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GLOBAL AND LOCAL VERSIONS FOR A PHÓNG VŨ THEOREM FOR PERIODIC EVOLUTION FAMILIES IN HILBERT SPACES

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Abstract. A theorem of Gearhart concerning strongly continuous semigroups in Hilbert spaces is extremely useful for stability analysis of concrete equations; see e.g. [20]), and for control theory [27] or [13, page 475]. Phóng Vũ introduced an equivalent condition in [23]. The aim of this article is to extend these results from the autonomous case to time dependent 1-periodic evolution equations in Hilbert spaces. Both cases (continuous and discrete) are analyzed and global and local versions of the Phóng Vũ theorem are provided.

1. Introduction

The following result is well known.

Theorem 1.1. *For any strongly continuous semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ that acts on a complex Hilbert space $\mathcal{H}$ the following three statements are equivalent:*

(1) $\mathbf{T}$ *is uniformly exponentially stable i.e. its growth bound*

$$\omega_0(\mathbf{T}) := \inf_{t\geq 0} \frac{\ln \|T(t)\|}{t} < 0. \tag{1.1}$$

(2) *For every vector b in $\mathcal{H}$ the solutions of the following Cauchy problems associated to the generator A of $\mathbf{T}$*

$$\begin{gathered} u'(t) = Au(t) + e^{i\mu t}b, \quad t \geq 0,\ \mu \in \mathbb{R} \\ u(0) = 0 \end{gathered} \tag{1.2}$$

are bounded on $\mathbb{R}_+$ (uniformly with respect to the parameter μ), or equivalently

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \| \int_0^t e^{i\mu s} T(t-s)x ds\| := M(x) < \infty \quad \forall x \in \mathcal{H}. \tag{1.3}$$

(3) *The resolvent operator $R(z, A)$ exists for $z \in \mathbb{C}_+ := \{\Re(z) > 0\}$ and*

$$\sup_{z\in\mathbb{C}_+} \|R(z, A\| < \infty. \tag{1.4}$$

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The proof of (2) $\Rightarrow$ (3) was given by Phóng Vũ in [23] and the implication (3) $\Rightarrow$ (1) is known as the Gearhart-Prüss Theorem, [15, 24]. Different proofs of this result were given in the literature by many authors. For example, the proof in [27] relies on the strong connection of this theorem and the half plane version of an old theorem of Paley and Wiener. The Gearhart-Prüss Theorem is not valid for semigroups acting on arbitrary Banach spaces, see for example [17] for a counterexample.

In [23] the equivalence between (1.3) and the negativeness of $\omega_0(\mathbf{T})$ was settled. We mention that the proof depends of the implication (3) $\Rightarrow$ (1). For information concerning strongly continuous semigroups we refer the reader to the monographs [13, 16, 18, 22] and the references therein.

Let X be a complex Banach space and denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms in X and in $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. Recall that a family $\mathcal{U} = \{U(t,s) : (t,s) \in \mathbb{R}^2, t \geq s\}$ of bounded linear operators acting on X is called a one periodic strongly continuous evolution family if $U(t,t) = I$- the identity operator on X, $U(t,s)U(s,r) = U(t,r)$ for all reals $t \geq s \geq r$, the map $(t,s) \mapsto U(t,s)$ is strongly continuous on the set $\Delta := \{(t,s) \in \mathbb{R}^2 : t \geq s\}$ and $U(t,s) = U(t+1, s+1)$ for every pair $(t,s) \in \Delta$. The evolution family $\mathcal{U}$ is called exponentially bounded if there exists a real number ω such that the map $(t,s) \mapsto e^{-\omega(t-s)}U(t,s)$ is bounded in $\mathcal{L}(X)$ and uniformly exponentially stable if there exists a negative ω with that property. Throughout the paper we assume that the evolution families are exponentially bounded. Clearly, if $\mathcal{U}$ verifies the convolution condition $U(t,s) = U(t-s,0)$ for every $(t,s) \in \Delta$ then the family $\mathbf{T} = \{U(t,0) : t \geq 0\}$ is a strongly continuous semigroup that acts on X. Thus it is natural to ask if the family $\mathcal{U}$ is uniformly exponentially stable provided (1.3) holds when $U(t,s)$ replaces $T(t-s)$. The proof of such a result cannot follow the ideas of Vũ because in the non-autonomous case an analogue for the implication (3) $\Rightarrow$ (1) does not exist. As expected, additional assumptions are required. The following result was obtained in [4].

Theorem 1.2. *A strongly continuous one periodic evolution family $\mathcal{U}$ of bounded linear operators acting on a Hilbert space $\mathcal{H}$ is uniformly exponentially stable provided that for every $x \in X$ the map $t \mapsto U(t,0)x$ satisfies a Lipschitz condition on $(0,1)$ and (1.3) holds when $U(t,s)$ replaces $T(t-s)$.*

Let X be a complex Banach space and $\alpha \in (0,1]$. Recall that a X-valued function f defined on a real closed interval $[a,b]$ is called α-Hölder continuous with the constant L_f if

$$\|f(t) - f(s)\| \leq L_f|t-s|^\alpha \quad \forall t,s \in [a,b]. \tag{1.5}$$

The next Lemma collects some facts concerning α-Hölder continuity. Its proof is obvious so we omit the details. As usual by $\|f\|_\infty$ we denote the uniform norm of f in the space $C([a,b],X)$.

Lemma 1.3. *Let $h{:}[a,b] \to \mathbb{C}$ and $H{:}[a,b] \to X$ be α-Hölder continuous functions on the closed interval $[a,b]$ with the constants L_h and L_H, respectively. Then one has:*

(i) *The map hH is α-Hölder continuous on $[a,b]$ with the constant $H_\infty L_h + h_\infty L_H$.*
(ii) *H is β-Hölder continuous for every $\beta \in (0,\alpha]$ (with the same constant).*

(iii) *The function* $t \mapsto \sin(\pi t) : [0,1] \to \mathbb{C}$ *is Lipschitz continuous (i.e. 1-Hölder continuous) with the constant* π.

(iv) *If, in addition,* $H(a) = H(b)$ *then its extension by periodicity to* $\mathbb{R}$ *is* α-*Hölder continuous with the constant* L_H.

Lemma 1.4. *Let* $\mathcal{U} = \{U(t,s), t \geq s \geq 0\}$ *be a strongly continuous 1-periodic evolution family of bounded linear operators acting on a complex Banach space* X *and let* $h(t) := \sin(\pi t)$, $t \in [0,1]$. *For each* $x \in X$ *set*

$$H_{xh}(t) := h(t)U(t,0)x, \quad t \in [0,1] \tag{1.6}$$

and denote by $\tilde{H}_{xh}$ *the extension by periodicity of* H_{xh} *to the entire real axis. Then (clearly)* $I(h,\mu) := \int_0^1 e^{i\mu t}h(t)dt \neq 0$ *for every* $\mu \in \mathbb{R}$. *Let* $T := U(1,0)$. *Then for any positive integer* n, *one has*

$$\sum_{j=1}^{n} e^{-i\mu j}T^j x = \frac{e^{-\mu jn}}{I(h,\mu)} \int_0^n e^{i\mu s}U(n,s)\tilde{H}_{xh}(s)ds. \tag{1.7}$$

Note that for every integer k one has

$$I(h,0) = \frac{2}{\pi}, I(h,(2k+1)\pi) = \pm\frac{1}{2}i \quad \text{and} \quad I(h,\mu) = \frac{\pi}{\pi^2 - \mu^2}(1 + e^{i\mu}), \tag{1.8}$$

elsewhere. To obtain (1.7) it is sufficient to write the integral in the right-hand side of (1.7) as

$$\sum_{k=0}^{n-1} \int_k^{k+1} e^{i\mu s}U(n,s)\tilde{H}_{xh}(s)ds. \tag{1.9}$$

and then to change the variable in the last integral.

Remark 1.5. With the notation from Lemma 1.4 assume that the map $U(\cdot,0)x$ is 1-Hölder continuous on $[0,1]$ with the constant $L(x)$. Then $\tilde{H}_{xh}$ is 1-Hölder continuous on $\mathbb{R}$ with the constant

$$L(\tilde{H}_{xh}) := \pi \|1_{[0,1]}(\cdot)U(\cdot,0)x\|_\infty + L(x), \tag{1.10}$$

where $1_{[0,1]}(\cdot)$ denotes the characteristic function of the interval $[0,1]$. In the particular case when the family $\{U(t,0) : t \geq 0\}$ is a strongly continuous semigroup generated by A and $x \in D(A)$, we can take $L(x) := \sup_{t\in[0,1]} ||U(t,0)|| \cdot ||Ax||$ and then there exists a positive c (independent of x) such that

$$L(\tilde{H}_{xh}) \leq c(||x||^2 + ||Ax||^2)^{\frac{1}{2}}. \tag{1.11}$$

Denote by $CP_1(\mathbb{R},X)$ the set of all X-valued continuous functions f defined on $\mathbb{R}$ which are periodic of period 1, i.e. $f(t+1) = f(t)$ for every $t \in \mathbb{R}$. The vectors in X

$$c_n(f) := \int_0^1 e^{2int\pi}f(t)dt, \quad n \in \mathbb{Z} \tag{1.12}$$

are called the Fourier-Bohr coefficients associated to f.

With $CP_1^1(\mathbb{R},X)$ we will denote the subset of $CP_1(\mathbb{R},X)$ consisting by all functions f for which

$$\|f\|_1 := \sum_{n\in\mathbb{Z}} \|c_n(f)\| < \infty. \tag{1.13}$$

Further details concerning the space of all continuous functions satisfying (1.13) (not necessarily periodic) can be found in the monograph [11] by Corduneanu.

For each $f \in CP_1^1(\mathbb{R}, X)$ consider its Fourier sum (in respect with the uniform norm) given by

$$s_f(t) = \sum_{n\in\mathbb{Z}} e^{-2int\pi} c_n(f), \quad t \in \mathbb{R}. \tag{1.14}$$

Clearly $s_f \in CP_1(\mathbb{R}, X)$ and thus $f = s_f$. The next lemma highlights an important subset of $CP_1^1(\mathbb{R}, X)$ when X is a complex Hilbert space.

Lemma 1.6. *Let $\mathcal{H}$ be a complex Hilbert space and let $f \in C_1(\mathbb{R}, \mathcal{H})$. If f is α-Hölder continuous on $[0, 1]$ for some $\alpha > \frac{1}{2}$ with a constant $L = L(f) > 0$, i.e.*

$$\|f(t) - f(s)\| \le L|t - s|^\alpha, \quad \forall t, s \in [0, 1]\,. \tag{1.15}$$

Then $f \in CP_1^1(\mathbb{R}, X)$, i.e.

$$f(t) = \sum_{n\in\mathbb{Z}} e^{-2int\pi} c_n(f), \quad t \in \mathbb{R}. \tag{1.16}$$

In addition, there exists a positive constant K depending only of α such that.

$$\|f\|_1 \le 3\|f\|_\infty + LK(\alpha). \tag{1.17}$$

Proof. Let $\rho > 0$ and set $g(t) := f(t + \rho) - f(t - \rho)$. A simple calculation shows that

$$c_n(g) = -2i \sin(2n\pi\rho) c_n(f), \tag{1.18}$$

$$4^\alpha \rho^{2\alpha} L^2 \ge \int_0^1 \|g(t)\|^2 dt = \sum_{n\in\mathbb{Z}} \|c_n(g)\|^2. \tag{1.19}$$

Let p be any positive integer and $\mathcal{A}_p := \{n \in \mathbb{Z} : 2^{p-1} < |n| \le 2^p\}$. For $\rho := \frac{1}{2^{p+2}}$ and $n \in \mathcal{A}_p$ one has $\frac{\pi}{4} < 2|n|\pi\rho \le \frac{\pi}{2}$ and then

$$2 < 4\sin^2(2n\pi\rho). \tag{1.20}$$

Thus, since $\operatorname{card}(\mathcal{A}_p) = 2^{p-1}$, the usual Hölder inequality yields

$$\Big(\sum_{n\in\mathcal{A}_p} \|c_n(f)\|\Big)^2 \le 2^{p-1} \sum_{n\in\mathcal{A}_p} \|c_n(f)\|^2. \tag{1.21}$$

Now (1.18), (1.19), (1.20) and (1.21) yield

$$2\Big(\sum_{n\in\mathcal{A}_p} \|c_n(f)\|\Big)^2 \le 2^{p-1} \sum_{n\in\mathcal{A}_p} \|c_n(g)\|^2 \le 2^{p-1}\rho^{2\alpha} 2^{2\alpha} L^2. \tag{1.22}$$

Thus

$$\sum_{p=1}^\infty \sum_{n\in\mathcal{A}_p} \|c_n(f)\| \le 2^{-1} L \sum_{p=1}^\infty 2^{(\frac{1}{2}-\alpha)p} = LK(\alpha). \tag{1.23}$$

Finally we obtain the estimate $\|f\|_1 \le 3\|f\|_\infty + LK(\alpha)$. □

Lemma 1.7 ([7])**.** *Let T be a bounded linear operator acting on a complex Banach space X. If for every real number μ one has*

$$\sup_{n\in\mathbb{Z}_+} \|\sum_{j=1}^n e^{-i\mu j} T^j\| := M(\mu) < \infty, \tag{1.24}$$

then the spectral radius of T is less than 1.

2. Main results

The following result was stated in [8, [Theorem 2.1].

Theorem 2.1. *Let $\mathcal{U} = \{U(t,s) : t \geq s\}$ be a strongly continuous and 1- periodic evolution family of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Assume that:*

1. *For every $x \in \mathcal{H}$ the trajectory $U(\cdot,0)x$ is Lipschitz continuous on $(0,1)$, and*
2. $$\sup_{\mu\in\mathbb{R}} \sup_{t\geq s\geq 0} \|\int_\tau^t e^{i\mu s} U(t,\tau)x d\tau\| := M(x) < \infty. \tag{2.1}$$

Then the family $\mathcal{U}$ is uniformly exponentially stable.

In our first result (using a different approach) we show that both assumptions in Theorem 2.1 can be relaxed.

Theorem 2.2. *Let $\mathcal{U} = \{U(t,s) : t \geq s\}$ be a strongly continuous and 1-periodic evolution family of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Assume that for some $\alpha > \frac{1}{2}$ and every $x \in \mathcal{H}$ the trajectory $U(\cdot,0)x$ is α-Hölder continuous on $(0,1)$ with the constant $L(x)$ and*

$$\sup_{\mu\in\mathbb{R}} \sup_{t\geq 0} \|\int_0^t e^{i\mu s} U(t,s)x ds\| := M(x) < \infty. \tag{2.2}$$

Then the family $\mathcal{U}$ is uniformly exponentially stable.

Proof. From the Uniform Boundedness Principle there exists a positive constant M such that $M(x) \leq M\|x\| < \infty$. We use again the notations in Lemma 1.4. The map $\tilde{H}_{xh}$, given in (1.6), can be represented as in (1.16) and in view of Lemma 1.3 and then of Lemma 1.6 it belongs to $CP_1^1(\mathbb{R}, X)$. Now for any positive integer n, using (1.7), (1.16), (1.23), (1.10) and (2.1), we obtain

$$\begin{aligned}
\|\sum_{j=1}^n e^{-i\mu j} T^j x\| &\leq \frac{1}{|I(h,\mu)|} \|\int_0^n e^{i\mu s} U(n,s) \tilde{H}_{xh}(s) ds\| \\
&\leq \frac{1}{|I(h,\mu)|} \sum_{k\in\mathbb{Z}} \|\int_0^n e^{i(\mu-2k\pi)s} U(n,s) c_k(\tilde{H}_{xh}) ds\| \\
&\leq \frac{M}{|I(h,\mu)|} \|\tilde{H}_{xh}\|_1 \\
&\leq \frac{M}{|I(h,\mu)|} (3\|1_{[0,1]}(\cdot)U(\cdot,0)x\|_\infty + K(\alpha)L(\tilde{H}_{xh})),
\end{aligned}$$

where $L(\tilde{H}_{xh})) = \pi\|1_{[0,1]}(\cdot)U(\cdot,0)x\|_\infty + L(x)$. The assertion follows using Lemma 1.7. □

The following particular case of Theorem 2.2 is an extension of [3, Theorem 3.2]] where only $\mathbb{C}^n$-valued functions was considered.

Corollary 2.3. *Let $\mathcal{U} = \{U(t,s) : t \geq s\}$ be a strongly continuous and 1-periodic evolution family of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. Assume that for every $x \in \mathcal{H}$ the map $U(\cdot,0)x$ is differentiable on $(0,1)$ and there* *exist a family of bounded linear operators* $\{A(t) : t \in [0,1]\}$ *that acts on* $\mathcal{H}$ *such that the map* $A(\cdot)U(\cdot,0)x$ *is bounded on* $(0,1)$ *and*

$$\frac{d}{dt}[U(t,0)x] = A(t)U(t,0)x, \quad \forall t \in (0,1), \quad \forall x \in \mathcal{H}. \tag{2.3}$$

Then (2.1) *holds true if and only if the evolution family* $\mathcal{U}$ *is uniformly exponentially stable.*

Proof. From (2.3) follows that the map $A(\cdot)U(\cdot,0)x$ is measurable. Thus for $t, s \in (0,1)$ one has

$$\begin{aligned}\|U(t,0)x - U(s,0)x\| &= \|\int_s^t A(\tau)U(\tau,0)x d\tau\| \\ &\le |t-s| \|1_{[0,1]}(\cdot)A(\cdot)U(\cdot,0)x\|_\infty,\end{aligned}$$

which shows that $U(\cdot,0)x$ is 1-Hölder continuous on $[0,1]$. Now we can apply Theorem 2.2 to complete the proof. □

Remark 2.4. Assume that the map $t \mapsto A(t)$ is continuous in the uniform operator topology of $\mathcal{L}(\mathcal{H})$ and that $A(0) = A(1)$ and let $\tilde{A}(\cdot)$ be the extension by periodicity of $A(\cdot)$ to the entire real axis. We can associate with the operator family $\{\tilde{A}(t) : t \in \mathbb{R}\}$ an evolution family $\mathcal{U}$ which satisfies all the assumptions in Corollary 2.3.

We also refer the reader to [22, Theorem 5.1]]. The next example considers the well-posedness in Corollary 2.3. A nice text concerning well-posedness which addresses in particular the framework of this paper provided by Schnaubelt is [13, pages 477-496], see also [12].

Example 2.5. Let $(\mathcal{H}, \langle , \rangle)$ be a separable complex Hilbert space and let $B = \{b_n, n = 1, 2, \dots\}$ be an orthonormal basis of it, i.e. $\langle b_n, b_m \rangle = 0$ when $m \neq n$, $\|b_n\|^2 := \langle b_n, b_n \rangle = 1$ and the linear span of B is dense in $\mathcal{H}$. Thus any $x \in \mathcal{H}$ can be represented uniquely as $x = \sum_{n=1}^{\infty} \langle x, b_n \rangle b_n$. Let (λ_n) be a sequence of real numbers with $\lambda_n \le -1$ for every positive integer n. For every $t \ge 0$ and every $x \in \mathcal{H}$ let

$$T(t)x := \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x, b_n \rangle b_n. \tag{2.4}$$

It is well-known, [28], and easy to prove that:

1. The family $\mathbf{T} := \{T(t) : t \ge 0\}$ is a strongly continuous semigroup that acts on $\mathcal{H}$.
2. $\omega_0(\mathbf{T}) = \sup_{n \ge 1} \lambda_n \le -1$.
3. $D(A) = \{x \in \mathcal{H} : \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, b_n \rangle|^2 < \infty\}$
4. $Ax := \sum_{n=1}^{\infty} \lambda_n \langle x, b_n \rangle b_n$ for all $x \in D(A)$.
5. The resolvent set of A, i. e. $\rho(A)$ consist by all complex numbers z for which $\inf_{n \ge 1} |z - \lambda_n| > 0$. In particular, $\{\lambda \in \mathbb{C} : \Re(\lambda) \ge 0\} \subset \rho(A)$.
6. $R(\lambda, A) = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle x, b_n \rangle b_n$, for all $\lambda \in \rho(A)$. Clearly,

$$\|R(\lambda, A)\| \le \frac{2}{|\lambda| + 1} \quad \forall \lambda \in \mathbb{C}, \text{ with } \Re(\lambda) \ge 0. \tag{2.5}$$

Let $a : \mathbb{R} \to [1, \infty)$ be a 1-periodic function having the property that there exists two constants $c > 0$ and $\alpha \in (0,1]$ such that

$$|a(t) - a(s)| \le c|t-s|^\alpha, \quad \forall t, s \in [0,1]. \tag{2.6}$$

Set $A(t) := a(t)A$ for all $t \in [0,1]$. Clearly $D(A(t)) = D(A)$ for all $t \in [0,1]$ and

$$\|R(\lambda, A(t))\| \le \frac{2}{|\lambda|+1} \quad \forall \lambda \in \mathbb{C}, \text{ with } \Re(\lambda) \ge 0. \tag{2.7}$$

Further, in view of (2.6) one has

$$\|(A(t)-A(s))A^{-1}(\tau)\| = \frac{1}{a(\tau)}|a(t)-a(s)| \le c|t-s|^{\alpha}, \quad \forall t,s \in [0,1]. \tag{2.8}$$

Thus, the family

$$U(t,s)x := T\Big(\int_s^t a(\tau)d\tau\Big)x, \quad 0 \le s \le t \le 1, x \in \mathcal{H} \tag{2.9}$$

solves the non-autonomous Abstract Cauchy Problem

$$u'(t) = A(t)u, \quad 0 \le s < t \le 1, \quad u(s) = x \in \mathcal{H}. \tag{2.10}$$

Clearly, the map $t \mapsto A(t)U(t,0)x$ is bounded on $(0,1)$ for every $x \in D(A)$. Further, if $\sum_{n=1}^{\infty} \frac{-1}{\lambda_n} < \infty$ then arguing exactly as in [4, Example 4.1] we can show that (2.1) holds for the 1-periodic evolution family $\{U(t,s) : t \ge s \ge 0\}$ obtained from (2.9) with extension by periodicity.

Let A be a bounded linear operator acting on a complex Banach space X. The following two statements are equivalent:

(i) There are the positive constants N and ν such that $\|e^{tA}\| \le e^{-\nu t}$ for all $t \ge 0$.
(ii) The solution of the Cauchy Problem $\dot{u}(t) = Au(t) + e^{i\mu t}$, $t \ge 0$, $\quad u(0) = 0$ is bounded on $[0,\infty)$ for each real number μ.

For a history of this result and its comparison with those in the non-autonomous case we refer the reader to [25], when further references can be found. The next Theorem shows that the result remains true for evolution semigroups acting on $CP_1(\mathbb{R}, X)$.

Evolution semigroups, was introduced in 1974 by Howland [19] and studied by Evans [14]. The modern theory of evolution semigroups was initiated in 1995 with the seminal paper of Latushkin and Montgomery-Smith [21]. For comprehensive information concerning evolution semigroups we refer the reader to [10, 26] and the references therein. Applications of the evolution semigroup theory to inequalities concerning the growth bound of evolution families of operators acting in Banach spaces are offered in [9].

Let X be a complex Banach space and $\mathcal{U}$ be a 1-periodic strongly continuous semigroup acting on X. Then the operator $\mathcal{T}(t)$ given by

$$(\mathcal{T}(t)f)(s) = U(s, s-t)f(s-t), \quad s \in \mathbb{R}, t \ge 0, \ f \in CP_1(\mathbb{R}, X) \tag{2.11}$$

is well defined and acts on $CP_1(\mathbb{R}, X)$.

Theorem 2.6. *A 1-periodic strongly continuous evolution family $\mathcal{U} = \{U(t,s) : t \ge s \in \mathbb{R}\}$ is uniformly exponentially stable if and only if for each $\mu \in \mathbb{R}$ and each $f \in CP_1(\mathbb{R}, X)$ one has*

$$\sup_{t\ge 0} \Big\| \int_0^t e^{i\mu s}\mathcal{T}(t-s)f ds\Big\|_{CP_1(\mathbb{R},X)} := N(\mu, f) < \infty \quad . \tag{2.12}$$

Proof. Obviously if the family $\mathcal{U}$ is uniformly exponentially stable then the evolution semigroup $\mathcal{T}$ associated to $\mathcal{U}$ on $CP_1(\mathbb{R}, X)$ is uniformly exponentially stable which yields (2.12). The proof of the converse statement is a little bit more difficult. Let $N = N(\mu) > 0$ be such that $N(\mu, f) \le N\|f\|_\infty$. For any $t \ge 0$, (2.12) yields

$$\|\Big(\int_0^t e^{i\mu s}\mathcal{T}(t-s)f ds\Big)(t)\| = \|\int_0^t e^{i\mu s}U(t,s)f(s)ds\| \le N\|f\|_\infty. \tag{2.13}$$

Now, writing the latter inequality for $t = n \in \mathbb{Z}_+$ and with f replaced by $\tilde{H}_{xh}$ (the extension by periodicity to the entire axis of the map H_{xh} defined in Lemma 1.4) and taking into account (1.7) we obtain

$$\begin{aligned}\|\sum_{j=1}^n e^{-i\mu j}T^j x\| &\le \frac{1}{|I(h,\mu)|}\|\int_0^n e^{i\mu s}U(n,s)\tilde{H}_{xh}(s)ds\| \\ &\le \frac{N}{|I(h,\mu)|}\|1_{[0,1]}(\cdot)U(\cdot,0)x\|_\infty.\end{aligned}$$

Since the above estimates hold for all $x \in X$ we can apply Lemma 1.7 to finish the proof. $\square$

3. Discrete case

Let $q \ge 2$ be a integer number, X be a complex Banach space and let $\mathcal{U} = \{U(n,m) : n \ge m \ge 0\} \subset \mathcal{L}(X)$ be a q-periodic discrete evolution family on X, that is $U(n,n)x = x, U(m,n)U(n,r) = U(m,r)$ and $U(n+q, m+q) = U(n,m)$ hold for all nonnegative integers m, n, r with $m \ge n \ge r$ and all $x \in X$. We denote by $T_q := U(q, 0)$ the monodromy operator associated to the evolution family $\mathcal{U}$. As is well known the family $\mathcal{U}$ is uniformly exponentially stable, that is, there exists the positive constants N and ν such that

$$\|U(n,m)\|_{\mathcal{L}(X)} \le Ne^{-\nu(n-m)} \quad \text{for all } n \ge m$$

if and only if the spectral radius of T_q

$$r(T_q) := \lim_{k\to\infty} \|T_q^k x\|^{\frac{1}{k}},$$

is less than 1.

Theorem 3.1. *Let T be a bounded linear operator acting on a Banach space X. If for a given $x \in X$ one has*

$$\sup_{\mu\in\mathbb{R}}\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^n e^{i\mu k}T^{n-k}x\| =: K(x) < \infty, \tag{3.1}$$

then

$$T^n x \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

Proof. Consider the sequence of holomorphic functions

$$h_n^x(z) := \sum_{j=1}^n \frac{T^j x}{z^{j+1}}, \quad |z| > \frac{1}{2}.$$

From the classical Abel-Dirichlet Theorem in Banach spaces, the sequence $(h_n^x(z))$ converges for each $z \in \mathbb{C}$ with $|z| > 1$. Moreover, when $|z| > \|A\| + 1$ we have that $h_n^x(z) \to (zI - T)^{-1}x$ as $n \to \infty$ and $\|h_n^x(z)\| \le \|T\|\|x\|$ for every $n \in \mathbb{Z}_+$. Then from the Phragmen-Lindelöf Theorem the sequence $(h_n^x(z))$ is uniformly bounded on $\{|z| \geq 1\}$ with the bound $\max\{K(x), \|A\|\|x\|\}$. The Vitali theorem ([1, Theorem 3.1], [18, Theorem 3.14.1]) assures us that $(h_n^x(z))_n$ converges for each z with $|z| \geq 1$. Thus $T^j x \to 0$ as $j \to \infty$ as well. □

Remark 3.2. (1) If in addition T is power bounded, i.e., $\sup_{n\in\mathbb{Z}_+} \|T^n\| < \infty$, and (3.1) is satisfied for x in a dense linear subspace D of X then T is strongly stable, i.e. (3.2) holds for all x in X.

(2) Let $b \in X$ be a given vector and let $\mathcal{T}(b)$ be the trajectory of T generated of b, i.e. the set $\{T^n b : n \in \mathbb{Z}_+\}$. With X_b we denote the smallest closed subspace of X which contains $\mathcal{T}(b)$. If for each $\tilde{b} \in X_b$ one has

$$\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k} T^{n-k}\tilde{b}\| := K(\tilde{b}) < \infty, \tag{3.3}$$

then every trajectory of T starting from X_b is exponentially stable, i.e. for every $x \in X_b$ there exist $\nu > 0$ and $N > 0$ such that

$$\|T^n x\| \leq N e^{-\nu n}\|x\|.$$

Indeed X_b is an invariant subspace of T (i.e. $T(X_b) \subseteq X_b$) and the assertion follows by applying Lemma 1.7 to the restriction of the operator T to X_b.

Theorem 3.3. *Let $q \geq 2$ be an integer, $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{U} = \{U(n,m) : n \geq m \geq 0\}$ be a q-periodic evolution family of bounded linear operators acting on $\mathcal{H}$. For a given nonzero vector $b \in \mathcal{H}$, let*

$$\mathcal{H}_b := \overline{\operatorname{span}}\{b, U(1,0)b, U(2,0)b, \dots, U(q,0)b\}. \tag{3.4}$$

If $\mathcal{H}_b$ is an invariant subspace for all operators $U(1,0), U(2,0), \dots, U(q,0)$ and if there exists an absolute positive constant K such that

$$\sup_{\mu\in\mathbb{R}} \sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k} U(n,k)v\| \leq K\|v\| \quad \forall v \in \mathcal{H}_b, \tag{3.5}$$

then for every $\tilde{b} \in \mathcal{H}_b$, the trajectory $U(\cdot,0)\tilde{b}$ is uniformly exponentially stable, that is there exist the positive constants N and ν such that $\|U(n,0)\tilde{b}\| \leq N^{-\nu n}\|\tilde{b}\|$ for all integers n.

Proof. **Step 1.** We prove that for every $(\mathcal{H}_b)$-valued, q-periodic sequence $w = (w_j)$, with $w_0 = 0$, one has

$$\sup_{\mu\in\mathbb{R}} \sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k} U(n,k)w_k\| := M(w) < \infty. \tag{3.6}$$

Let f be the continuous function verifying $f(j) = w_j$ for every $j = 0, 1, \dots, q-1, q$ and such that f is linear on each of the intervals $[0,1], [1,2], \dots, [q-1,q]$. The function is $(\mathcal{H}_b)$-valued since it is piecewise linear and satisfies a Lipschitz condition on $[0,q]$. We denote by $\tilde{f}$ its extension by periodicity to the entire real axis which satisfies a Lipschtiz condition on $\mathbb{R}$, as well. Taking into account that $\mathcal{H}_b$ is a complex Hilbert space and arguing as in Lemma 1.6, we obtain

$$w_j = \sum_{n\in\mathbb{Z}} e^{2\pi i n \frac{j}{q}} c_n(f), \tag{3.7}$$

where the Fourier coefficients associated to f, given by

$$c_n(f) := \frac{1}{q}\int_0^q e^{-2i\pi n\frac{t}{q}} f(t)dt, \tag{3.8}$$

belong to $\mathcal{H}_b$. In addition, one has

$$\sum_{n\in\mathbb{Z}} \|c_n(f)\| := c(w) < \infty.$$

Now, (3.5) and (3.7) yield

$$\begin{aligned}
&\sup_{\mu\in\mathbb{R}}\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k} U(n,k) w_k\| \\
&= \sup_{\mu\in\mathbb{R}}\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k} U(n,k) \sum_{j\in\mathbb{Z}} e^{2\pi i j\frac{k}{q}} c_j(f)\| \le Kc(w).
\end{aligned}$$

Hence (3.6) is satisfied with $M(w) := Kc(w)$.

Step 2. We show that all the hypothesis in Lemma 1.7 are fulfilled when $\mathcal{H}_b$ replaces X. Indeed, let $v \in \mathcal{H}_b$ be randomly chosen and let w be the q-periodic sequence defined by

$$w_k := k(q-k)U(k,0)v, \quad k = 0, 1, \dots, q-1.$$

By assumption (w_k) is an $(\mathcal{H}_b)$-valued sequence. As in [6, Theorem 2] we can show that (1.24) holds with $(\mathcal{H}_b)$ instead of X and T_q instead of T (The fact that $(\mathcal{H}_b)$ is an invariant subspace of T_q was used as well). Thus from Lemma 1.7, the spectral radius of the restriction of T_q to $(\mathcal{H}_b)$ is less than 1 and we obtain the conclusion. □

The following example shows that the uniformity of boundedness in respect to the parameter μ in (3.5) cannot be removed.

Example 3.4. Let $\mathcal{U}$ be a 2-periodic evolution family on $X = \mathbb{C}^2$ (endowed with the usual inner product) given by

$$U(1,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}, \quad U(2,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}, \quad U(2,1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.9}$$

Set $b = (0,1)^T$. Thus $U(1,0)b = -b$ and $U(2,0)b = b$ and $\mathcal{H}_b = \{\lambda b, \lambda \in \mathbb{C}\}$ is an invariant subspace for the operators $U(1,0)$ and $T_2 := U(2,0)$. Moreover, for each real number μ one has

$$\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{2n} e^{i\mu k} U(2n,k)b\| = \sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{2i\mu k} T_q^{n-k}(U(2,1) + e^{i\mu} b)\| < \infty. \tag{3.10}$$

See [2, Example 1] for further details. On the other hand

$$\sum_{k=0}^{2n+1} e^{i\mu k} U(2n+1,k)b = e^{i\mu(2n+1)} + U(1,0)\sum_{k=0}^{2n} e^{i\mu k} U(2n,k)\cdot b. \tag{3.11}$$

Hence, in view of (3.10) the sequence

$$\Big(\sum_{k=0}^{2n+1} e^{i\mu k} U(2n+1,k)b\Big)_n \tag{3.12}$$

is bounded as well. Finally for a certain $0 < M(\mu) < \infty$ we have

$$\sup_{n\in\mathbb{Z}_+} \|\sum_{k=0}^{n} e^{i\mu k}U(n,k)b\| \le M(\mu)\|b\| < \infty. \tag{3.13}$$

and $T_2^n b = \pm b$ for all $n \in \mathbb{Z}_+$, that is the trajectory $U(n,0)b$ is not asymptotically stable.

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References

[1] W. Arendt; *Vector-valued holomorphic and harmonic functions*, Concrete Operators, 2016, No. **3**, 68-76.

[2] S. Arshad, C. Buşe, A. Nosheen, A. Zada; *Connections between the stability of a Poincare map and boundedness of certain associate sequence*, Electronic J. Qual. Theory of Differential Equ., 2011, No. **16**, 1-12.

[3] S. Arshad, C. Buşe, O. Saierli; *Connections between exponential stability and boundedness of solutions of a couple of differential time depending and periodic systems*, Electron. J. Qual. Theory Differential Equ., **90**(2011), 1–16.

[4] D. Barbu, J. Blot, C. Buşe, O. Saierli; *Stability for trajectories of periodic evolution families in Hilbert spaces* Electron. J. Differential Equ. (2014), no. 1, 13 pp.

[5] D. Barbu, C. Buşe; *Asymptotic stability and uniform boundedness with respect to parameters for discrete non-autonomous periodic systems*, Journal of Difference Equations and Applications, Vol. **18**, Issue 9 (2012), 1435-1441.

[6] C. Buşe, P. Cerone, S. S. Dragomir, A. Sofo; *Uniform stability of periodic discrete systems in Banach spaces*, Journal of Difference Equations and Applications, Vol. **11**, No. 12, October 2005, 1081-1088.

[7] C. Buşe, S. S. Dragomir, V. Lupulescu; *Characterizations of stability for strongly continuous semigroups by boundedness of its convolutions with almost periodic functions*, Int. J. Differ. Equ. Appl. 2 (2001), no. 1, 103–109.

[8] C. Buşe, D. Lassoued, Lan Thanh Nguyen, O. Saierli; *Exponential stability and uniform boundedness of solutions for nonautonomous periodic abstract Cauchy problems. An evolution semigroup approach*, Integral Equations Operator Theory, **74** (2012), no. 3, 345–362.

[9] C. Buşe, D. O'Regan, O. Saierli; *An inequality concerning the growth bound of an evolution family and the norm of a convolution operator*, Semigroup Forum, Vol. **94** (2017), 618-631.

[10] C. Chicone, Y. Latushkin; *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, **70**, American Mathematical Society, Providence RI, (1999).

[11] C. Corduneanu; *Almost Periodic oscillations and waves*, Springer, New York, (2009).

[12] Daniel Daners, Pablo Koch Medina; *Abstract evolution equations, periodic problems and applications*, Pitman Research Notes in Mathematics Series, 279, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, (1992).

[13] K. Engel, R. Nagel; *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics 194, Springer, 2000.

[14] D. E. Evans; *Time dependent perturbations and scattering of strongly continuous groups on Banach spaces*, Math. Ann., **221** (1976), 275-290.

[15] L. Gearhart; *Spectral theory for contraction semigroups on Hilbert spaces*, Trans. Amer. Math. Soc. **236**(1978), 385-394.

[16] J. Goldstein; *Semigroups of Linear Operators and Applications*, Oxford University Press, New York, 1985.

[17] G. Greiner, J. Voight, M. P. Wolff; *On the spectral bound of the generator of semigroups of positive operators*, J. Operator Theory, **5** (1981), no. 2, 245-256.

[18] E. Hille, R. S. Philips; *Functional Analysis and Semi-Groups*, Coll. Publ. Am. Math. Soc. XXXI, Providence, RI, 1957.
[19] J. S. Howland; *Stationary scattering theory for time-dependent Hamiltonians*, Math. Ann., **207** (1974), 311-335.
[20] F. Huang; *Characteristic conditions for exponential stability of linear dynamical system in Hilbert spaces*, Ann. Diff. Equations, **1** (1985), 43–55.
[21] Y. Latushkin, S. Montgomery-Smith; *Evolutionary semigroups and Lyapunov theorems in Banach spaces*, J. Funct. Anal., **127** (1995), 173-197.
[22] A. Pazy; *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, 1983.
[23] Vũ Quoc Phong; *On stability of C_0-semigroups*, Proceedings of the American Math. Soc., Vol. 129, No. 10 (2001), 2871-2879.
[24] Jan Prüss; *On the spectrum of C_0-semigroups*, Transactions of the American Mathemathical Society, Vol. 284, No.2 (Aug. 1984), 847-657.
[25] O. Saierli; *Stability for periodic evolution families of bounded linear operators*, Survey in Mathematics and its Applications, Volume **10** (2015), 61-95.
[26] R. Schnaubelt; *Well-posedness and asymptotic behaviour of non-autonomous linear evolution equations*, Evolution equations, semigroups and functional analysis (Milano, 2000), 311–338, Progr. Nonlinear Differential Equations Appl., 50, Birkhäuser, Basel, 2002.
[27] G. Weiss; *Weak L^p-stability of linear semigroup on a Hilbert space implies exponential stability*, J. Diff. Equations, **76** (1988), 269-285.
[28] J. Zabczyk; *Mathematical control theory: An introduction*, Birkhäuser, Systems and Control, 1992.

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