\le r\le n, \\ u(x)=0, x\in S\big\}. \end{aligned}$$

Here $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are known and unknown respectively abstract functions defined on $[0,1]$ with the value in $H=L_2(\bar\Omega)$. So, estimates (3.7)-(3.8) follow from estimates (2.10)-(2.11) and from the coercivity inequality for the solution of the elliptic differential problem in $L_2(\bar\Omega)$. □

Theorem 3.3. *For the solution of the elliptic differential problem*

$$-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}=w(x),\quad x\in\Omega,\quad u(x)=0,\ x\in S$$

the coercivity inequality

$$\sum_{r=1}^{n}\|u_{x_rx_r}\|_{L_2(\overline{\Omega})}\le M\|w\|_{L_2(\overline{\Omega})}$$

is valid [26]*. Here M does not depend on $w(x)$.*

Third application. We consider the boundary-value problem for a third-order partial differential equation

$$\begin{gathered} \frac{\partial^3 u(t,x)}{\partial t^3}-\sum_{r=1}^{n}(a_r(x)u_{x_r}(t,x))_{x_r}+\delta u(t,x)=f(t,x), \\ x=(x_1,\dots,x_n)\in\Omega,\quad 0<t<1, \\ u(0,x)=\varphi(x),\quad u(1,x)=\psi(x),\quad u_t(1,x)=\xi(x),\quad x\in\bar\Omega, \\ \frac{\partial u}{\partial\vec n}(t,x)=0,\quad x\in S,\ 0\le t\le 1, \end{gathered} \tag{3.10}$$

where $a_r(x)$, $x\in\Omega$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $x\in\bar\Omega$ and $f(t,x)$ $(x\in[0,1])$, $x\in\Omega$ are given smooth functions and $a_r(x)>0$, $\vec n$ is the normal vector to S.

Theorem 3.4. *For the solution of* (3.10)*, the following two stability inequalities hold:*

$$
\begin{aligned}
\max_{0\le t\le 1}\|u(t,\cdot)\|_{L_2(\bar\Omega)} \le M_3\Big[&\max_{0\le t\le 1}\|f(t,\cdot)\|_{L_2(\bar\Omega)}+\|f_t(1,\cdot)\|_{L_2(\bar\Omega)}\\
&+\|\varphi\|_{L_2(\bar\Omega)}+\|\psi\|_{L_2(\bar\Omega)}+\|\xi\|_{L_2(\bar\Omega)}\Big],
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\max_{0\le t\le 1}\|u(t,\cdot)\|_{W_2^2[0,1]}+\max_{0\le t\le 1}\|\frac{\partial^3 u}{\partial t^3}(t,\cdot)\|_{L_2(\bar\Omega)}\\
&\le M_3\Big[\max_{0\le t\le 1}\|f_t(t,\cdot)\|_{L_2(\bar\Omega)}+\|f(0,\cdot)\|_{L_2(\bar\Omega)}+\|\varphi\|_{W_2^2(\bar\Omega)}\\
&\quad+\|\psi\|_{W_2^2(\bar\Omega)}+\|\xi\|_{W_2^2(\bar\Omega)}\Big],
\end{aligned}
\tag{3.12}
$$

where M_3 does not depend on $f(t,x)$ and $\varphi(x)$, $\psi(x)$, $\xi(x)$.

Proof. Problem (3.10) can be written in the abstract form (3.4) in the Hilbert space $L_2(\bar\Omega)$ with a self-adjoint positive definite operator $A=A^x$ defined by the formula

$$
A^x u(x) = -\sum_{r=1}^{m}(a_r(x)u_{x_r})_{x_r}+\delta u(x) \tag{3.13}
$$

with domain

$$
\begin{aligned}
D(A^x)=\Big\{&u(x): u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r}\in L_2(\bar\Omega),\ 1\le r\le m,\\
&\frac{\partial u}{\partial\vec n}=0,\ x\in S\Big\}.
\end{aligned}
$$

Here $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are respectively known and unknown abstract functions defined on $[0,1]$ with the value in $H=L_2(\bar\Omega)$. So, estimates (3.11)-(3.12) follow from estimates (2.10)-(2.11) and from the coercivity inequality for the solution of the elliptic differential problem in $L_2(\bar\Omega)$. □

Theorem 3.5. *For the solution of the elliptic differential problem*

$$
-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u(x)=w(x), x\in\Omega, \frac{\partial}{\partial\vec n}u(x)=0,\ x\in S
$$

the coercivity inequality

$$
\sum_{r=1}^{n}\|u_{x_r x_r}\|_{L_2(\overline\Omega)}\le M\|w\|_{L_2(\overline\Omega)}
$$

is valid [26]*. Here M does not depend on $w(x)$*

Conclusions. This article is devoted to the stability of the boundary value problem for a third order partial differential equation. Theorem on stability estimates for the solution of this problem is established. Three applications of the main theorem to a third order partial differential equations are given. Theorems on stability estimates for solutions of these partial differential equations are obtained.

In papers [9], [6], three step difference schemes generated by Taylor's decomposition on three points for the numerical solution of local and nonlocal boundary value problems of the linear ordinary differential equation of third order were investigated.

Note that Taylor's decomposition on four points is applicable for the construction of difference schemes of problem (1.1). Operator method of [10] permits to establish the stability of these difference problem for the approximation problem of (1.1).

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References

[1] Yu. P. Apakov; On the solution of a boundary-value problem for a third-order equation with multiple characteristics, *Ukrainian Mathematical Journal,* 64 (1) (2012), 1-12.
[2] Yu. P. Apakov, B. Yu. Irgashev; Boundary-value problem for a degenerate high-odd-order equation, *Ukrainian Mathematical Journal*, 66 (10) (2015), 1475-1490.
[3] Yu. P. Apakov, S. Rutkauskas; On a boundary value problem to third order PDE with multiple characteristics, *Nonlinear Analysis: Modelling and Control,* 16 (3) (2011), 255-269.
[4] M. Ashyraliyev; A note on the stability of the integral-differential equation of the hyperbolic type in a Hilbert space, *Numerical Functional Analysis and Optimization,* 29 (7-8)(2008), 750–769.
[5] A. Ashyralyev, N. Aggez; A note on the difference schemes of the nonlocal boundary value problems for hyperbolic equations, *Numerical Functional Analysis and Optimization,* 25 (5-6) (2004), 439-462.
[6] A. Ashyralyev, N. Aggez, F. Hezenci; Boundary value problem for a third order partial differential equation, *AIP Conference Proceedings*, 1470 (2012), 130-132.
[7] A. Ashyralyev, D. Arjmand; A note on the Taylor's decomposition on four points for a third-order differential equation, *Applied Mathematics and Computation,* 188 (2) (2007), 1483-1490.
[8] A. Ashyralyev, D. Arjmand, M. Koksal; Taylor's decomposition on four points for solving third-order linear time-varying systems, *Journal of the Franklin Institute Engineering and Applied Mathematics,* 346 (2009), 651-662.
[9] A. Ashyralyev, S. N. Simsek; Nonlocal boundary value problems for a third order partial differential equation, *AIP Conference Proceedings*, 1611 (1)(2014), 234-240.
[10] A. Ashyralyev, P. E. Sobolevskii; *New Difference Schemes for Partial Differential Equations*, Birkhauser Verlag, Basel, Boston, Berlin, 2004.
[11] A. Ashyralyev, O. Yildirim; On multipoint nonlocal boundary value problems for hyperbolic differential and difference equations, *Taiwanese Journal of Mathematics,* 14 (1) (2010), 165-194.
[12] Kh. Belakroum, A. Ashyralyev, A. Guezane-Lakoud; A note on the nonlocal boundary value problem for a third order partial differential equation, *AIP Conference Proceedings,* 1759, 020021 (2016), http://dx.doi.org/10.1063/1.4959635.
[13] M. Denche, A. Memou; Boundary value problem with integral conditions for a linear third-order equation, *J. Appl. Math,* 11 (2003), 533-567.
[14] Z. Direk, M. Ashyraliyev; FDM for the integral-differential equation of the hyperbolic type, *Advances in Difference Equations*, 2014 (2014), DOI:10.1186/1687-1847-2014-132.
[15] H. O. Fattorini; *Second Order Linear Differential Equations in Banach Spaces*, Elsevier Science Publishers B.V., Amsterdam, 1985.
[16] S. A. Gabov, A. G. Sveshnikov; *Problems of the Dynamics of Stratified Fluids*, Nauka, Moscow, 1986. (in Russian).
[17] A. Guezane-Lakoud, N. Hamidane, R. Khaldi; On a third-order three-point boundary value problem, *International Journal of Mathematics and Mathematical Sciences,* 2012 (2012), Article ID 513189, 7 pages.
[18] T. S. Kalmenov, D. Suragan; Initial-boundary value problems for the wave equation, *Electronic Journal of Differential Equations,* 2014, no. 48 (2014): 1-6.
[19] A. I. Kozhanov; Mixed problem for one class of quasilinear equation of third order, *Boundary Value Problems for Nonlinear Equations*, Novosibirsk, 1982, pp. 118-128. (in Russian).
[20] A. I. Kozhanov; Mixed boundary value problem for some classes of third order differential equations, *Mathematics of the USSR* -Sbonik, 118 (4) (1982), 507-525. (in Russian).
[21] A. I. Kozhanov, L. S. Pulkina; On the solvability of boundary value problems with a nonlocal boundary condition of integral form for multidimensional hyperbolic equations, *Differential Equations,* 42 (9) (2006), 1233-1246. DOI: 10.1134/S0012266106090023.
[22] M. Kudu, I. Amirali; Method of lines for third order partial differential equations, *Journal of Applied Mathematics and Physics,* 2 (2) (2014), 33-36.

[23] Jing Niu, Ping Li; Numerical algorithm for the third-order partial differential equation with three-point boundary value problem, *Abstract and Applied Analysis*, 45 (2014),1-7. DOI: 10.1155/2014/630671.

[24] L. S. Pulkina; Initial-boundary value problem with a nonlocal boundary condition for a multidimensional hyperbolic equation, *Differential Equations,* 44 (8) (2008), 1119-1125. DOI: 10.1134/S0012266108080090.

[25] A. L. Skubachevskii; Boundary value problems for elliptic functional-differential equations and their applications, *Russian Mathematical Surveys*, 71(5)(431) (2016), 3–112. DOI: http://dx.doi.org/10.4213/rm9739 (in Russian).

[26] P.E. Sobolevskii; *Difference Methods for the Approximate Solution of Differential Equations*, Voronezh State University Press, Voronezh, Russia, 1975. (in Russian).

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